The physics of Curling

Non central elastic collision of two identical circular stones

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1. Curling and the physics of collisions

Curling is an excellent example of demonstrating a non central elastic collision of two identical bodies on an (almost) frictionless underlay.
The non central collision is, however, much more physically and mathematically intriguing, than the central collision, which was earlier an integrated part of the Danish curriculum in the 9-12 grade high school.
We shall therefore begin discussing the central collision with two bodies with different masses to introduce general concepts and notations.

2. The central elastic collision

We shall consider a central collision of two bodies (1) and (2). Since the movements are along a straight line, we shall drop the vector symbols, and instead use signed variables.

Velocities before the collision are consequently designated with a low letter *u*, and velocities after the collision are consequently designated with a low letter *v*. The two bodies are designated with subscript (1) and (2). So for example: v_1, is the velocity of body (1) after the collision.
To ease the mathematics, we shall initially perform the calculation with the assumption that body (2) is at rest before the collision, so that $u_2 = 0$. Afterwards we shall deal with the general case.

For the elastic central collision we have conservation of energy as well as conservation of momentum.

(2.1) $$I: \quad m_1 u_1 = m_1 v_1 + m_2 v_2$$ (Conservation of momentum, where $u_2 = 0$)

$$II: \quad \tfrac{1}{2} m_1 u_1^2 = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2$$ (Conservation of the kinetic energy)

We shall then solve these two (nonlinear) equations to find v_1 and v_2 , the velocities after the collision. This is done by some mathematical rearrangement.

(2.2) $$\begin{aligned} I: &\quad m_1(u_1 - v_1) = m_2 v_2 \\ II: &\quad \tfrac{1}{2} m_1(u_1^2 - v_1^2) = \tfrac{1}{2} m_2 v_2^2 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} I: &\quad m_1(u_1 - v_1) = m_2 v_2 \\ II: &\quad m_1(u_1 - v_1)(u_1 + v_1) = m_2 v_2^2 \end{aligned}$$

In the latter expressions we then divide *II* with *I*, but keeping *I* for the sake of doing one to one calculations.

$$\begin{aligned} I: &\quad m_1(u_1 - v_1) = m_2 v_2 \\ II: &\quad (u_1 + v_1) = v_2 \;\vee\; v_2 = 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} I: &\quad m_1(u_1 - v_1) = m_2 v_2 \\ II: &\quad u_1 + v_1 = v_2 \;\vee\; v_2 = 0 \end{aligned}$$

$$I: \quad u_1 - v_1 = 0 \qquad \vee \qquad I + II: \quad m_1(u_1 - v_1) = m_2(u_1 + v_1)$$

From the last expression, we may find v_1, and subsequently find v_2 by inserting it in $u_1 + v_1 = v_2$.

$$(2.3) \qquad v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1 \quad \wedge \quad v_2 = \frac{2m_1}{m_1 + m_2} u_1 \qquad \vee \qquad v_2 = 0 \ \wedge \ v_1 = u_1$$

From (2.3) it appears that the velocities after the collision v_1 and v_2 can be evaluated when the velocity u_1 together with the masses m_1 and m_2 are known.

The solution $v_2 = 0$, although mathematically correct, has no physical interest since it means that the two bodies passing each other without interacting.

From the solution it is also evident that v_2 (the velocity of (2) after the collision) is always unidirectional to u_1, (the velocity of (1) before the collision), while v_1 (the velocity of (1) after the collision) is unidirectional to u_1 if : $m_1 > m_2$, and opposite directed u_1 if $m_1 < m_2$ (it is reflected)
In the case $m_1 = m_2$, we can see that $v_2 = u_1$ and $v_1 = 0$. This means that the two bodies swap velocities, a phenomenon, which is well known for any pool player.

