The physics of rockets
For educational purposes

This is an article from my home page: www.olewitthansen.dk

Ole Witt-Hansen 2010

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1. Explanatory introduction (may be bypassed)

Since 2005, the students in the Danish 9-12 grade high school (gymnasium), must deliver a project in the middle of the third year. And for ideological reasons the project must comply two subjects one from science and one from humanities (e.g. chemistry and English). And yes , it is absurd.
At the same time the academic (theoretical) level has vastly deteriorated in the Danish high school from acquiring objective knowledge and skills from text books and learning scientific thinking, it has turned into humanistic critical analysis of material found on the Internet done in an almost scholastic framework.
The knowledge that the students acquire in mathematics and physics are no longer theoretical but lexical. Proofs in mathematics and derivation in physics are no longer a part of their curriculum. Even in the third year most student are unable to solve a linear equation with one unknown without resorting to a computer, and they are not even near to be able to differentiate or integrate by hand. However, when they get their project, it is mandatory that it should be formally a typographically formed as a scientific article or rather an essay, (with an abstract in English). Of course this is utterly crazy, but this is how it is.
Often the formulations of the projects imply theoretical considerations, and the students are lost of course, because theoretical explanations are seldom found on the Internet.
I stopped teaching in 2013, but until then, I had to write some notes to the students to whom I formulated their 3 year projects, to insure, that the project the student delivered should not be entirely superficially lexical transcript from Wikipedia and others.
In 2010 I had a student, who had chosen rockets as the subject for his project. About the same time the school had acquired a water rocket.
Being a physicist, I am not entirely satisfied by a lexical essay. I want real numbers based on theory from first principles. This article is the result of my endeavour.

2. The rocket equation

The water rocket is a relatively new invention in the teaching of physics in the Danish high school. Basically it is a (rocket shaped) container with a hole at the bottom. In the hole is placed a tight cork with a tube connected to an air pump to raise the pressure in the container. At a certain pressure the cork in the bottom is released, because of the pressure in the container, and if the rocket is half filled with water, the water is pressed out through the hole at a considerable speed delivering a force to lift the rocket.

The rocket equation is well established, and it can be derived from the conservation of momentum
We shall assume that we have a rocket (of any dimension) with mass $m = m(t)$.
The rocket is driven forward by spouting a constant mass μ per unit of time, with a velocity u relative to the rocket, such that $\mu = -dm/dt$. The minus sign because $m(t)$ is decreasing.
Hereby the velocity of the rocket is increased from v to $v+dv$.
From an observer at rest compared to the rocket, that is, where the rocket has velocity v, the mass dm has the velocity $v - u$.
Applying the conservation of momentum $p(t) = p(t+dt)$ on the system consisting of the rocket and the spouted mass, we have:

(2.1)
$$mv = (m + dm)(v + dv) + (-dm)(v - u) \quad \Leftrightarrow$$
$$mv = mv + mdv + vdm + dmdv + udm - vdm$$

We discard the term *dmdv*, since it is of second order in the differential quantities, and it will go to zero, when dividing with *dt*, and letting *dt* go to zero.
After reduction, we are left with the equation:

(2.2) $$mdv + udm = 0 \quad \Leftrightarrow \quad dv = -u\frac{dm}{m}$$

And this can be integrated to the rocket equation.

(2.3) $$v - v_0 = u\ln\frac{m_0}{m}$$, where $m = m_v = m_0 - \mu t$

2.1 Kinematics considerations, when launching a rocket from the earth

If we perceive m and v, as functions of time the equation (2.2) after division by dt becomes:

(2.4) $$\frac{dv}{dt} = -u\frac{1}{m}\frac{dm}{dt}$$

If the rocket is launched vertically, we must include the influence of gravity g in the calculation of the speed v.

(2.5) $$\frac{dv}{dt} = -\frac{u}{m}\frac{dm}{dt} - g$$

Which is integrated to:

(2.6) $$v - v_0 = u\ln(\frac{m_0}{m(t)}) - gt$$

If we want to determine the height the rocket reaches (the distance passed), we must evaluate:

$$s - s_0 = \int_0^t v(t)dt = \int_0^t (u\ln(\frac{m_0}{m(t)}) - gt)dt = u\int_0^t (\ln(m_0) - \ln(m(t))dt - g\int_0^t tdt$$

