

# Simulations of three body motion

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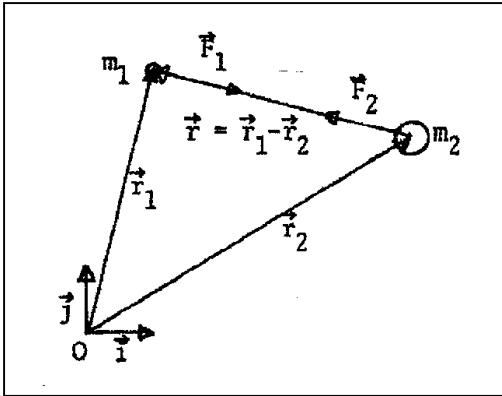
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## 1. Newton's law of gravitation

The gravitational force between two masses  $m_1$  and  $m_2$  separated by the distance  $r$  is according to Newton's law of gravitation:

$$(1.1) \quad F_G = G \frac{m_1 m_2}{r^2}$$



Writing the force in vector form, we have according to the figure to the left:

$$\vec{F}_1 = -G \frac{m_1 m_2}{r^2} \frac{\vec{r}}{r} = -G \frac{m_1 m_2}{r^3} \vec{r}$$

If the two masses have the coordinates:  $(x_1, y_1)$  and  $(x_2, y_2)$ , then the force, written in coordinates takes the form:

$$F_{1x} = G \frac{m_1 m_2 (x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3} \quad \text{and} \quad F_{1y} = G \frac{m_1 m_2 (y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3}$$

And subsequently the acceleration of the mass  $m_1$  becomes:

$$\ddot{x}_1 = G \frac{m_2 (x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3} \quad \text{and} \quad \ddot{y}_1 = G \frac{m_2 (y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3}$$

We shall repeatedly apply these expressions in the following.

## 2. The gravitational equations of motion for a three body system

For a three body system having the three masses  $m_1$ ,  $m_2$  and  $m_3$ , and where the masses placed in  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  the equations of motion become six coupled second order differential equations.

$$\ddot{x}_1 = Gm_2 \frac{(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3} + Gm_3 \frac{(x_3 - x_1)}{\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}^3}$$

$$\ddot{y}_1 = Gm_2 \frac{(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3} + Gm_3 \frac{(y_3 - y_1)}{\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}^3}$$

$$\ddot{x}_2 = Gm_1 \frac{(x_1 - x_2)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3} + Gm_3 \frac{(x_3 - x_2)}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}^3}$$

$$\ddot{y}_2 = Gm_1 \frac{(y_1 - y_2)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3} + Gm_3 \frac{(y_3 - y_2)}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}^3}$$

$$\ddot{x}_3 = Gm_1 \frac{(x_1 - x_3)}{\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}^3} + Gm_2 \frac{(x_2 - x_3)}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}^3}$$

$$\ddot{y}_3 = Gm_1 \frac{(y_1 - y_3)}{\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}^3} + Gm_2 \frac{(y_2 - y_3)}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}^3}$$

That there is no way to find a general analytic solution to the three body problem is well known, and it has already been proved by the French physicist and mathematician Poincaré.

In finding the solution to the two body problem, it was transformed into a central motion of the reduced mass orbiting around the centre of mass of the two bodies. A similar approach is, however, not possible, with regard to the three body problem. Almost 300 years of mathematical effort has ever resulted in presenting an analytical solution to the three body problem, neither establishing condition for stability of such a system.

The only way to handle celestial 3 body problems has been using numerical methods – and until the mid sixties – doing it by hand. When I started at the H.C. Oersted institute in 1964, the institute had one and only one (Danish built) computer, having 1 kb magnetic core store, working together with a magnetic drum.

The computer consisting of 4 large cupboards occupied about 40 square meters.

I believe that the computer worked more 1000 times slower, than an ordinary Laptop computer nowadays.

Although it was a formidable progress no longer to solve differential equation by hand (using at mechanical calculator), however, it lasted another 20 years before the graphic colour screens were at large.

