The Wave Equation

The speed of propagation of transversal and longitudinal waves on strings, in solids and in gasses

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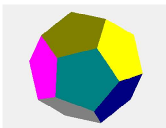

Ole Witt-Hansen (2008) 2016

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1. The wave equation

Any one dimensional plane harmonic wave, which propagates along an x–axis, can be described by the following expression.

(1.1) $$u(x,t) = A\cos(\omega t - kx + \varphi_0)$$

$u(x,t)$ is the displacement at x and time t. A is the amplitude, ω is the angular frequency, and k is the wave number. $\omega t - kx + \varphi_0$ is called the phase, and φ_0 is the initial phase.
The equation (1.1) is based on two simple observations on waves:

1. They propagate with a constant speed v.
2. the shape of the wave is unchanged in its propagation

The expression for a harmonic wave can be derived, if we assume that we have a harmonic oscillation, at $x = 0$:

$$u(0,t) = A\cos(\omega t + \varphi_0)$$

According to (1) and (2) we therefore state that the displacement at position x and time t *is the same displacement that we had at $x = 0$, at an earlier time, namely the time it takes the wave to propagate from $x = 0$ to x.* Written formally:

(1.2) $$u(x,t) = u(0,t - \frac{x}{v}) \quad \Rightarrow \quad u(x,t) = A\cos(\omega(t - \frac{x}{v}) + \varphi_0) = A\cos(\omega t - \frac{\omega}{v}x + \varphi_0)$$

Introducing the wave number: $k = \frac{\omega}{v}$ we arrive at (1.1).
There are some relations between the various symbols: v (the speed), λ (The wavelength) , ω (the angular frequency), ν (the frequency) and T (the period) that can easily be verified.

(1.2) $$\omega = \frac{2\pi}{T}, \quad k = \frac{2\pi}{\lambda}, \quad v = \lambda \cdot \nu \Leftrightarrow v = \frac{\lambda}{T} \Leftrightarrow v = \frac{\omega}{k}$$

If v is the speed of propagation, then any more general (not necessarily harmonic) wave phenomena, where $u(0,t) = f(t)$, it can be represented by a function,

(1.3) $$u(x,t) = u(0,t - \frac{x}{v}) = f(t - \frac{x}{v})$$

Where f is the "shape" of the wave.

All one dimensional wave phenomena satisfy the so called *wave equation*.

(1.4) $$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$$

The symbol ∂ denotes partial differentiation of a function of several variables.

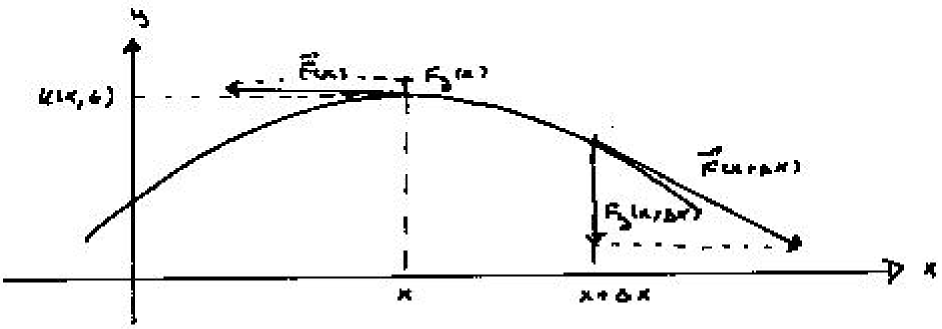
Partial differentiation with respect to one variable is the same as ordinary differentiation, but where all the other variables are considered as constants.
That both of the expressions (1.1) and (1.3) satisfy the wave equation is straightforward to verify by differentiation.
On the other hand, if a physical phenomena, which can be described by a function, $u(x,t)$ satisfying (1.4) written on the form

(1.5) $$\frac{\partial^2 u}{\partial x^2} - k^2 \frac{\partial^2 u}{\partial t^2} = 0$$

Then it is a wave phenomena propagating with a speed $v = 1/k$.

In the following, we shall consider various physical phenomena and show that they satisfy the wave equation (1.4), and in this manner, we shall determine their speed of propagation.

