The two-body problem
and
The laws of Kepler

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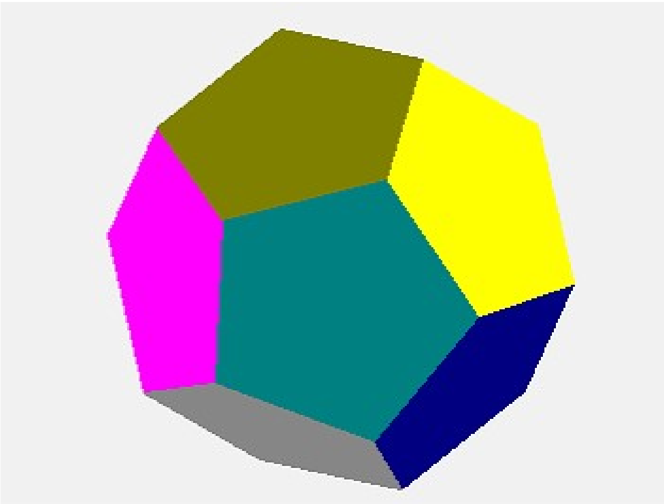

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1. The two-body problem. Reduction to a motion in a central field

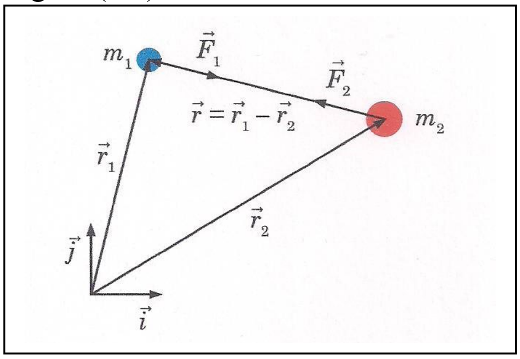
The two-body problem deals with the description of two bodies, where their mutual attraction (or repulsion) is given by a force, which depends only on the distance r between the two bodies.
The force can either be the gravitational force F_G between to masses m_1 and m_2 given by Newton's law of gravitation, where r is the distance between the (centres of the two spherical) bodies.

(1.1) $$F_G = G\frac{m_1 m_2}{r^2}, \quad \text{where } G = 6.67\ 10^{-11}\ Nm^2/kg^2,$$

Or the force may be the Coulomb force F_C between to charges Q_1 and Q_2.

(1.2) $$F_C = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r^2}, \quad \text{where } \frac{1}{4\pi\varepsilon_0} = 9.010^9\ Nm^2/C^2$$

It turns out that the two body problem can be reduced by solving a differential equation for the motion of a particle in a central field. This means that the force on the particle depends only on the distance of a fixed point.
If one of the two bodies is much heavier than the other, the motion of the smaller body will, (to a good approximation) correspond to a motion in a central field e.g. the moon and the earth or the planets and the sun. In the following, we shall often designate differentiation with respect to time with a bullet over the variable e.g. $dx/dt = \dot{x}$.
Figure (1.3)



The centre of mass of the two bodies is designated G. If $\vec{r}_1$ and $\vec{r}_2$ designates the vectors from the origin to the two masses, m_1 and m_2 , then the vector to the centre of mass is given by.

(1.4) $$\vec{r}_G = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

(1.5) $\vec{r} = \vec{r}_1 - \vec{r}_2$ is the vector from m_2 to m_1 as shown in the figure. The two equations (1.4) and (1.5) are then solved to express $\vec{r}_1$ *and* $\vec{r}_2$ *by* $\vec{r}$ *and* $\vec{r}_G$.
$\vec{r} = \vec{r}_1 - \vec{r}_2$ is therefore multiplied by m_2 .

(1.6) $$\begin{matrix} m_1\vec{r}_1 + m_2\vec{r}_2 = (m_1 + m_2)\vec{r}_G \\ m_2\vec{r}_1 - m_2\vec{r}_2 = m_2\vec{r} \end{matrix} \quad \Leftrightarrow \quad \vec{r}_1 = \vec{r}_G + \frac{m_2}{m_1 + m_2}\vec{r} \ \wedge\ \vec{r}_2 = \vec{r}_G - \frac{m_1}{m_1 + m_2}\vec{r}$$

The force between the two bodies, as well as their mutual potential energy depend only on their relative distance r.