The Danish computer scientist and programmer Anders Hejlsberg was the first to make a MS-DOS based Pascal compiler (PolyPascal) in the mid 80'ties, but few years later he was headhunted to Borland in the US, where he became the chief architect on the famous Borland Turbo-Pascal series 2.0 – 7.0, which had full graphic screen (and from the 5.5 version also colour) capacities, as well as a most effective compiler and running on a IDE (Integrated Development Environment). Something that the Java community 10 years later never achieved

Later Hejlsberg developed the notorious Delphi (2.0 – 8.0) presented in 1995, with a IDE (integrated developing environment) having built in –and ready to use - MS Windows components, which were the same that appeared in Windows 95 and later releases of Windows including (the rather well working) Windows XP.

The reason for this brief historical survey, is that for a physics and mathematical student from the mid sixties it was a revelation to be able to make graphs of functions, doing differentiation and evaluate integral, solving transcendent equation, and especially to solve several coupled differential equations of first or second order and presenting the solutions directly on the graphics screen in a matter of seconds.

However the graphics tool could only set points, draw line sections, squares, and ellipses filled or not filled, together with drawing text of variable size. So before making a general program, one had to make a windows interface with menus by hand, if the program should be able to handle not only one problem, but any function or differential equation. All input had to be text files, which had to be interpreted before they were processed.

It took me about a year to accomplish such a program, which finally had ten 64 kb units and filled 75 pages.

The program ran perfectly on Win95 and Win98. Especially because it was possible to take a screen dump from a DOS-screen, and insert it directly in a Word document. The program could still run on Windows XP, but no longer with the capacity of making a screen dump, which of course raised a problem. But after Windows XP, DOS-graphics program can no longer run, and neither can Delphi by the way.

Fortunately I have always kept a stationary computer from the nineteen's running on Windows 98, so I can still get the graphics screens into Word.

For more than two decades I have used this more than 20 years old program in my articles on physics and mathematics. It can (among many other things) solve up to 6 second order differential equations, draw proper coordinate system, whatever the axis are, and display in both 2 and three dimensions.

Although I only programmed in Delphi from the mid nineteen's, I never had the urge to transform the old DOS program into Delphi.

Although there has been developed several mathematical programs since 2000, I have never encountered one which has the same capabilities, as my turbo Pascal program from the beginning of the 90'ties. All the graphs in his article are done with my Turbo Pascal program.

It is well known for example from the motion of the planets and their moons that the three body (or he many body problem) is stable if one of the bodies is much larger than the others and if the trajectories of the minor bodies do not come to close to each other.

For example the Helium atom (from a classical calculation) is only stable, if the two electrons have the same radius in their orbits and move diametrically opposite. The reason for this is of course that the mutual repulsion between the electrons is of the same magnitude as their attraction from the nucleus.

This is, however, very different from gravitation, since the attractive force is proportional to the mass.

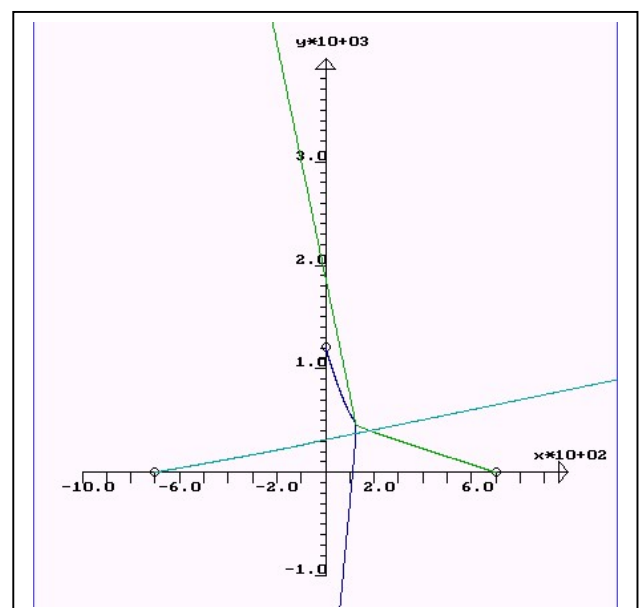
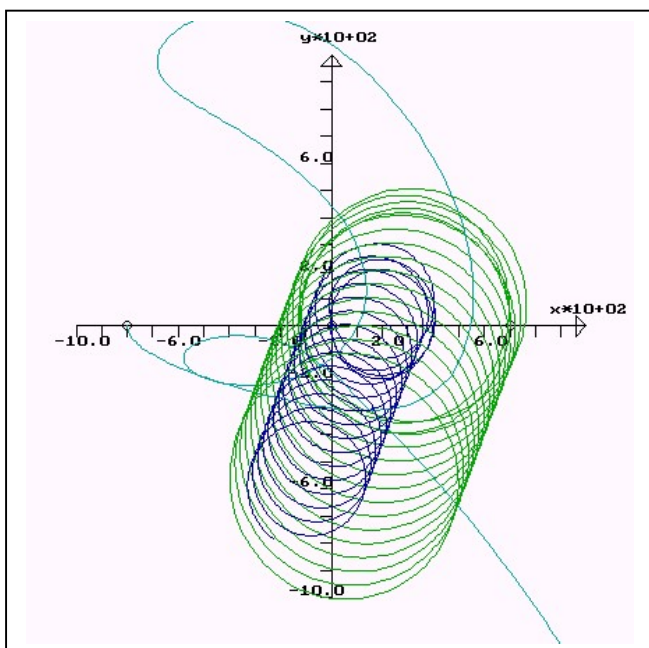
Below we shall present several examples of stable as unstable gravitational three body systems.