If we assume that m_2 is infinitely larger than m_1, (e.g. a ball hits the floor), then we know the result from experience, but the equations also give the answer, if we divide (1.7) by m_2 both in the numerator and denominator, and uses that $m_1 : m_2$ is practically zero.

$$v_1 = \frac{\frac{m_1}{m_2} - 1}{\frac{m_1}{m_2} + 1} u_1 = -u_1 \quad \wedge \quad v_2 = \frac{2\frac{m_1}{m_2}}{\frac{m_1}{m_2} + 1} u_1 = 0$$

The floor rests where it is, and the ball returns with the same but opposite velocity.
We have included this result, because it is imperative for developing the kinetic theory of gasses.

The general case, where both bodies are moving before the collision can be treated similar to the derivations above, requiring only a little more mathematical cunning.

$$I: \ m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 \qquad \text{(Conservation of momentum, where } u_2 <> 0\text{)}$$
$$II: \ \tfrac{1}{2} m_1 u_1^2 + \tfrac{1}{2} m_2 u_2^2 = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 \qquad \text{(Conservation of the kinetic energy)}$$

$$I: \ m_1(u_1 - v_1) = m_2(v_2 - u_2)$$
$$II: \ m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2)$$

$$I: \ m_1(u_1 - v_1) = m_2(v_2 - u_2)$$
$$II: \ m_1(u_1 - v_1)(u_1 + v_1) = m_2(v_2 - u_2)(v_2 + u_2)$$

Dividing *II* by *I*:

$$I: \ m_1(u_1 - v_1) = m_2(v_2 - u_2)$$
$$II: \ u_1 + v_1 = v_2 + u_2$$

$$I: \ m_2 v_2 + m_1 v_1 = m_1 u_1 + m_2 u_2 \qquad II: \ v_2 - v_1 = u_1 - u_2$$

Multiplying II by m_1 and solving the two linear equations:

$$I:\ m_2v_2 + m_1v_1 = m_1u_1 + m_2u_2 \qquad II:\ m_1v_2 - m_1v_1 = m_1u_1 - m_1u_2$$

$$(m_1 + m_2)v_2 = 2m_1u_1 + (m_2 - m_1)u_2 = 2(m_1u_1 + m_2u_2) - (m_1 + m_2)u_2$$

When solving for v_2 , we find the expression below. The expression for v_1 is found in the same manner.

(2.4) $$v_1 = \frac{2(m_1u_1 + m_2u_2)}{m_1 + m_2} - u_1 \qquad \wedge \qquad v_2 = \frac{2(m_1u_1 + m_2u_2)}{m_1 + m_2} - u_2$$

If we put u_2 =0, we retrieve (after a minor reduction) the expressions (2.3).

2.1. Introducing the centre of mass (CM) system.

For two particles, having the velocities $\vec{u}_1$ and $\vec{u}_2$, and masses m_1 and m_2, the velocity of the CM-system $\vec{v}_{CM}$ (It is the velocity of the Centre of mass) is given by the equation:

(2.5) $$\vec{v}_{CM} = \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2}$$

Velocities in the CM are consequently designated with a hyphen. So we have:

(2.6) $$\vec{u}_1 = \vec{u}_1' + \vec{v}_{CM} \quad \Rightarrow \quad \vec{u}_1' = \vec{u}_1 - \vec{v}_{CM} = \vec{u}_1 - \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2}$$

$$\vec{u}_1' = \frac{m_2(\vec{u}_1 - \vec{u}_2)}{m_1 + m_2} \text{ And in a similar manner: } \vec{u}_2' = -\frac{m_1(\vec{u}_1 - \vec{u}_2)}{m_1 + m_2}$$

We notice that to total momentum of the system is zero, since:

(2.7) $$\vec{p}_1' + \vec{p}_2' = m_1\vec{u}_1' + m_2\vec{u}_2' = \frac{m_1m_2(\vec{u}_1 - \vec{u}_2)}{m_1 + m_2} - \frac{m_2m_1(\vec{u}_1 - \vec{u}_2)}{m_1 + m_2} = \vec{0}$$, which implies that:

$$\vec{u}_2' = -\frac{m_1}{m_2}\vec{u}_1' \quad \text{and if the masses are equal: } \vec{u}_2' = -\vec{u}_1'$$

In the CM the two masses have opposite directed velocities. If the masses are equal, so are the opposite directed velocities. After a collision, we have exactly the same equations if we replace u by v.