2. The speed of propagation for waves on an elastic string



We begin by considering a small piece of a string Δx, between x and $x+\Delta x$.
The string is stretched along the x-axis, and the vibrating displacement is along the y–axis, so $u(x,t) = y(x,t)$.
If we neglect gravity, the piece Δx of the string is affected only by the forces $F(x)$ *and* $F(x+\Delta x)$, both directed along the tangent to the string.
The component of the forces in the y-direction, can be determined as $F\sin\theta$, where $\tan\theta$ is the slope of the tangent. For small θ, however $\sin\theta \cong \tan\theta = \frac{\partial u}{\partial x}$ (the tangents slope).
For the resulting force on Δx, in the y-direction, we thus find:

(2.1) $$F_y = \Delta F = F_y(x+\Delta x) - F_y(x)$$

We have ignored the small differences in the tension F that there might be at the end points of the piece Δx of the string. Therefore the components of F in the x-direction cancel. So we have:

(2.2)

$$\Delta F = F\frac{\partial u}{\partial x}\bigg|_{x+\Delta x} - F\frac{\partial u}{\partial x}\bigg|_{x} \approx F\frac{\partial^2 u}{\partial x^2}\Delta x$$

According to Newton' second law, the resulting force is equal to the mass of a body times its acceleration. If the mass per unit length is μ, then $m = \mu\Delta x$. We thus find:

(2.3)
$$\begin{aligned} &Force \quad = mass\cdot acceleration \\ &F\frac{\partial^2 u}{\partial x^2}\Delta x = \mu\Delta x\frac{\partial^2 u}{\partial t^2} \Rightarrow \\ &\frac{\partial^2 u}{\partial x^2} - \frac{\mu}{F}\frac{\partial^2 u}{\partial t^2} = 0 \end{aligned}$$

Comparing this expression to the wave equation (1.4), we see that, the speed of propagation v on the string must be:

(2.3)
$$\frac{1}{v^2} = \frac{\mu}{F} \quad \Rightarrow \quad v = \sqrt{\frac{F}{\mu}}$$

Which was what, we set out to establish.

3. The speed of propagation for longitudinal waves in solids

The formula for the speed of propagation, for longitudinal waves in elastic solids is based on Hooke's law.

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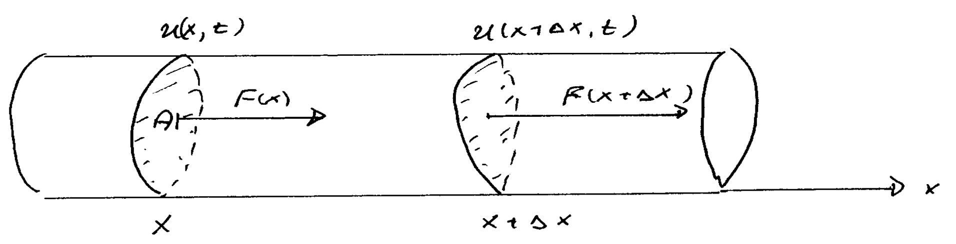
If we have an elastic material with a cylindrical cross section A, length L, which is affected by a force F, and therefore is compressed/prolonged an amount x, Hooke's law states:

(3.1)
$$F = E\frac{A}{L}x$$

E is a material constant, and is called Young's module. (Sometimes written as Y)
We then consider a small piece of such a material, which is supposed to comply with Hooke's law. If a wave is propagating we have a dynamic state, and the forces $F(x)$ and $F(x+\Delta x)$, are not the same. See the figure below.



The displacement from the position of stability, we shall as before denote as $u(x,t)$.
The displacement is now along the direction of propagation of the wave, that is, in the x direction. Otherwise the reasoning is the same as for the transversal waves on a string.

We calculate the resulting force on the segment Δx, and puts it equal to the mass times the acceleration of the piece of material that is situated between x and $x+\Delta x$.

To establish the equation we apply Hooke's law with $L = \Delta x$ and $x = u(x+\Delta x,t) - u(x,t)$
The length of material in consideration is Δx, and the deformation of Δx is the difference between the displacements at $x+\Delta x$ and x

(3.2)
$$F(x) = \frac{EA}{\Delta x}(u(x+\Delta x,t) - u(x,t)) \approx EA\frac{\partial u}{\partial x}$$
$$F_{res} = F(x+\Delta x) - F(x) = \frac{\partial F}{\partial x}\Delta x = EA\Delta x\frac{\partial^2 u}{\partial x^2}$$

If ρ is the density of the material, and m is the mass situated between x and $x+\Delta x$, then $m = \rho A\Delta x$.
Newtons 2. law: $F_{res} = ma$ therefore gives::

$$F_{res} = EA\Delta x\frac{\partial^2 u}{\partial x^2} = \rho A\Delta x\frac{\partial^2 u}{\partial t^2} \Rightarrow$$
(3.3)
$$\frac{\partial^2 u}{\partial x^2} - \frac{\rho}{E}\frac{\partial^2 u}{\partial t^2} = 0$$

If we compare (3.3) with the wave equation (1.4), we find the wanted expression for the speed of propagation of waves in elastic solids.

(3.4)
$$v = \sqrt{\frac{E}{\rho}}$$

4. The speed of sound in gasses

A more direct derivation of the formula for the speed of sound is rather complicated. Here we shall adapt a derivation, using an analogy and comparison to result for the speed propagation in solids.