It is not possible, however, to give strict conditions for stability, since unstable three body problems tend to develop chaotically.

All distances are measured in  $Mm$  and velocities in  $Mm/h$ .

In these units the value of  $GM_{earth} = 5169$  and  $GM_{moon} = 63.6$

## 2.1 Unstable three body problems

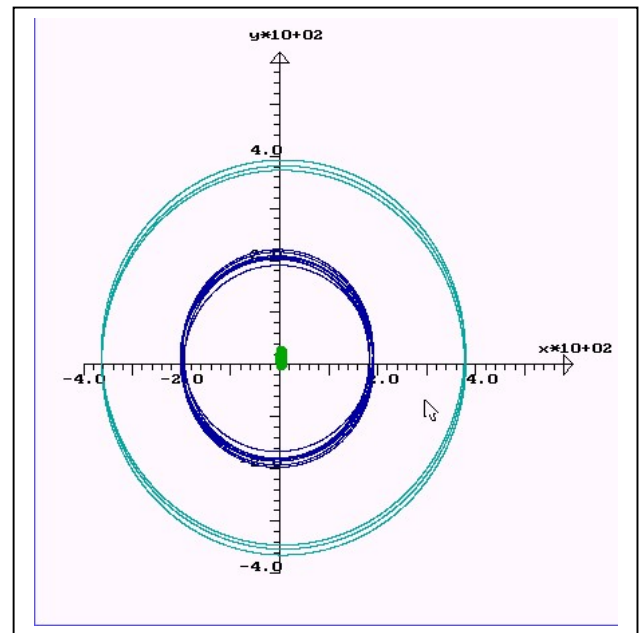
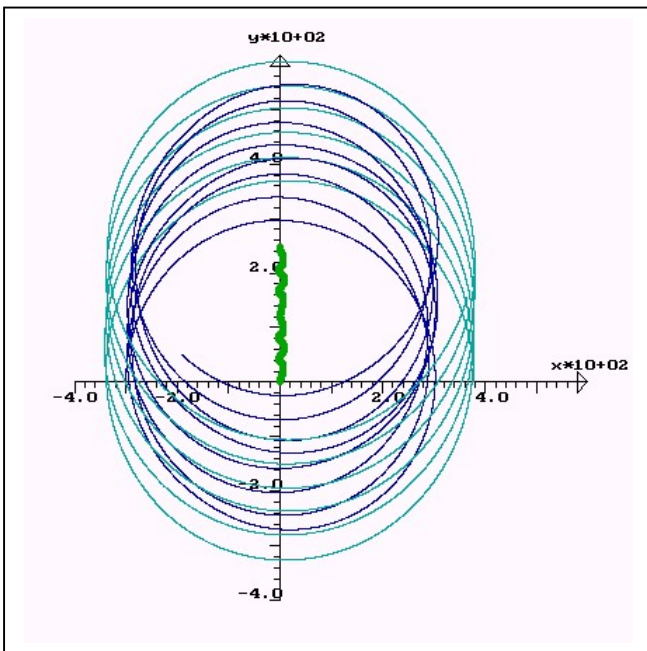