Notice that the equations above applies, whether the collision is central or not.

If the particle (2) is at rest before the collision the formulas take a simpler form:

$$\vec{u}_1' = \frac{m_2 \vec{u}_1}{m_1 + m_2} \quad \text{and} \quad \vec{u}_2' = -\frac{m_1 \vec{u}_1}{m_1 + m_2}$$

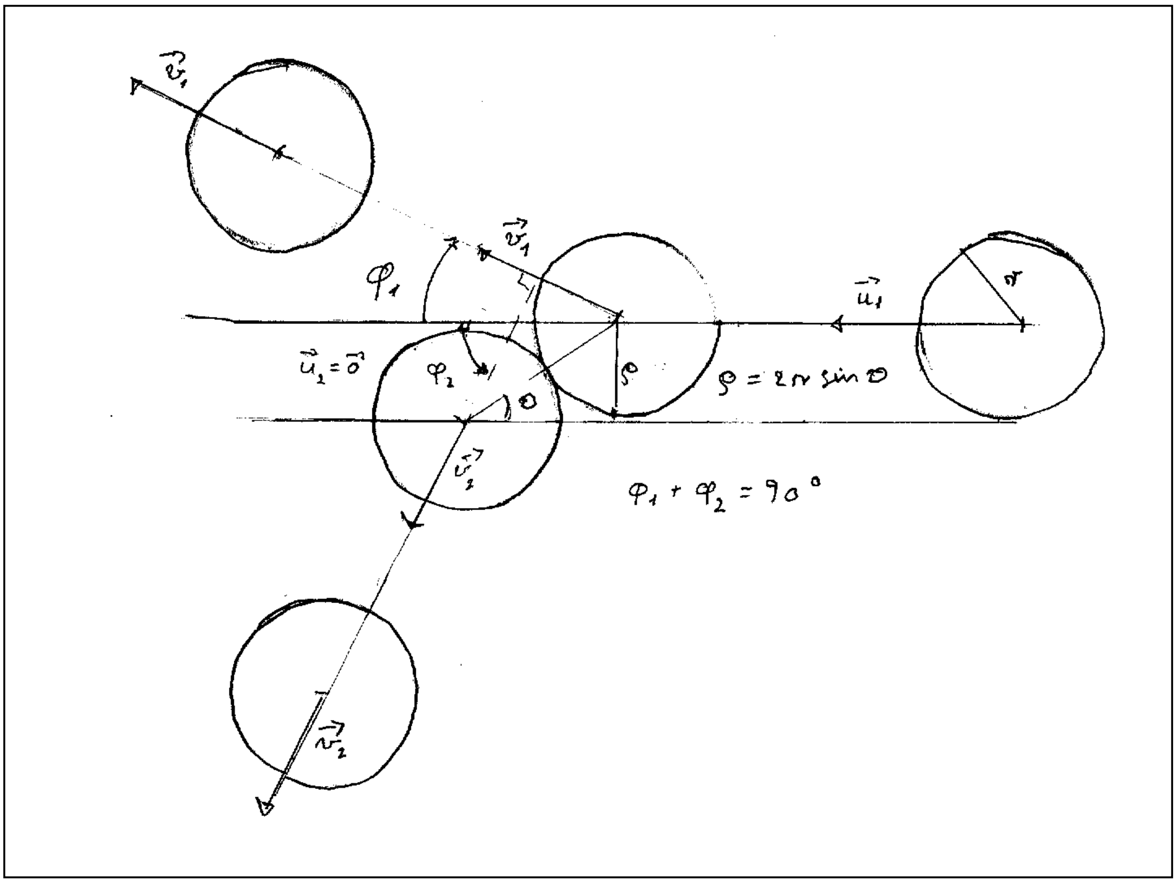
3. The physics of curling

We shall then consider two identical balls (or two curling stones) colliding in a non central collision, where one of them at rest before the collision. (Without these conditions the mathematics simply becomes incomprehensible). Our aim is to find the deflection angles for both bodies, and their velocities after the collision in the Lab-system.
In the lab-system the velocities of two equal masses must be perpendicular after the collision. This follows from conservation of momentum and energy:

$$m\vec{v}_1 + m\vec{v}_2 = m\vec{u}_1 \quad and \quad \tfrac{1}{2}mv_1^{\,2} + \tfrac{1}{2}mv_2^{\,2} = \tfrac{1}{2}mu_1^{\,2} \quad \Rightarrow$$
$$\vec{v}_1 + \vec{v}_2 = \vec{u}_1 \quad and \quad v_1^{\,2} + v_2^{\,2} = u_1^{\,2}$$

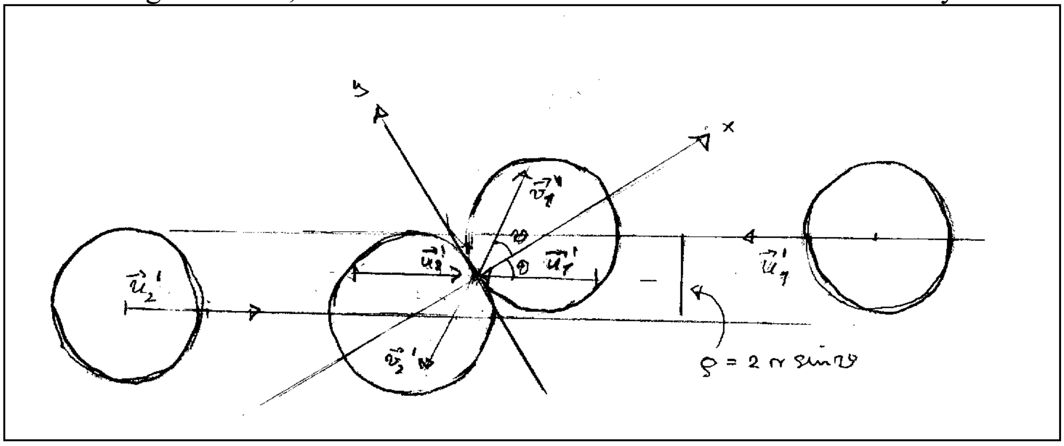
We can see that the three vectors form a triangle for which Pythagoras theorem holds, and therefore: $\vec{v}_1 \perp \vec{v}_2$. This applies for example for curling stones.

In the figure below is sketched the collision in the Lab-system.