In the evaluation of $\int_0^t \ln(m(t)dt$, we use the formula: $\int \ln x dx = x\ln x - x$

$$\int_0^t \ln(m(t)dt \; = -\frac{1}{\mu}\int_0^t \ln(m)dm = \left[-\frac{1}{\mu}(m(t)\ln(m(t)) - m(t))\right]_0^t =$$

$$\int_0^t \ln(m(t)dt = -\frac{1}{\mu}(m(t)\ln(m(t)) - m(t) - (m_0\ln(m_0) - m_0)$$

After some reduction an expression for the height is found:

(2.7) $$s - s_0 = u\frac{m(t)}{\mu}\ln(\frac{m_0}{m(t)}) - ut - ½gt^2 \qquad ; \quad where\ m(t) = m_0 - \mu t$$

3. The air pressure rocket. The water rocket.

We assume that we are dealing with a rocket, where the "engine" consists of water that is pressed backwards out of the rocket with velocity u through a small opening with cross section A

We shall assume that the pressure under which the water (or gas) is exhausted is the same during the launch. This is not realistic of course, but without this assertion things simply get to complicated.

The atmospheric pressure is p_0 , while the pressure inside the rocket is p. To determine the velocity with which the water or gas is exhausted, we may apply *Bernoulli's law*, which apply along a streamline running from (1) to (2). Using the notations: p = pressure, ρ = density and v = velocity, we can write Bernuilli's law.

(3.1) $$p_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \tfrac{1}{2}\rho v_2^2$$

The position (1) = "is inside the rocket" og (2) = "outside the rocket", and therefore:
$v_1 = 0$, $p_2 = p_0$ and $v_2 = u$, which gives the equation: $\Delta p = p_1 - p_0 = \tfrac{1}{2}\rho u^2$.
The equation may be solved with respect to u:

(3.2) $$u = \sqrt{\frac{2\Delta p}{\rho}}$$

The mass dm that is exhausted through the opening A in the time dt , is the mass in a mathematical "tube" having the length udt and cross section A, and therefore the volume: $dV = uAdt$, so that $dm = \rho dV = \rho uAdt$. Thus dm/dt is determined by the formula:

(3.3) $$\mu = -\frac{dm}{dt} = \rho uA$$

3. 1 Numerical example with the water rocket

We put: $\Delta p = 1.0\ atm = 1.0\ 10^5\ Pa$. $\rho = 1.0\ 10^3\ kg/m^3$. $A = \pi\ (1.0\cdot 10^{-2})^2\ m^2 = 3.14\cdot 10^{-4}\ m^2$, then we have, according to (3.2) and (3.3).

(3.4) $$u = \sqrt{\frac{2\cdot 10^5}{10^3}} m/s = 14.1 m/s \quad \text{and} \quad \frac{dm}{dt} = 10^3 \cdot 14.1 \cdot 3.14 \cdot 10^{-4} kg/s = 4.4\ kg/s$$

If the mass of the rocket without water is 50.0 g, and it is filled with 0.5 l water, the rocket will obtain a vertical velocity v, and with:

$\frac{dm}{dt} = 4.4\ kg/s$, it will take $\frac{0.50\ l}{4.4\ l/s} = 0.11\ s$ to empty the container.

For the velocity we have according to (2.6):

(2.3.2)

$$v - v_0 = u \ln(\frac{m_0}{m}) - gt \quad \Rightarrow$$

$$v = u \ln(\frac{m_0}{m}) - gt = 14{,}1 \cdot \ln(11) - 9.82 \cdot 0{,}11\, m/s = 32{,}8\, m/s$$

The height that the rocket reaches can be determined from one of the equations regarding motion with constant acceleration:

$$v^2 - {v_0}^2 = 2a(s - s_0),$$

which in this case becomes:

$$0 - {v_0}^2 = -2gh \;\Rightarrow\; h = \;55\;\; m$$

This should of course be compared with experiments with the water rocket, which gave: 14 *m*.

There is really noting allarming in the large discrepancy between theory and experiment. Firstly we have assumed a constant pressure in the rocket container, but more severely, we have ignored any loss of energy from dissipative forces, which we cannot ignore, but neither we can take them into account. Finally we se that the height is proportional to the square of the velocity, with which the rocket is launched, and if we cut this into one half, we almost arrive at a height of 14 *m*. Nevertheless we have made a quantitative prediction of the behaviour of a rocket from first principles.

4. The Lighter gas rocket

The lighter gas rocket is just an empty plastic soda pop bottle, where a small hole has been drilled in the capsule. With the correct mixture of Butane and oxygen in the bottle, one may ignite the mixture in the bottle, causing the mixture to explode.

The fuel to the lighter gas rocket is butane: C_4H_{10} (M_{butane} = 58 g/mole), which reacts with oxygen O_2 (M_{oxygen} = 32 g/mol) after the reaction equation.