In the figure to the left there are three masses, which are comparable to the mass of the earth. The three masses are placed in  $(-800,0)$ ,  $(700,0)$  and  $(0,0)$  and given initial velocities so they might exhibit circular motions around  $(0,0)$ . But as you can see the mass to the left enters from the beginning in a chaotic motion and is no longer bound to the two others masses. The two heavier masses continue in circular motions with a common linear velocity.

With this choice of masses, there is no way to adjust the initial velocities, so that the system remains stable.

In the figure to the right, the three masses are placed in the same positions, but without initial velocities. Not surprisingly they move by straight lines except, when two of the masses come near to each other.

## 2.2 Examples of stable planetary motion



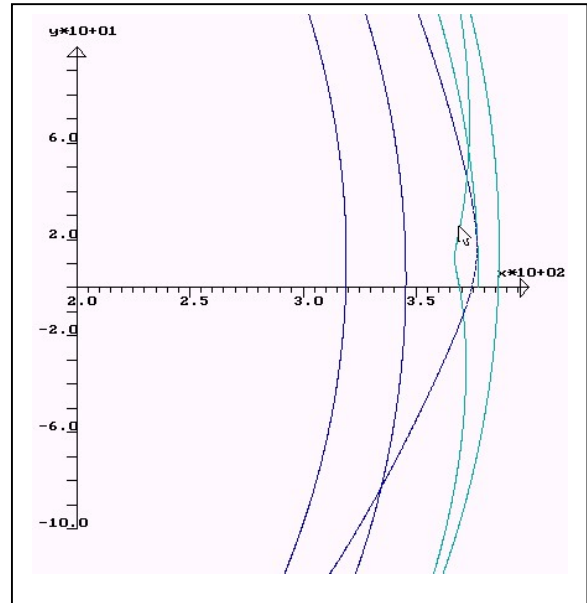
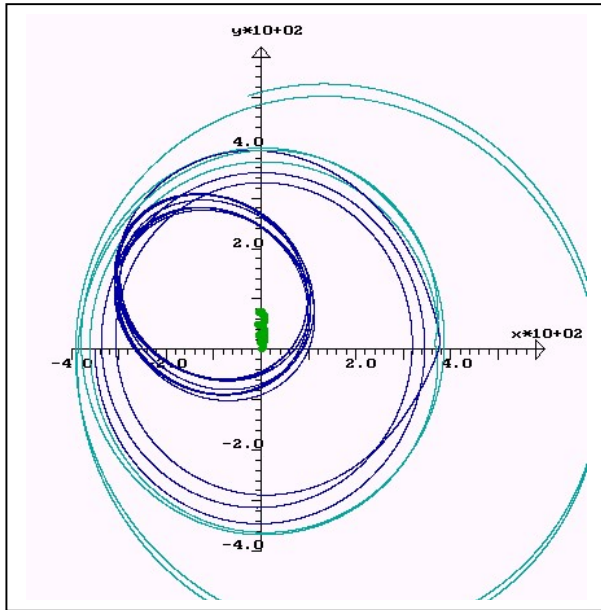
In the figures above we have a system consisting of the earth (the thick green dot) and two planets with masses of the moon and half the moon, each beginning a circular motion with opposite directions. Notice that the earth, initially at rest, also moves upward as the system as a whole. The system is stable, since the two satellite trajectories never cross.

In the figure to the right the two satellite trajectories are farther apart, and they circulate the same way. Since one of the satellites is above, while the other is below, the motion of the earth is less significant than in the former case. The system is stable.