Our aim is to find the angle φ_1 ,($\varphi_2 = 90 - \varphi_1$) together with v_1 and v_2 expressed by u_1 and the impact parameter ρ which is the distance between the two centres, when they collide



It turns out that it is not really possible to solve the problem in the Lab-system, so we shall look at the collision in the CM-system, and take advantage of the symmetry of this system.

So in the figure below, we show how the collision looks like in the CM-system



We have supplied with a coordinate system, which has its origin at the point of impact, and where the *x*-axis connects the two centres, and the *y*-axis is tangent to the two circular bodies at that point. Since the masses are the same, then: $\vec{u}_2' = -\vec{u}_1'$.
Since the collision for each ball, corresponds to the reflection against a wall the only thing that happens is that the *x*-coordinate changes sign. This can also be seen from the equations:
(Momentum conservation for equal masses): $\vec{v}_2' = -\vec{v}_1'$ and $\vec{u}_2' = -\vec{u}_1'$.
Further we must have $v_2' = v_1' = u_1'$.
The last equation may be understood from conservation of energy. (for typographical reasons we drop the hyphen)

$$\tfrac{1}{2}mu_1^2 + \tfrac{1}{2}mu_2^2 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}mv_2^2 \quad \Leftrightarrow \quad u_1^2 + u_2^2 = v_1^2 + v_2^2 \quad \Leftrightarrow \quad 2u_1^2 = 2v_1^2 \quad \Leftrightarrow \quad v_1 = u_1$$

