The physics of windmills

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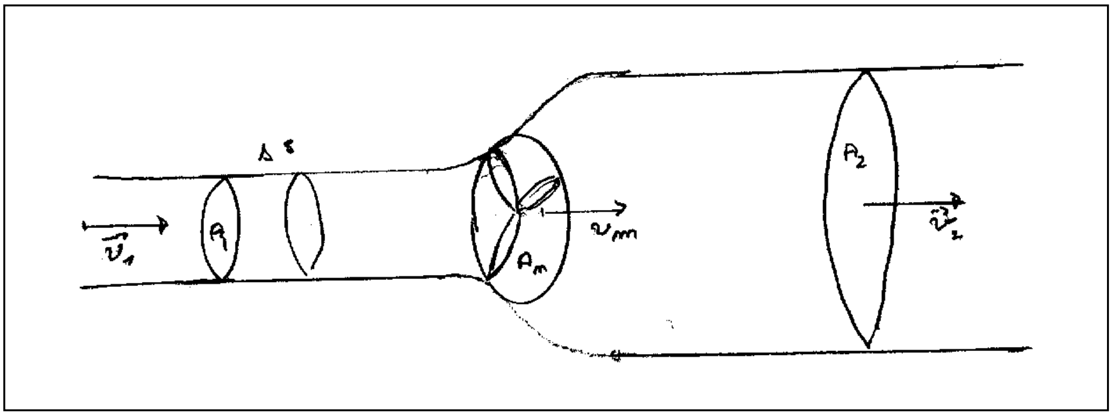

Our aim for this article is to determine the theoretical maximum power that can be exploited from a windmill.
The practical exploit depends, however, of the design of the windmill, depending on its construction based on engineering ingenuity, which we shall not go into at all.

1.1 Stationary flow

A stationary flow of a liquid or gas is characterized so that even when the liquid or gas particles move, the velocity of a fluid or gas particle is the same at any point and at any time.

In the following, we shall mostly apply differential changes, e.g. ds or dt instead of finite changes Δs and Δt, which also directly makes $\frac{ds}{dt}$ a differential quotient.

Initially we shall establish the *continuity equation*, which in this case becomes the same as the conservation of mass.

For a stationary flow in the tube, having varying cross section, the mass that passes through a cross section with thickness ds, density ρ, velocity v, and area A in the time dt , must be the same in the two arbitrary positions in the wind tube (1) and (2).

$$dm_1 = \rho_1 ds_1 A_1 = \rho_1 v_1 A_1 dt \quad \text{and} \quad dm_2 = \rho_2 ds_2 A_2 = \rho_2 v_2 A_2 dt$$

(1.1) $$\rho_1 v_1 A_1 dt = \rho_2 v_2 A_2 dt \quad \Leftrightarrow \quad \rho_1 v_1 A_1 = \rho_2 v_2 A_2$$

Or, if the density is constant

(1.2) $$\rho v A = \text{const} \qquad \text{(at any cross section of the tube).}$$

In the following, we shall assume that the density is constant, which is not entirely obvious, when dealing with a gas. However, according to the equation of state of gasses, we have:

$$PV = n_M RT$$

The number of moles in a substance is equal to the mass divided by the mole mass: $n_M = \frac{m}{M}$,

and the density is the mass divided by the volume of the mass: $\rho = \frac{m}{V}$. We then find:

$n_M = \frac{\rho V}{M}$, so that equation of state can also be written: $PV = \frac{\rho V}{M} RT \quad \Rightarrow \quad \rho = \frac{PM}{RT}$.

So if both temperature and the pressure are constant, we may proceed, assuming that the density is also constant. Anyway a deviation from this conjecture, will not change the basic outcome.

2. Finding the theoretical efficiency of a windmill

Since the wind in the wind tube must continue, (but with a slower velocity and a larger cross section of the tube), after passing the mill, the mill can never absorb all the energy of the wind.

In any case we have for the mass *dm* that passes through the cross section *A*.

(2.1) $$dm = \rho v A dt, \qquad \text{and thus:} \qquad \frac{dm}{dt} = \rho v A$$

For the momentum, and the kinetic energy of the mass *dm*, we therefore get:

(2.2) $$p = (dm)v = \rho A v^2 dt \qquad \text{and} \qquad dE = \tfrac{1}{2}(dm)v^2$$

For the power we have:

$$P = \frac{dE}{dt} = \tfrac{1}{2}\frac{dm}{dt}v^2$$

Inserting the expression for *dm/dt*, we get:

(2.3) $$P = \frac{dE}{dt} = \tfrac{1}{2}\rho v A v^2 = \tfrac{1}{2}\rho v^3 A$$

We notice that the power of the wind is proportional to the third power of the velocity.

We now consider three cross sections, and the corresponding velocities of the wind. Before the mill denoted by (1), at the mill, denoted by (*m*), and after the mill denoted by (2). $(A_1, v_1), (A_m, v_m), (A_2, v_2)$. See the figure above.

The power P_m submitted to the mill is the force F_m on the mill times the velocity of the wind at the mill.