## 2.3 An example of an unstable two planetary motion.

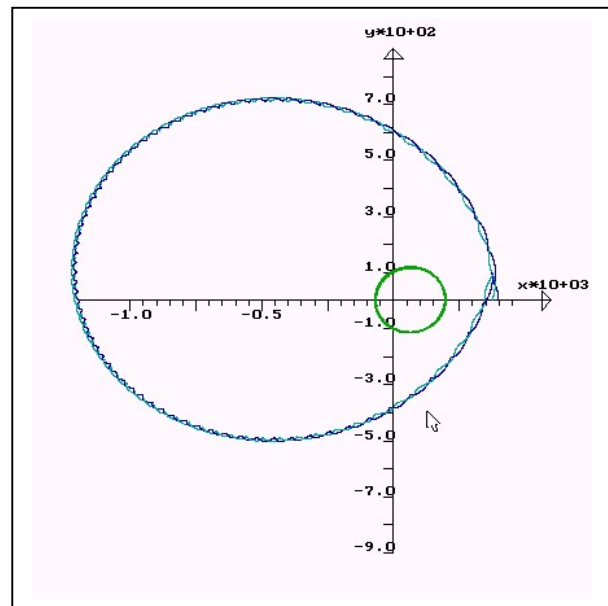
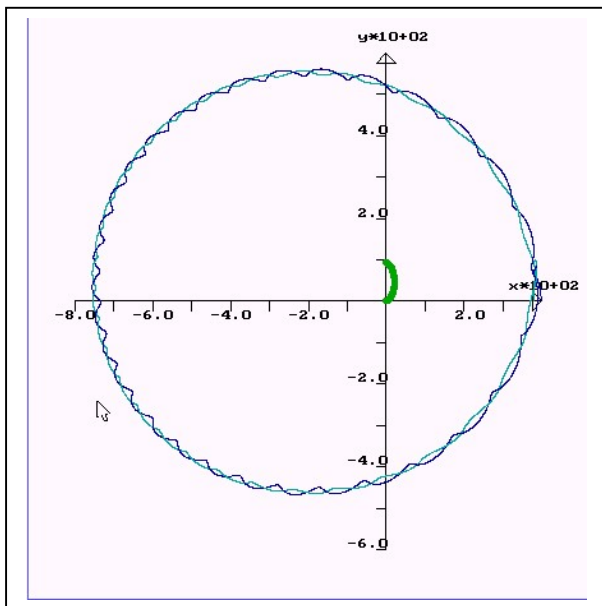
In the figure below two satellites are launched on each side of the earth, such that both perform a uniform circular motion. In this case however, the two trajectories are rather close, and as you can see from the figure the two planets meet and interact, and the trajectories are distorted, the blue planet moves in to an orbit closer to the earth, whereas the green planet takes a larger orbit. Eventually the green planet will disappear to infinity.

The point of interaction between the two planets is enlarged in the figure to the right, where both planets make a turn, as a consequence of their mutual attraction



### 2.4 Earth with the moon, which itself has a satellite

Below are two simulations with the earth, and the moon, with itself has a satellite having half the mass of the moon. There are only minor differences in the two simulations. In the figure to the right is also shown the relative motion of the moon and its satellite, scaled by a factor 100.