This equation actually also holds in the CM if the masses are not equal: It follows from conservation of momentum and energy:

$$m_1\vec{u}_1 + m_2\vec{u}_2 = \vec{0} \ \wedge \ m_1\vec{v}_1 + m_2\vec{v}_2 = \vec{0} \quad \Rightarrow \quad u_2 = \frac{m_1}{m_2}u_1 \ \wedge \ v_2 = \frac{m_1}{m_2}v_1$$

$$\tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 \quad \Rightarrow$$

$$m_1u_1^2 + m_2(\frac{m_1}{m_2}u_1)^2 = m_1v_1^2 + m_2(\frac{m_1}{m_2}v_1)^2 \quad \Rightarrow$$

$$\frac{m_1(m_1+m_2)}{m_2}u_1^2 = \frac{m_1(m_1+m_2)}{m_2}v_1^2 \quad \Rightarrow \quad v_1 = u_1$$

For an elastic collision in the CM-system, the masses are reflected as if they hit a wall.

For the elastic central collision in the Lab system where $u_2 = 0$, the velocities after the collision are given by (2.3). We shall then find the velocities in the CM-system.

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1 \quad \wedge \quad v_2 = \frac{2m_1}{m_1 + m_2} u_1 \quad \text{and} \quad v_{CM} = \frac{m_1 u_1}{m_1 + m_2}$$

$$u_1 = u_1' + v_{CM} \quad \Rightarrow \quad u_1' = u_1 - v_{CM} = u_1 - \frac{m_1 u_1}{m_1 + m_2} = \frac{m_2 u_1}{m_1 + m_2}$$

$$u_2 = u_2' + v_{CM} \quad \Rightarrow \quad u_2' = u_2 - v_{CM} = 0 - \frac{m_1 u_1}{m_1 + m_2} = -\frac{m_1 u_1}{m_1 + m_2}$$

$$v_1 = v_1' + v_{CM} \quad \Rightarrow \quad v_1' = v_1 - v_{CM} = \frac{m_1 - m_2}{m_1 + m_2} u_1 - \frac{m_1 u_1}{m_1 + m_2} = -\frac{m_2 u_1}{m_1 + m_2}$$

$$v_2 = v_2' + v_{CM} \quad \Rightarrow \quad v_2' = v_2 - v_{CM} = \frac{2m_1}{m_1 + m_2} u_1 - \frac{m_1}{m_1 + m_2} u_1 = \frac{m_1 u_1}{m_1 + m_2}$$

From this we conclude that:

$$m_1 u'_1 + m_2 u'_2 = 0 \quad \wedge \quad m_1 v'_1 + m_2 v'_2 = 0 \quad \textit{and} \quad v_1' = -u_1' \quad \wedge \quad v_2' = -u_2'$$

3.1 Non central collision in the CM-system

For a central collision in the CM, the two balls are reflected as if they hit a wall, and the same thing happens for the non central collision in the CM-system. From the figure above, we can see:

$$\vec{u}_1' = u_1'(-\cos\theta, \sin\theta) \quad \textit{and} \quad \vec{u}_2' = u_2'(\cos\theta, -\sin\theta) \quad \text{and since} \quad v_2' = v_1' = u_1' = u_2'$$

$$\vec{u}_1' = u_1'(-\cos\theta, \sin\theta) \quad \textit{and} \quad \vec{u}_2' = u_1'(\cos\theta, -\sin\theta) \qquad (\vec{u}_2' = -\vec{u}_1')$$

$$\vec{v}_1' = u_1'(\cos\theta, \sin\theta) \quad \textit{and} \quad \vec{v}_2' = u_1'(-\cos\theta, -\sin\theta) \qquad (\vec{v}_2' = -\vec{v}_1')$$

The Lab system has its x-axis along the incoming stone.