(1.7) $$F = G\frac{m_1 m_2}{r^2} \qquad E_{pot} = -G\frac{m_1 m_2}{r}$$

The kinetic energy is: $E_{kin} = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2}$

By inserting the expressions found for $\vec{r}_1$ *og* $\vec{r}_2$ we get:

$$E_{kin} = \tfrac{1}{2} m_1 (\dot{\vec{r}}_G + \frac{m_2}{m_1 + m_2} \dot{\vec{r}})^2 \ + \tfrac{1}{2} m_2 (\dot{\vec{r}}_G - \frac{m_1}{m_1 + m_2} \dot{\vec{r}})^2$$

Putting $\vec{v}_G = \dot{\vec{r}}_G$ *and* $\vec{u} = \dot{\vec{r}}$ we find for the energy:

$$E_{kin} = \tfrac{1}{2} m_1 (\vec{v}_G + \frac{m_2}{m_1 + m_2} \vec{u})^2 \ + \tfrac{1}{2} m_2 (\vec{v}_G - \frac{m_1}{m_1 + m_2} \vec{u})^2$$

The double product from the two parenthesis disappear, and we get :

$$E_{kin} = \tfrac{1}{2}(m_1 + m_2) v_G^{\,2} + \tfrac{1}{2}\left(\frac{m_1 m_2^{\,2}}{(m_1 + m_2)^2} + \frac{m_2 m_1^{\,2}}{(m_1 + m_2)^2} \right) u^2 \qquad \Leftrightarrow$$

$$E_{kin} = \tfrac{1}{2}(m_1 + m_2) v_G^{\,2} + \tfrac{1}{2}\left(\frac{m_1 m_2 (m_1 + m_2)}{(m_1 + m_2)^2} \right) u^2 \qquad \Leftrightarrow$$

(1.8) $$E_{kin} = \tfrac{1}{2}(m_1 + m_2) v_G^{\,2} + \tfrac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)} u^2$$

Thus we have thus expressed the kinetic energy by the velocity of the centre of mass, and the relative velocity.

The expression $m = \frac{m_1 m_2}{(m_1 + m_2)}$ is called the reduced mass..

If the motion of a system is only caused by internal forces, the centre of mass remains at rest, and then we may choose our coordinate system with its origin in *G*, so that $\vec{v}_G = 0$.

From the expression of the reduced mass follows that: $\frac{m_1 m_2}{m_1 + m_2} = \frac{m_2}{1 + \frac{m_2}{m_1}} \approx m_2 \quad when \ m_1 >> m_2$,

and from this is seen, that if m_1 is much larger than m_2 , then the system of the two masses will correspond to the system: earth/moon or sun/earth.

The energy is the sum of the kinetic and potential energy, and while we are concerned with gravitational forces, it is given by:

(1.9) $$E = E_{kin} + E_{pot} = \tfrac{1}{2} m u^2 - G \frac{m_1 m_2}{r}$$

$m = \frac{m_1 m_2}{(m_1 + m_2)}$ is the reduced mass, *r* is the distance between the centres of the bodies, and $\vec{u} = \dot{\vec{r}}$.

2. Solution of the differential equations for a motion in a central field

It is common knowledge that the solution to the differential equations for motion in a central field are an elliptic motion, a hyperbolic motion or the limiting case of a parabolic motion.
The derivation is however not quite elementary. Several methods can be applied, but I have (qua my studies at the university of Copenhagen) chosen the Lagrange formalism adapted from the book “Mechanics” by Landau and Lifschitz from 1960.

In the **Lagrange formalism**, the kinetic energy T and potential energy U are expressed in generalized coordinates: $q_1, q_2, q_3, ..q_n$,with generalized velocities $\dot{q}_1, \dot{q}_2, \dot{q}_3, ..\dot{q}_n$, where $\dot{q} = \frac{dq}{dt}$
The Lagrange function is defined as:

(2.1) $$L = T - U$$
And the equations of motion are:
(2.2) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

To solve these equations, we express the kinetic and potential energy in polar coordinates (r,θ).
The kinetic energy is: $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$, and the polar coordinates are
$x = r\cos\theta \quad and \quad y = r\sin\theta \quad \Rightarrow \quad \dot{x} = \dot{r}\cos\theta - \dot{\theta}r\sin\theta \quad and \quad \dot{y} = \dot{r}\sin\theta - \dot{\theta}r\cos\theta$
When taking the square of these expressions, and using $\cos^2\theta + \sin^2\theta = 1$ we obtain:

(2.3) $$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

This can also be illustrated “geometrically”, since an infinitesimal displacement ds from (r,θ) to $(r + dr, \theta + d\theta)$ consists of a tangential displacement $rd\theta$ along the arc with radius r, followed by a radial displacement dr. The displacements $rd\theta$, dr, og ds are sides in a right angle triangle, so: $ds^2 = dr^2 + r^2 d\theta^2$. The expression (2.3) then follows when dividing by dt^2.
From Newton’s law of gravitation, the potential energy is $U(r) = -G\frac{mM}{r}$, where M is the mass of the central body, m is the mass of the satellite and r is the distance between their centres.
To simply the calculation we put α = GmM, so: $U(r) = -\frac{\alpha}{r}$, and then the Lagrange function becomes:

(2.4) $$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\alpha}{r}$$

And this yield the two equations of motion :

(2.5) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \quad \Leftrightarrow \quad m\ddot{r} - mr\dot{\theta}^2 + \frac{\alpha}{r^2} = 0$$

(2.6) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \quad \Leftrightarrow \quad \frac{d}{dt}(mr^2\dot{\theta}) = 0 \quad \Leftrightarrow \quad \frac{1}{2}r^2\dot{\theta} = \dot{A} \text{ (= constant)}$$

The last equation represents Kepler's law of areas. The radius vector to the satellite sweeps equal areas in equal times. This can be seen, since if $d\theta$ is the infinitesimal angle on an arc with radius r, then the area is the area of a triangle with "height" $rd\theta$ and "baseline" r , equal to $\frac{1}{2}r^2 d\theta$, from which the equation (2.6) follows by dividing by dt.

From (2.6) it also follows, that the tangential velocity $v = r\dot{\theta}$ is constant, so Kepler's law of areas is equivalent to the conservation af angular momentum $M = mr^2\dot{\theta} = mrv_{\tan}$. For an isolated system, that is, without external forces M remains constant, as we already know from Newtonian mechanics.
To determine the trajectory of the satellite, the Lagrange's equations of motion are not quite adequate, because they are second order differential equations. Instead we shall use the energy equation, which is a first order equation.

(2.7) $$E = T + U = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\alpha}{r}$$

The first step is to eliminate $\dot{\theta}^2$ using the expression for the angular momentum: $M = mr^2\dot{\theta}$,
isolating $\dot{\theta}^2 = \dfrac{M^2}{m^2 r^4}$ when inserted in (2.7) gives:

$$E = \tfrac{1}{2}m\dot{r}^2 + \frac{M^2}{2mr^2} - \frac{\alpha}{r} \qquad \Rightarrow$$

$$\frac{dr}{dt} = \sqrt{\frac{2E}{m} - \frac{M^2}{m^2r^2} + \frac{2\alpha}{mr}} \qquad \Rightarrow$$

(2.8) $$dt = \frac{dr}{\sqrt{\dfrac{2E}{m} - \dfrac{M^2}{m^2r^2} + \dfrac{2\alpha}{mr}}}$$

Finally we shift integrating variable from t to θ .

$$M = mr^2\frac{d\theta}{dt} \quad \Rightarrow \quad dt = \frac{mr^2}{M}d\theta \ ,$$

When inserted in (2.8) gives:

(2.9)
$$d\theta = \frac{M}{mr^2}\frac{dr}{\sqrt{\dfrac{2E}{m} - \dfrac{M^2}{m^2r^2} + \dfrac{2\alpha}{mr}}}$$
$$d\theta = \frac{M}{r^2}\frac{dr}{\sqrt{2mE - \dfrac{M^2}{r^2} + \dfrac{2m\alpha}{r}}}$$

When integrating (2.9), it gives, (expressed by θ and r) the trajectory curve of the satellite.

(2.10) $$\theta = M\int \frac{\frac{1}{r^2}}{\sqrt{2mE - \frac{M^2}{r^2} + \frac{2m\alpha}{r}}}dr$$

Using the substitution: $z = \frac{1}{r} \quad \Rightarrow \quad dz = -\frac{1}{r^2}dr$, we can rewrite the integral as:

(2.11) $$\theta = M\int \frac{-dz}{\sqrt{2mE - M^2z^2 + 2m\alpha z}}$$

Then we remodel the denominator, such as it can be written in the form: $\sqrt{1-x^2}$.

