Vertical motion with turbulent resistance in liquids and gasses

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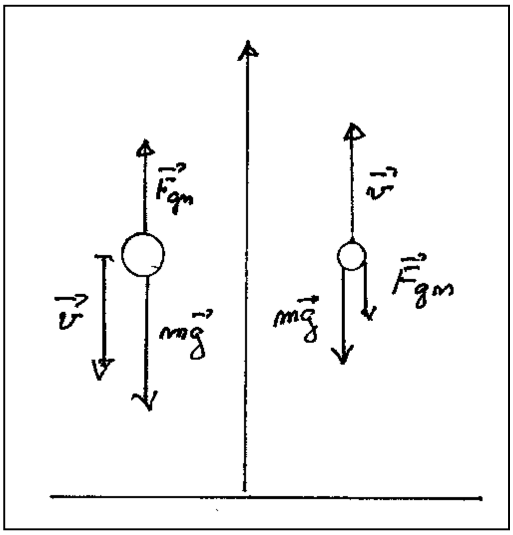
3. Vertical motion with turbulent resistance in liquids and gasses

In general the equations of motion cannot be solved in two dimensions, mostly because of the non linear $-v^2$ term in the drag force, but they can actually be solved analytically for a vertical motion.



We shall look at a vertical motion of a spherical body in a liquid or a gas, either falling, or moving upwards, as a consequence of buoyancy.
The body is in all cases influenced by

1. The gravitational force. (Always directed downwards)
2. The buoyancy. (Always directed upwards)
3. The viscous resistance. (Always directed against the velocity)

Whether the ball (bullet) sinks or moves upward depends on whether the density of the ball is larger or smaller than the density of the liquid (water) or gas (air).
For the viscous drag force, we shall apply the semi empiric expression.

(3.8) $$F_{visc} = \tfrac{1}{2} c_w \rho A v^2$$

ρ is the density of the liquid/gas, A is the area of cross section of the body, v is the velocity, and c_W is the so called dimensionless form factor. For convenience we put: $F_{visc} = cv^2$

An estimate of c_W can be found in a standard table of physical constants, where one can also look up the kinematics viscosity ν and the dynamic viscosity η.
The connection between the two viscosities is η = νρ.
c_W depends on the shape of the body, and the Reynold's number is defined as: $R = \frac{v \cdot D}{\nu}$.
Where v in the numerator denotes the speed, and ν i the denominator is the kinematics viscosity.
D is the linear extension of the body.

As we have already seen in section 3, the equations of motion can be solved, if we apply Stoke's law for the drag force $F_{stoke} = 6\pi\eta r v$, but the result for bodies with a diameter larger than a few centimetres and having a weight more than a 100 grams does not yield results in accordance with experience.
If the body moves upwards, as a consequence of the buoyancy, it will be influenced by the buoyancy, the gravity and the viscous resistance, the two latter having the same direction.

(3.9) $$F_{res} = F_T + F_{up} + F_{visc} \quad \Leftrightarrow \quad ma = -mg + m_v g - cv^2$$

In contrast, if the body sinks it is influenced by the same thee forces, but now the gravity and the drag force have opposite directions.

(3.10) $$F_{res} = F_T + F_{op} + F_{visc} \quad \Leftrightarrow \quad ma = -mg + m_v g + cv^2$$

m is the mass of the body and m_v is the mass of the displaced liquid according to Archimedes law.

3.2.1 Upward movement

(3.11) $$a = \frac{dv}{dt} = \frac{m_v - m}{m} g - \frac{c}{m} v^2$$

We put: $\mu = \frac{m_v - m}{m}$ in (3.11), and the equation is simplified to: $a = \frac{dv}{dt} = \mu g - \frac{c}{m} v^2 \quad \Leftrightarrow$

(3.12) $$\frac{dv}{dt} = \mu g \left(1 - \frac{c}{\mu g m} v^2\right)$$

The equation can be solved in a usual manner by separating the variables v and t followed by doing some integrals and rearranging the terms, but it is easier to notice that $(\tanh x)' = 1 - \tanh^2 x$

If we put: $k^2 = \frac{c}{\mu g m}$ the equation takes the form.

(3.13) $$\frac{dv}{dt} = \mu g (1 - (kv)^2)$$

And it is seen to have the solution

(3.14) $$v = \frac{1}{k} \tanh(\mu g k t)$$

Or, when the expressions for k and μ are reinserted.