In this coordinate system we have: $\vec{v}_{CM} = \dfrac{m_1 \vec{u}_1 + m_2 \vec{u}_2}{m_1 + m_2} = \tfrac{1}{2}\vec{u}_1 = \tfrac{1}{2} u_1 (1,0)$

and in the Lab-system system: Notice caused by the orientation of the Lab-system the scattering angle is $2\theta - 180$

$$\vec{u}_1' = u_1'(1,0) \ , \ \vec{u}_2' = u_1'(-1,0) \ ,$$
$$\vec{v}_1' = u_1'(-\cos(2\theta - 180), -\sin(2\theta - 180)) \ ,$$
$$\vec{v}_2' = u_1'(\cos(2\theta - 180), \sin(2\theta - 180))$$
$$\vec{v}_1' = u_1'(\cos 2\theta, \sin 2\theta) \ , \ \vec{v}_2' = u_1'(-\cos 2\theta, -\sin 2\theta)$$

We then transform the velocities to the Lab-system:

$$\vec{u}_1 = \vec{u}_1' + \vec{v}_{CM} \quad \Rightarrow \quad \vec{u}_1' = \vec{u}_1 - \tfrac{1}{2}\vec{u}_1 = \tfrac{1}{2}\vec{u}_1 = \tfrac{1}{2} u_1 (1,0)$$

$$\vec{u}_2 = \vec{u}_2' + \vec{v}_{CM} \quad \Rightarrow \quad \vec{u}_2' = \vec{u}_2 - \tfrac{1}{2}\vec{u}_1 = \vec{0} - \tfrac{1}{2}\vec{u}_1 = \tfrac{1}{2} u_1 (-1,0)$$

If we put $\varphi = 2\theta$, then we have:

$$\vec{v}_1' = u_1'(\cos\varphi, \sin\varphi) \quad og \quad \vec{v}_2' = -u_1'(\cos\varphi, \sin\varphi)$$

And in the Lab-system:

$$\vec{v}_1 = \vec{v}_1' + \vec{v}_{CM} = \tfrac{1}{2}u_1(\cos\varphi, \sin\varphi) + \tfrac{1}{2}u_1(1,0) = \tfrac{1}{2}u_1(\cos\varphi + 1, \sin\varphi)$$

$$\vec{v}_2 = \vec{v}_2' + \vec{v}_{CM} = -\tfrac{1}{2}u_1(\cos\varphi, \sin\varphi) + \tfrac{1}{2}u_1(1,0) = -\tfrac{1}{2}u_1(\cos\varphi - 1, \sin\varphi)$$

We may verify that the two velocities are perpendicular to each other, since:

$$\vec{v}_1 \cdot \vec{v}_2 = -\tfrac{1}{4}u_1^2((\cos\varphi - 1)(\cos\varphi + 1) + \sin^2\varphi) = \cos^2\varphi - 1 + \sin^2\varphi = 0$$

The angle of deflection for (1) is given by:

$$\tan\varphi_1 = \frac{v_{1y}}{v_{1x}} = \frac{\sin\varphi}{\cos\varphi + 1} = \frac{\sin 2\theta}{\cos 2\theta + 1} = \frac{2\sin\theta\cos\theta}{1 - 2\sin^2\theta - 1} = -\frac{1}{\tan\theta} = -\frac{\sqrt{1 - \sin^2\theta}}{\sin\theta} \quad \text{or}$$

$$\sin\varphi_1 = \frac{v_{1y}}{\sqrt{v_{1x}^2 + v_{1y}^2}} = \frac{\sin 2\theta}{\sqrt{(\cos 2\theta + 1)^2 + \sin^2 2\theta}} = \frac{\sin 2\theta}{\sqrt{\cos^2 2\theta + 1 + 2\cos 2\theta + \sin^2 2\theta}} = \frac{2\sin\theta\cos\theta}{\sqrt{2 + 2\cos 2\theta}}$$

$$\sin\varphi_1 = \frac{2\sin\theta\cos\theta}{\sqrt{2 + 2(2\cos^2\theta - 1)}} = \frac{2\sin\theta\cos\theta}{2\sqrt{\cos^2\theta)}} = \sin\theta$$