If M^2z^2 should be the square of the first term, (in a quadratic expression), and $-2m\alpha z$ is expected to be the double product, then the second term should be $2m\alpha z = 2Mzy$ when solved for y gives $y = m\alpha / M$

(2.12) $$\theta = -M\int \frac{dz}{\sqrt{2mE - (Mz - \frac{m\alpha}{M})^2 + \left(\frac{m\alpha}{M}\right)^2}}$$

If we set: $2mE + \left(\frac{m\alpha}{M}\right)^2 = \beta^2$ and moving this term outside the square root:

And further setting

$$x = \frac{1}{\beta}(Mz - \frac{m\alpha}{M}) \quad \Rightarrow \quad dz = \frac{\beta}{M}dx$$

the integral is reduced to:

(2.13) $$\theta = -\int \frac{dx}{\sqrt{1-x^2}} \quad \Rightarrow \quad \theta = \cos^{-1} x \quad \Leftrightarrow \quad x = \cos\theta$$

And when we substitute backwards :

(2.14) $$\frac{1}{\beta}(Mz - \frac{m\alpha}{M}) = \cos\theta \;\Rightarrow\; (\frac{M}{r} - \frac{m\alpha}{M}) = \beta\cos\theta$$

Introducing the parameters p and e *by*:

$$p = \frac{M^2}{m\alpha} \quad \text{and} \quad e = \sqrt{1 + \frac{2EM^2}{m\alpha^2}}$$

We finally arrive at the equation for curve for the trajectory of the satellite.

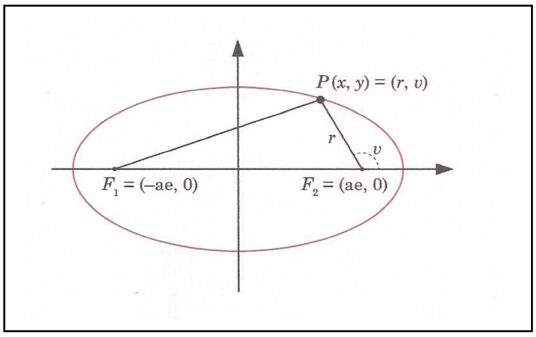
(2.15) $$r = \frac{p}{1 + e\cos\theta}$$

This equation is recognized as the equation for a conic section in polar coordinates.
From its definition $e > 0$.

If $0 \leq e < 1$, then r will be bound within the limits : $\frac{p}{1+e}$ *og* $\frac{p}{1-e}$ and the trajectory will be an ellipse, or a circle for $e = 0$.

If $1 \leq e$, then r is no longer bounded, since: $1 + e\cos\theta = 0$ has the solution $\cos\theta = -1/e$, and the trajectory will be a hyperbola or a parabola for $e = 1$.

3. The Elliptic equation in polar coordinates



The great axis of the ellipse is $2a$. The point P on the ellipse has coordinates (x,y), and the definition of the polar coordinates (r,θ) are shown in the figure.
The two points of foci F_1 and F_2, have coordinates $(-ae,0)$ and $(ae,0)$, where e is the eccentricity.
In any elementary derivation of the equation of the elliptic equation, one has the following expressions for the two distances

$$|F_1P| = a + ex \quad \text{and} \quad |F_2P| = a - ex$$

At the same time $|F_2P| = r$, and it is seen from the figure that:

$$x - ae = r\cos\theta \quad \Rightarrow \quad x = r\cos\theta + ae$$

and when inserted in $|F_2P| = r = a - ex$, it gives: $r = a - e(r\cos\theta + ae) \Rightarrow r(1+e\cos\theta) = a(1-e^2)$.
Solving for r:

(3.1) $$r = \frac{a(1-e^2)}{1 + e\cos\theta} \quad \Leftrightarrow \quad r = \frac{p}{1 + e\cos\theta}$$

This is in accordance with what we found in (2.15) when solving the equation of motion.

4. The laws of Kepler

First law of Kepler: Each planet moves around the sun, with the sun at one focus.

The solution to the equation of motion

$$r = \frac{p}{1 + e\cos\theta}$$

presents for $0 \leq e < 1$ an ellipse, which is the content of the first law of Kepler.

Second law of Kepler: The radius vector from the sun to the planet sweeps out equal areas in equal intervals of time. (The area-speed is constant)

This follow from (2.6): $\frac{d}{dt}(mr^2\dot{\theta}) = 0 \quad \Leftrightarrow \quad \frac{1}{2}r^2\dot{\theta} = \dot{A}$ (constant)

Third law of Kepler: The square of the periods of any two planets are proportional to the cubes af the semi-major axis of their respective orbits.

$$\frac{a^3}{T^2} = c \quad \text{(Is the same for all planets)}$$

We take our starting point in the expression for the angular momentum $M = mr^2\dot{\theta}$, which we rewrite as:

$$Mdt = mr^2 d\theta \quad \Rightarrow \quad M\int_0^T dt = 2m\int_0^{2\pi} \tfrac{1}{2}r^2 d\theta$$

The first integral equals the period *T*, and the second is the area of the ellipse, which is πab, so we have the equality $MT=2m\pi ab$. (b = semi-minor axis).