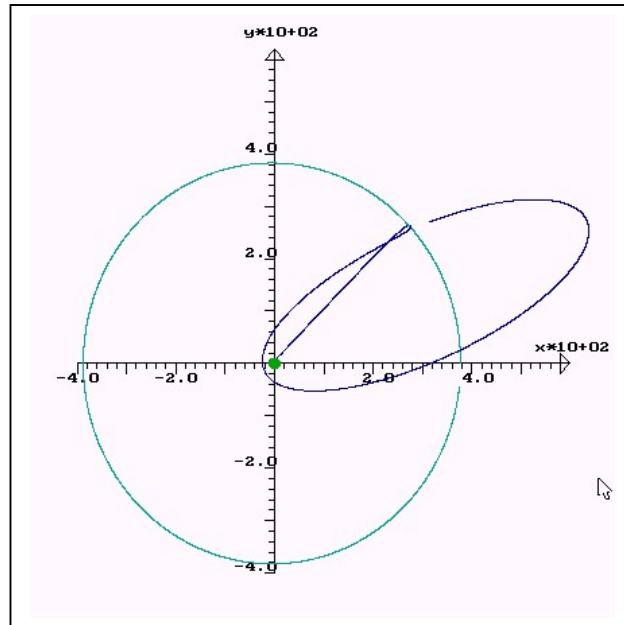


### 2.5 A ballistic rocket to the moon

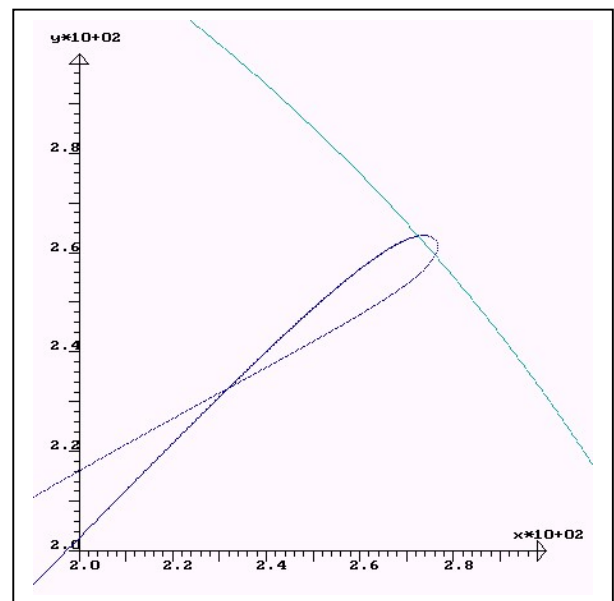
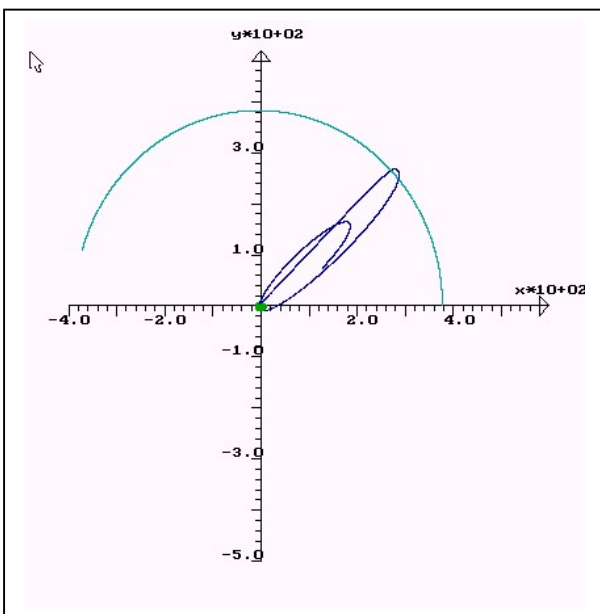
It is an amusing fact, which can hardly be supported theoretically, that it is possible to launch a ballistic rocket from the earth which encounters the moon, make a circle around the moon and returns to the earth.

The computer simulation below shows, however, that it actually possible, although it takes many attempts adjusting the velocity of the rocket, and the launching angle of inclination with the earth.

In the first figure the rocket does not return to the earth, but enters an orbit in a perfect ellipse, (as it should). The motion of the earth is due to the moon of course.



Below in the figure to the left the rocket actually hits the earth, but it has no damaging consequence since the earth is represented by a point in the differential equations. However the gravitational attraction becomes very strong near the centre of the earth. In the figure to the right is enlarged the rockets interaction with the moon.





## 2.6 Almost stable three body motion, having three equal masses

The articles about the three body problem are actually remarkably scarce, when you search on the Internet. (They are almost drowned in references to the novel by the same name).