But these expressions are not so interesting, if they are not linked to the impact parameter:

The impact parameter is: $\rho = 2r\sin\theta \;\Leftrightarrow\; \sin\theta = \frac{\rho}{2r} = x$

$$\tan\varphi_1 = -\frac{1}{\tan\theta} = -\frac{\sqrt{1 - x^2}}{x} \quad \text{and} \quad \sin\varphi_1 = \sin\theta = x$$

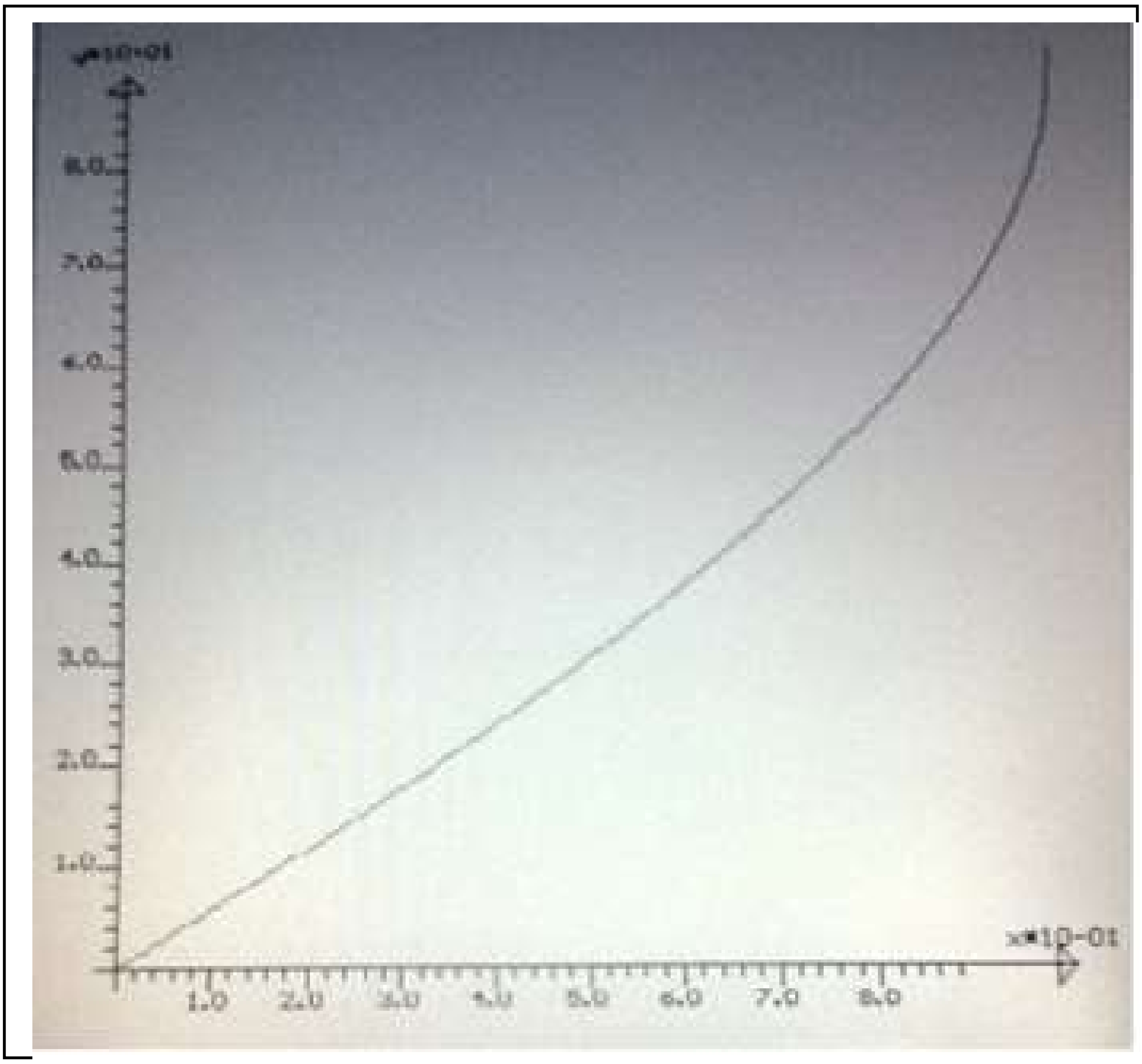
$$\tan\varphi_1 = -\frac{1}{\tan\theta} = -\frac{\sqrt{1 - x^2}}{x} \quad \text{and} \quad \sin\varphi_1 = \cos\theta = \sqrt{1 - x^2}$$