For an arbitrary ellipse the following formulas hold:

$$p = a(1-e^2) \quad and \quad \frac{b^2}{a^2} = (1-e^2) \Rightarrow b = \frac{p}{\sqrt{(1-e^2)}}$$

The parameter: $p = \frac{M^2}{m\alpha} \quad \Rightarrow \quad M^2 = pm\alpha \quad \Rightarrow \quad M = \sqrt{pm\alpha}$

This is inserted in $MT=2m\pi ab$, and solved for *T*:

$$T = \frac{2\pi mab}{M} = \frac{2\pi mab}{\sqrt{pm\alpha}} = \frac{2\pi ap}{\sqrt{pm\alpha}\sqrt{1-e^2}}$$

When we then insert: $\sqrt{p} = \sqrt{a(1-e^2)}$ we find:

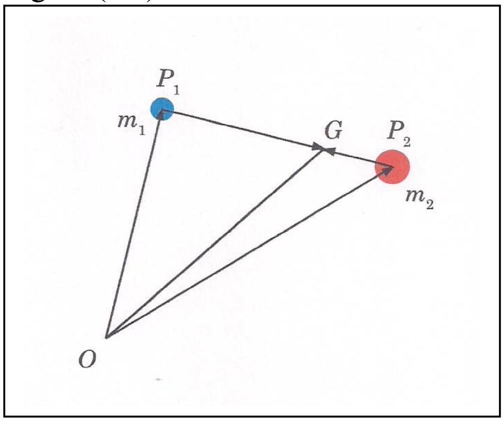
$$T = \frac{2\pi ap}{\sqrt{pm\alpha}\sqrt{1-e^2}} = \frac{2\pi a\sqrt{p}}{\sqrt{m\alpha}\sqrt{1-e^2}} = \frac{2\pi a\sqrt{a}}{\sqrt{m\alpha}}$$

(4.4) $$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{m\alpha}} \quad \Leftrightarrow \quad \frac{a^3}{T^2} = \frac{m\alpha}{4\pi^2}$$

In accordance with the third law of Kepler.

5. Returning to the two-body problem

Figure (5.1)



When the trajectory (2.15) for the reduced mass moving in a central field has been determined, we can subsequently find the trajectory for each of the two bodies from the expressions (1.6)
This requires however that we know the position of G the centre of mass, which is situated on the line connecting the centres of the two bodies.
The position of G is most elegantly determined by using plane vectors as shown in the figure (5.1)

(1.6) $$\vec{OG} = \frac{1}{m_1 + m_2}(m_1 \vec{OP_1} + m_2 \vec{OP_2}) \Leftrightarrow$$

$$(m_1 + m_2)\vec{OG} = m_1 \vec{OP_1} + m_2 \vec{OP_2} \Leftrightarrow$$

$$m_1(\vec{OG} - \vec{OP_1}) = -m_2(\vec{OG} - \vec{OP_2}) \Leftrightarrow$$

(5.2) $$m_1 \vec{P_1G} = -m_2 \vec{P_2G} \Rightarrow \frac{|P_1G|}{|P_2G|} = \frac{m_2}{m_1}$$

From above, we conclude that the centre of mass G is situated on the line connection the centres of the two bodies, such that it divides the line from P_1 to P_2 in the inverse proportion of the masses of the two bodies.
This may be applied to find the location of G for a two body system. For example the system of the earth and the moon.

The proportion of the masses are: $$\frac{M_{moon}}{M_{earth}} = 0.0123$$

The distance between the earth and the moon is $60R$, where R is the radius of the earth.

If r_G is the distance from the centre of the earth to the centre of mass G, then it follows from above:

(5.2) $$\frac{r_G}{60R - r_G} = \frac{M_{moon}}{M_{earth}} = 0.0123 \quad \Rightarrow \quad r_G = 0.729R$$

Consequently the centre of mass is situated within the surface of the earth.
As a consequence both the earth and the moon move in elliptic orbits with the same period.
The interaction of the moon with the earth gives rise to minor irregularities in the earths motion around the sun, the so called perturbations.