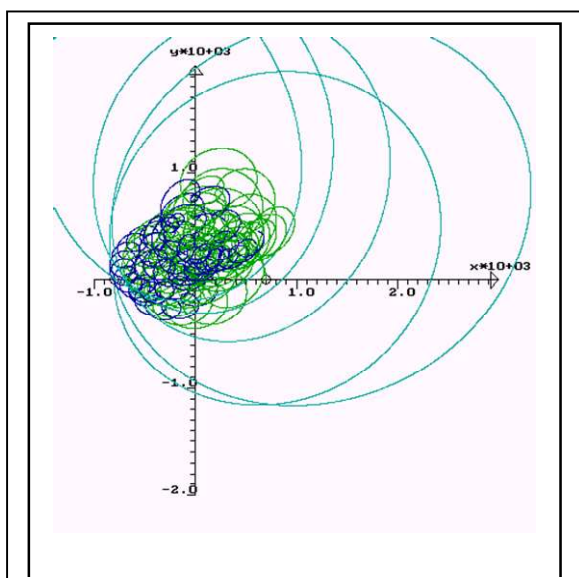
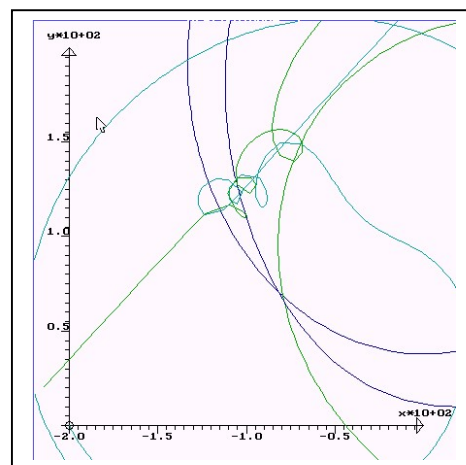
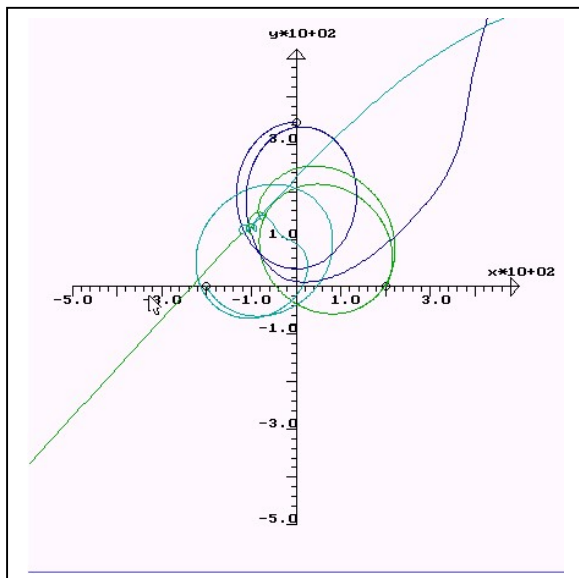
It seems, however, to be an established (theoretical?) fact that for three different masses of the same magnitude there exist no stable configuration.

However, already Euler found theoretically a stable configuration for three equal masses placed in the corners of an equilateral triangle, and the literature about the three body problem is mostly occupied with the many (academic, theoretical) stable solutions of the three body problem for three equal masses.

I have therefore tried to make a computer simulation of such configuration. My experience is, however, that they are unstable equilibriums, as when you balance a pendulum upside down.

Even a very small deviation from the equilibrium state develops chaotically, which is also the essence of the (now almost abandoned interest in) chaos theory, which was otherwise a popular hype in the 70'ties and in the 80'ties.

In my first attempt I placed three equal masses in the corners of an equilateral triangle and gave them an initial velocity to perform a uniform circular motion from the attraction from the centre of mass of the other two masses. The simulation is shown below.



The first round for the three bodies seem to work (almost) but already in the second round the motion becomes chaotic, as seen on the zoom in, on the figure to the right. An adjustment of the velocities do not substantially improve the general picture.

After a while the motion of the three masses are as shown to the left. Chaotic Yes!

Another idea, where the symmetry is complete, is to let the three masses rotate uniformly on the circumscribed circle of the equilateral triangle keeping their relative positions. This means that the triangle itself rotate uniformly.

This configuration gives a result that is certainly better than the former simulation, but after two rounds the minor differences in positions and speed will eventually lead to chaotic motions.

Below is shown the first two rounds, but as you can see the green and the light blue trajectory are already going off piste.

In the next figure the motion is already chaotic, since also the blue trajectory leaves the circular motion. An enlarged figure of where the chaotic motion is initiated is shown in the figure below to the left.

In the figure to the right, there is no longer any reminiscence of a uniform circular motion.