First we rewrite Hooke's law in differential form, since our aim is to obtain a link between a small change ΔV of the volume, resulting from a small change of the pressure ΔP.

$$F = E\frac{A}{L}x \quad \Rightarrow \quad \Delta F = -E\frac{A}{x}\Delta x \quad \Rightarrow \quad \frac{\Delta F}{A} = -E\frac{A\Delta x}{xA} \quad \Rightarrow \quad \Delta P = -E\frac{\Delta V}{V} \Rightarrow$$

(4.1)
$$\frac{\partial P}{\partial V} = -\frac{E}{V} \quad \Leftrightarrow \quad E = -V\frac{\partial P}{\partial V}$$ (Where E is "Young's module")

The minus sign in the expression comes about, because ΔP and ΔV have opposite signs.

From the last equation, we get an expression for the module E. (Which no longer can be identified with Young's module). Inserting (4.1) into (3.3), we obtain an equation, from which we may determine the "speed of sound".

But first we shall then apply (4.1) to an ideal gas, which has the familiar equation of state.

(4.2) $$PV = n_M RT$$

The relation $PV = const$ between pressure P and volume V, holds good for isotherm changes. It is called Boyle-Mariotte's law:

For several decades however, this was a challenge for the physicists since, if you apply Boyle-Mariotte's law in (4.1), you get a result about 20% off for the speed of sound, when compared to measurements.

But as Newton pointed out, the cause of the discrepancy with experiment was that when sound propagates in air the changes in pressure and volume are so fast that they are not *isotherm*, but *adiabatic*, that is, heat isolated. (No time for countervailing of temperature)

Therefore we cannot use Boyle-Mariotte's law, but must apply the corresponding, and slightly more complicated relation between pressure P and volume V, valid for *adiabatic* changes.

To derive that relation we take as a starting point the first law of thermodynamic, written in differential form. The change of energy of a system is the added heat plus the work done on the system.

(4.3) $$dE = dQ + dA$$

For an ideal gas, the energy only depends on the temperature T, and is given by the expression:

(4.4) $$E_{kin} = \gamma NkT = \gamma n_M RT \quad \Rightarrow \quad dE = \gamma n_M RdT$$

N is the number of molecules, k is Boltzmann's constant, n_M is the mole number and R is the gas constant. γ is a constant, which depends of the nature of the gas. It has the value 3/2 for one atomic gasses.

"The piston work" is $dA = -PdV$, and adiabatic means that $dQ = 0$. From this follows:

(4.5) $$dE = dA \quad \Rightarrow \quad \gamma n_M RdT = -PdV \quad \Rightarrow \quad \gamma n_M RdT + PdV = 0$$

Inserting (4.5) in the equation of state for ideal gasses in differential form

$$d(PV) = n_M RdT \quad \Leftrightarrow \quad PdV + VdP = n_M RdT \quad \Leftrightarrow \quad \gamma PdV + \gamma VdP = \gamma n_M RdT$$

we get

$$\gamma PdV + \gamma VdP + PdV = 0 \quad \Rightarrow \quad \frac{(\gamma+1)}{\gamma}\frac{dV}{V} + \frac{dP}{P} = 0 \quad \Rightarrow$$

(4.6) $$\kappa\frac{dV}{V}+\frac{dP}{P}=0 \qquad \text{where } \kappa=\frac{(\gamma+1)}{\gamma}$$

Integrating the last equation:

(4.7) $$\ln V^{\kappa}+\ln P=const \quad \Leftrightarrow \quad PV^{\kappa}=const$$

This constitutes the wanted adiabatic relation to be applied in (4.1) $E=-V\dfrac{\partial P}{\partial V}$.
Taking the differential of (4.7) it gives:

(4.8) $$d(PV^{\kappa})=V^{\kappa}dP+\kappa PV^{\kappa-1}dV=0 \quad \Rightarrow \quad \frac{\partial P}{\partial V}=-\kappa\frac{P}{V}$$

And finally:

(4.9) $$E=-V\frac{\partial P}{\partial V}=\kappa P$$ (Where E is "Young's module")

Then we express the pressure P, with the help of the equation of state for gasses.

(4.10) $$P=\frac{n_M RT}{V}=\frac{n_M MRT}{VM}=\frac{m}{V}\frac{RT}{M}=\rho\frac{RT}{M}$$

(4.9) and (4.10) are then used to insert in the formula (3.3) for the speed of propagation in solids.

(4.11) $$v=\sqrt{\frac{E}{\rho}}=\sqrt{\frac{\kappa P}{\rho}}=\sqrt{\frac{\kappa\rho\frac{RT}{M}}{\rho}}=\sqrt{\frac{\kappa RT}{M}}$$

We have then arrived at the wanted expression for the speed of sound, which turns out to be in perfect accordance, with measurements.

(4.11) $$v=\sqrt{\frac{\kappa RT}{M}} \qquad \text{where} \quad \kappa=\frac{\gamma+1}{\gamma}$$

γ can be found in a table of physical constants. For atmospheric air it is $\gamma=1.4$