(3.1) $1\ C_4H_{10} + 6.5\ O_2 \ \text{->}\ 4\ CO_2 + 5\ H_2O$

If the reaction shall develop explosively it is necessary that the amounts of butane and oxygen are carefully tuned, so for each mole of butane, there should be 6.5 mole of oxygen.
According to Avogrado's law (or the equation of state of ideal gasses) then: Equal numbers of moles of different gasses occupy at the same pressure and temperature, the same volume.

The volume of butane should there for be 1:6.5 = 2:13 of the volume of oxygen.
If the volume of the container is V, then the volume of oxygen is $0.20V$, since the atmosphere consists of roughly 20% oxygen), the volume of butane should be: 0.20:6.5 V = 3.08 10^{-2} V.
So if the volume of the container is V = 0.5 *l* = 500 *ml* it gives 15.4 *ml* butane.
The calorific value for butane is 45.8 *MJ/kg*.
15.4 *ml* of butane corresponds to = 15.4 *ml*/24 *l /mol* = 6.42 10^{-4} *mol*.
And the mass is therefore: $m = n_M M$ = 6.42 10^{-4} *mol* 58 *g/mol* = 3.72 10^{-2} *g* = 3.72 10^{-5} *kg*.

The calorific value of the amount of butane is: Q = 3.72 10^{-5} *kg*· 45.8 *MJ/kg* = 1.70 *kJ*.

4.1 Pressure conditions for the lighter gas rocket

For gasses the density depends on the pressure of the gas, and this may complicate the calculations severely. But in the same manner as we assumed that the pressure was constant during the exhaustion, we shall assume that the density of the gasses is constant during the exhaustion.
As it is often the case in physics one must compromise to obtain a theoretical result.
The derivation requires four theorems:

1. The law of Bernoulli for the flow of liquid: $\Delta p = p_1 - p_0 = \frac{1}{2}\rho u^2 \quad \Rightarrow \quad u = \sqrt{\frac{2\Delta p}{\rho}}$

2. $m = n \cdot M$ (The mass of a gas equals the number of moles times the mass of one mole)

3. The equation of state for ideal gasses: $P = n\frac{RT}{V} \Rightarrow dP = \frac{RT}{V}dn$

4. The continuity equation: $\mu = -\frac{dm}{dt} = \rho u A \Leftrightarrow dm = -\rho u A dt$.

As it was the case for the water rocket, *dn* moles are ejected from a hole with cross section *A*.

(3.2) $$dP = \frac{RT}{V}dn = \frac{RT}{V}\frac{dm}{M} = -\frac{RT}{VM}\rho u A dt \quad \text{and} \quad u = \sqrt{\frac{2\Delta p}{\rho}} \quad \text{gives:}$$

$$dp = -\frac{RT}{VM}\sqrt{2(p-p_0)\rho}\,A dt$$

$$dp = -\frac{\sqrt{2\rho}RT}{VM}\sqrt{p-p_0}\,dt$$

(3.3) $$dp = -k_1\sqrt{p-p_0}\,dt \quad \Leftrightarrow \quad \frac{dp}{\sqrt{p-p_0}} = -k_1 dt$$

The atmospheric pressure is p_0. Inserting some suitable values:
$A = 1.26\ 10^{-5}\ m^2$, $T = 500\ K$, $p_0 = 1.0\ atm$, $V = 0.50\ l$, $M = 29\ g/mol$ and $\rho = 1.29\ kg/m^3$, we obtain a value for $k_1 = 5.80\ 10^3$.

We have separated the differential equation, so it is quite easy to integrate. We stipulate no value (for the present unknown) initial pressure p_1.

$$\int_{p_1}^{p}\frac{dp}{\sqrt{p-p_0}} = -k_1\int_0^t dt \quad \Leftrightarrow \quad 2\sqrt{p-p_0} - 2\sqrt{p_1-p_0} = -k_1 t$$

The equation above, however, is of little use, since we still do not know the initial pressure p_1.
In the next calculation, we shall try to calculate p_1 finding the temperature from the heat generated from the chemical reaction.