(3.15) $$v = \sqrt{\frac{(m_v - m)g}{c}} \tanh \sqrt{\frac{c(m_v - m)g}{m^2}}\, t$$

tanh approaches rather quickly asymptotic to 1, e.g. *tanh*(1) =0.76 og *tanh*(2) =0.96.
The end velocity is seen to be:

(3.16) $$v_\infty = \frac{1}{k} = \sqrt{\frac{(m_v - m)g}{c}} \;,$$

as can also be inferred directly by putting: $\frac{dv}{dt} = 0$ in (3.13) , implying:

$$\mu g (1 - (kv)^2) = 0 \quad \Leftrightarrow \quad v = \frac{1}{k}$$

We may then apply the results to a beach ball, with a diameter of 0.30 *m*, and estimate how high it will jump, when it is hold under water and released.
You may find the form factor in a table of physical constants, and for a ball it is: $c_W = 0.2$.
For the ball in consideration it gives the value $c = 7.07$ *kg/m*, in the formula:

$$F_{visc} = \tfrac{1}{2} c_w \rho A v^2 = c v^2 .$$

If we in the formula for the velocity:

$$v = \sqrt{\frac{(m_v - m)g}{c}} \tanh \sqrt{\frac{c(m_v - m)g}{m^2}} t$$

We solve the equation $\sqrt{\frac{c(m_v - m)g}{m^2}} t = 2$, corresponding to 96% of the end velocity, we se that the ball will reach this value in fractions of a second, so we may safely use the end velocity in the calculations.

$$v_{end} = \sqrt{\frac{(m_v - m)g}{c}} = 4.4\ m/s\ .$$

So when a beach ball is held under water and released, the calculations show that it will jump to:

$$h = \frac{v^2}{2g} = 0.98\ m$$

For a ping pong ball with a radius 2 *cm*, and the mass 3.0 *g* the calculations goes as follows:
The Reynold's number: $R = \frac{v \cdot D}{\nu} = \frac{4 \cdot 0{,}04}{1.0 \cdot 10^{-6}} = 1.6\,10^7$ gives the form factor $c_W = 0.2$
$A = \pi\,(0.02)^2\ m^2 = 1.26\ 10^{-3}\ m^2$, $\rho = 10^3\ kg/m^3$.
$c = \tfrac{1}{2} c_w \rho A =$ $0.1 \cdot 10^3 \cdot 1.26\ 10^{-3}$ *kg/m* = 0.126 *g/m*. $m_v = \tfrac{4}{3}\pi r^3 \rho = 0.0335\ kg$, which gives the end velocity:

$$v = \sqrt{\frac{(m_v - m)g}{c}} = \sqrt{\frac{0.0305 \cdot 9.82}{0.126}}\ m/s\ =\ 1.54\ m/s$$

With this velocity the ping pong ball will, however, only jump: $h = \frac{v^2}{2g} = 0.12\ m$

3.2.2 Downward movement:

We shall look into the case of a body sinking in water. The equation of motion is.

(3.17) $$F_{res} = F_T + F_{op} + F_{visc} \Leftrightarrow \quad ma = -mg + m_v g + cv^2$$

The difference from above is, that the density of the body is larger than the density of water, so that $m > m_v$ i.e. the mass of the body is larger than the mass of the displaced water.
The equations of motion are otherwise the same, apart from a minus sign. The acceleration is:

$$a = \frac{dv}{dt} = \frac{m_v - m}{m} g + \frac{c}{m} v^2 = -\frac{m - m_v}{m} g + \frac{c}{m} v^2$$

We put: $\mu = \frac{m - m_v}{m}$.

(3.18) $$a = \frac{dv}{dt} = -\mu g + \frac{c}{m} v^2 \quad \Leftrightarrow \quad \frac{dv}{dt} = -\mu g (1 - \frac{c}{\mu g m} v^2)$$

And we get the same solution as in (3.15), apart from a change in sign.