(2.4) $$P_m = F_m v_m$$

The power, which the wind practices on the area A_m, swept by the wings of the mill is:

(2.5) $$P_1 = \tfrac{1}{2}\rho {v_1}^3 A_m$$

The power that the mill delivers is denoted P_m, and the efficiency of the mill is therefore defined as:

(2.6) $$\varepsilon_{mill} = \frac{P_m}{P_1} = \frac{F_m v_m}{P_1} \quad => \quad \varepsilon_{mill} = \frac{F_m v_m}{\frac{1}{2}\rho v_1^3 A_m}$$

The force F_m of the wind on the windmill is equal to the rate of change of the momentum of the wind.

(2.7) $$F_m = \frac{\Delta p_m}{\Delta t} = \frac{p_1 - p_2}{\Delta t} = \frac{\Delta m v_1 - \Delta m v_2}{\Delta t} = \frac{\Delta m}{\Delta t}(v_1 - v_2) \quad \text{where} \quad \frac{\Delta m}{\Delta t} = \rho v_m A_m$$

For a frictionless system, the power is equal to the rate of change of the kinetic energy.
This follows from the Work Theorem:
The work done by the resulting force is equal to the change of the kinetic energy.

$$\frac{\Delta E_{kin}}{\Delta t} = \frac{1}{2}\frac{\Delta m}{\Delta t}(v_1^2 - v_2^2)$$

Holding this equation together with:

$$F_m v_m = \frac{\Delta p_m}{\Delta t} v_m = \frac{\Delta m}{\Delta t}(v_1 - v_2) v_m$$

We get:

(2.8) $$\frac{1}{2}\frac{\Delta m}{\Delta t}(v_1^2 - v_2^2) = \frac{\Delta m}{\Delta t}(v_1 - v_2) v_m \quad => \quad v_m = \frac{1}{2}(v_1 + v_2)$$

This will now enable us to obtain an expression for the efficiency of the mill:

(2.9)
$$\varepsilon_{mill} = \frac{F_m v_m}{\frac{1}{2}\rho v_1^3 A_m} = \frac{\Delta m}{\Delta t}\frac{(v_1 - v_2)\frac{1}{2}(v_1 + v_2)}{\frac{1}{2}\rho v_1^3 A_m}$$
$$\varepsilon_{mill} = \frac{\rho v_m A_m (v_1 - v_2)\frac{1}{2}(v_1 + v_2)}{\frac{1}{2}\rho v_1^3 A_m} = \frac{\rho \frac{1}{2} A_m (v_1 + v_2)(v_1 - v_2)\frac{1}{2}(v_1 + v_2)}{\frac{1}{2}\rho v_1^3 A_m}$$
$$\varepsilon_{mill} = \frac{\rho \frac{1}{2} A_m (v_1 - v_2)\frac{1}{2}(v_1 + v_2)^2}{\frac{1}{2}\rho v_1^3 A_m} = \frac{1}{2}\frac{(v_1 - v_2)(v_1 + v_2)^2}{v_1^3}$$

If we put $x = \frac{v_2}{v_1}$, we obtain, after reducing the fraction by v_1^3.

(2.10) $$\varepsilon_{mill} = \frac{1}{2}(1 + x)^2(1 - x)$$

To find the theoretical max for the efficiency of the mill we differentiate ε_{mill} with respect to *x*.

$$\varepsilon_{mill}'(x) = \frac{1}{2}(2(1 + x)(1 - x) - (1 + x)^2) = \frac{1}{2}(1 + x)(2(1 - x) - (1 + x))$$

$$\varepsilon_{mill}'(x) = 0 \quad \Leftrightarrow \quad (x = -1) \vee \quad x = \frac{1}{3}$$

The theoretical max efficiency is thus obtained for $x = \frac{v_2}{v_1} = \frac{1}{3}$.

Inserting this value into $\varepsilon_{mill} = \frac{1}{2}(1+x)^2(1-x)$ we get $\varepsilon_{mill}(\max) = \frac{1}{2}(1+\frac{1}{3})^2(1-\frac{1}{3}) = \frac{1}{2}\frac{16}{9}\frac{2}{3} = \frac{16}{27} = 0.59$

No matter how the wind mill is designed, it is only possible to extract 59% of the energy from the wind.
One should, however, expect that in practice the actual gain is considerably lower, as it is the case in any mechanical device that produces work. However!

Let us for example assume that the length of the mill wing is 10 *m*. This correspond to a circular swept area of $\pi r^2 = 100\pi\ m^2$. The density of atmosphere near to the surface of the earth is 1.29 kg/m^3.

The maximum power submitted to the mill is according to the equation above:

$$P_m = \tfrac{1}{3}P_1 = \tfrac{1}{3}(\tfrac{1}{2}\rho v_1^{\ 3} A_m)$$

Assuming that the wind has the velocity: 10 *m/s*, we find:

$$P_m = 67.544\ W \approx 68\ kW\ .$$

If the wings are 15 *m* long, the result should be multiplied by a factor $1.5^2 = 2.25$ to give:

$$P_m = 151.974\ W \approx 152\ kW$$

If the lengths of the wings are 15 *m*, and the wind blows with 15 m/s the result should be multiplied by a factor 2.25 times $1.5^3 = 3.375$ to give:

$$P_m = 512{,}914\ W \approx 513 kW$$

This should be compared to the data for a so called standard 2*MW* mill under optimal conditions, which means that it delivers 2*MWh* in an hour. The reference, however, does neither supply the dimension of the mill, neither the optimal strength of the wind. We may only notice that the mill must be considerably larger than the first two examples, since the Power of a 2*MW* mill becomes:

$$\frac{2MWh}{3600s} = 556 kW\ .$$