This implies, however, that we must also stipulate a reaction time for the chemical reaction that generates the heat. We shall assume that the reaction time is from 0.50 to 1.0 seconds, and let the outcome of the calculation decide which is most likely.
We have earlier found that the heat generated for the rocket in question is: Q = 1.70 kJ. Also we put A =1.26 10^{-5} m^2. c_{gas} is the heat capacity of the gas. It then follows:

(3.4) $$dQ = \frac{Q}{t_r}dt = m_{gas}c_{gas}dT \quad \Rightarrow \quad dT = \frac{Q}{t_r m_{gas} c_{gas}}dt = k_2 dt$$

From the equation of state for ideal gasses: $PV = nRT$, we find:

(3.5) $$dP = \frac{nR}{V}dT$$

(Valid, when n *and* V are held constant). Actually the number of moles n is not constant, since gas is streaming out of the rocket, but we have to compromise if we want to obtain theoretical result, that can be interpreted. We therefore insert (3.4) into (3.5) to obtain.

(3.6) $$dP = \frac{nR}{V}dT = \frac{nR}{V}\frac{Q}{t_r m_{gas} c_{gas}}dt = k_2 dt$$

(3.7) $$k_2 = \frac{nR}{V}\frac{Q}{t_r m_{gas} c_{gas}} = \frac{2{,}1\cdot 10^{-2}\,mol \cdot 8{,}31\,J/(mol\,K)\cdot 1{,}7\cdot 10^3\,J}{5{,}0\cdot 10^{-4}\,m^3\cdot 1{,}0\,s\cdot 0{,}604\cdot 10^{-3}\,kg\cdot 1{,}0\cdot 10^3\,J/kgK} = 9{,}8\cdot 10^5 \text{ SI}$$

The differential equation is hereafter:

(3.8) $$\frac{dp}{dt} = -k_1\sqrt{p-p_0} + k_2 \qquad \frac{dp}{dt} = -5{,}801\cdot 10^3\sqrt{p-p_0} + 9{,}8\cdot 10^5$$

The differential equation can be separated, and (with only a little effort) be integrated.

(3.9) $$\int_{p_0}^{p}\frac{dp}{k_2 + k_1\sqrt{p-p_0}} = -\int_0^t dt$$

The integral has the form: $\int \frac{dx}{a+\sqrt{x}}$.

It is solved by the substitution: $\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow dx = 2tdt$

Resulting in: $\int \frac{dx}{a+\sqrt{x}} = \int \frac{2tdt}{a+t}$, and followed by the substitution $a + t = z.$

(3.10) $$\int \frac{2tdt}{a+t} = 2\int \frac{z-a}{z}dz = 2z - 2a\ln z = 2(a+\sqrt{x}) - 2a\ln(a+\sqrt{x})$$

If we insert: $x = p - p_0$, but keeping $a = k_2/k_1$, we get the solution to the original integral (3.9).

$$2(a+\sqrt{p-p_0})-2a\ln(a+\sqrt{p-p_0})-2a+2a\ln a=-k_1t \quad \Leftrightarrow$$

$$2a\ln\frac{a+\sqrt{p-p_0}}{a}-2\sqrt{p-p_0}=k_1t$$

The solution is, however, not immediately applicable, since we have no way of solving for p

To investigate how the pressure grows with time, we have to resort to numerical methods, but the result depends heavily on the duration of the chemical reaction.

What we are interested in is actually the momentum transferred to the rocket.

$$d(mv)=udm=u\rho uAdt=\rho Au^2dt \quad \text{and} \quad u=\sqrt{\frac{2(p-p_0)}{\rho}} \quad \text{it gives:}$$

$$dp_{momemtum}=d(mv)=2A(p-p_0)dt$$

The last equation can be solved numerically at the same time as we do the pressure equation numerically.

Assuming that the reaction time $t_r = 0.5\ s$, then the numerical calculation gives a pressure of 2.1 *atm*, and the momentum transferred to the rocket is 1.15 *kg m/s*.

Assuming that the reaction time $t_r = 1.0\ s$, then the numerical calculation gives a pressure of 1.3 *atm*, and the momentum transferred to the rocket is 0.65 *kg m/s*.

A pressure of 2.1 *atm* is hardly realistic, so we carry on with the latter value.

$P_{generated}$ = 0.65 *kgm/s*. Putting it equal to the momentum transferred to the rocket.
$P_{generated} = m_{rocket}v$ = 0.65 *kgm/s* and m_{rocket} = 22.3 *g* , we get:

$$v=\frac{0,65}{2,23\cdot 10^{-2}}=29\ m/s\ .$$

Experimenting with the lighter gas rocket, where it was launched on a ramp with an elevation of 15^0, it was found from measuring the range, that it had an initial velocity of about 12 *m/s*.
Again this is not entirely discouraging, since we, (as it was the case with the water rocket), have made several simplifying assumptions, and more important have ignored any dissipative forces.

Still it is to be considered as an theoretical explanation of the performance of a (lighter gas) rocket based in first principle of physics.