As before we put: $k^2 = \frac{c}{\mu g m}$, which gives the equation:

(3.19) $$\frac{dv}{dt} = -\mu g (1 - (kv)^2)$$

Having the solution:

$$v = -\frac{1}{k} \tanh(\mu g k t) \qquad \Leftrightarrow \qquad v = -\sqrt{\frac{(m - m_v) g}{c}} \tanh \sqrt{\frac{c(m_v - m) g}{m^2}} t$$

If we for example look at an iron ball with radius 5 cm , and density $\rho = 7.8\ 10^3\ kg/m^3$, the constant $c = \frac{1}{2} c_w \rho A = 0.79$ (SI-units), $m_v = \rho_{water} V_{ball} = 1.0\ 10^3 \cdot 4/3(5\ 10^{-2})^3\ kg = 0.524\ kg$, $m = \rho_{iron} V_{ball} = 7.8\ 10^3\ 4/3(5\ 10^{-2})^3\ kg = 4{,}1\ kg$. From which we get the end velocity.

$$v_\infty = -\sqrt{\frac{(m - m_v) g}{c}} = -6{,}7\ m/s = -24\ km/h$$

3.2.3 Vertical motion in air

For motion of a body in air, we need not to be concerned with the buoyancy, since it is vanishing compared to the gravity and drag forces. The equations of motion are therefore.

Upward: $$F_{res} = F_T + F_{luft} \quad \Leftrightarrow \quad ma = -mg - cv^2$$

(3.20)

Downward: $$F_{res} = F_T + F_{luft} \quad \Leftrightarrow \quad ma = -mg + cv^2$$

First we solve for the upward movement:

$$a = \frac{dv}{dt} = -g - \frac{c}{m} v^2 \quad \Leftrightarrow \quad \frac{dv}{dt} = -g (1 + \frac{c}{mg} v^2)$$

Setting $k = \sqrt{\frac{c}{mg}}$ gives:

(3.21) $$\frac{dv}{dt} = -g (1 + (kv)^2)$$

As mentioned earlier, (3.21) can be solved by separating the variables followed by integration, but it is easier to notice that (tan x)' =1+tan^2x, (which correspond to the right hand side), and then try with a parametized solution:

$$v = -a\tan bt$$

Differentiating we find:

$$\frac{dv}{dt} = -a(1+\tan^2 bt)b$$

Which we compare this to: $\frac{dv}{dt} = -g(1+(kv)^2)$

we see that

$$kv\tan bt \;\Rightarrow\; v = \frac{1}{k}\tan bt$$

so

$$a = -\frac{1}{k} \quad \text{and consequently:} \quad ab = -g \;\Rightarrow\; b = kg\,.$$

The solution then becomes:

(3.22) $$v - v_0 = -\frac{1}{k}\tan(kgt)$$

where $k = \sqrt{\frac{c}{mg}}$ and $c = \frac{1}{2}c_w\rho A$

If $kgt << 1$ then tan(kgt) $\approx kgt$, and the formula goes into $v = v_0 - gt$, as it should.

If we have a ball with radius r = 0.05 m and mass m = 250 g, c = 2.0 10^{-3} kg/m and k = 0.0285 s/m,
And if the ball has an initial velocity 5,0 m/s, we can determine the max height, by solving the equation
$v = 0 \Leftrightarrow \tan gkt = kv_0$, which gives: t =0.51 s.
If we want to find the max height, we must integrate (3.22) to give

(3.23) $$s - s_0 = -\frac{1}{k^2 g}\ln(\cos(gkt))$$

When we calculate the distance, with v_0 = 5.0 m/s, it is only a correction on the second decimal compared to a vertical throw without air resistance.

We shall then deal with a free fall with air resistance.

$$F_{res} = F_T + F_{luft} \;\Leftrightarrow\; ma = -mg + cv^2 \quad \Leftrightarrow$$

$$-\frac{dv}{dt} = -g + \frac{c}{m}v^2 \quad \Leftrightarrow \quad \frac{dv}{dt} = -g(1 - \frac{c}{mg}v^2) \quad \Leftrightarrow \quad \frac{dv}{dt} = -g(1-(kv)^2)$$

Where we have put: $k^2 = \frac{c}{mg}$.

The last equation has the same solution as (3.15) and (3.19).

(3.24) $$v = -\frac{1}{k}\tanh(gkt)$$

With the end velocity $v_\infty = -\frac{1}{k} = -\sqrt{\frac{mg}{c}}$.

Inserting c = 8.1 10^{-3}, corresponding to a ball with radius r = 0.10 m, and density 1.0 10^3 *kg/m*^{3}, we find: v_{end} =226 *m/s.*

The distance completed can be calculated by integrating (3.24) to give: $s - s_0 = \frac{1}{gk^2}\ln(\cosh(gkt)$

The end velocity occurs when: gkt =2, which gives: $t = 1/gk$ = 46 s, and this corresponds to the distance: $s - s_0$ = 5200 m.

I will not be held responsible, whether these results are in accordance with reality. Firstly the v^2 dependence is not necessarily correct, and the form factor is only fixed by a factor of 2.