Finally we shall find the speed of the two stones after the collision:

Below is shown a graph, where the deflection angle φ_1 is mapped as a function of $\frac{\rho}{2r} = x$.

$$\vec{v}_1 = \tfrac{1}{2}u_1(\cos 2\theta + 1, \sin 2\theta) \;\Rightarrow\; v_1 = \tfrac{1}{2}u_1\sqrt{(\cos 2\theta + 1)^2 + \sin^2 2\theta} = \tfrac{1}{2}u_1 2\cos\theta = u_1\sqrt{1 - x^2}$$

$$\vec{v}_2 = -u_1(\cos 2\theta - 1, \sin 2\theta) \Rightarrow\; v_2 = \tfrac{1}{2}u_1\sqrt{(\cos 2\theta - 1)^2 + \sin^2 2\theta} = \tfrac{1}{2}u_1 2\sin\theta = u_1 x$$



3.2 Example

The stones have a circumference 91.4 cm, so the radius $r = \dfrac{91.4\,cm}{2\pi} = 14.5\,cm$

If the incoming stone has a speed of 2.0 m/s and an impact parameter, equal to the radius, then it correspond to $x = 0.5$, and we find the deflection angle from:

$$\sin\varphi_1 = \sin x : \quad \varphi_1 = \sin^{-1} 0.5 = 30^0 \text{, and therefore: } \varphi_2 = 60^0$$

The speed of the two stones after the collision are:

$$v_1 = u_1\sqrt{1-x^2} = \sqrt{3}\, m/s \;\; og \quad v_2 = u_1 x = 1{,}0 m/s$$

And if the impact parameter $\rho = 0.9 \cdot 2r : x = 0.9$ then the results are:

$$\varphi_1 = 34.2^0 \,, \quad v_1 = u_1\sqrt{1-x^2} = 0.872\, m/s \quad and \quad v_2 = u_1 x = 1.8 m/s$$

Curling is about an extreme visually based precision. We shall try to calculate how much the angle changes if x changes with $\Delta x = 0.01$ at x =0.5 corresponding to $\rho = 14.5\, cm \quad \Delta\rho = 0.29\, cm$

We have: $\sin\varphi_1 = \sin\theta = x$, so that $\varphi_1 = \sin^{-1} x$ and $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$,

Thus:

$$\frac{d\varphi_1}{dx} = \frac{x}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \quad \Delta\varphi_1 = \frac{180}{\pi}\frac{1}{\sqrt{1-x^2}}\Delta x = 0{,}66^0$$

Gliding 1 *m*, it will correspond to a deviation perpendicular to the motion of 1.15 cm.

We may also find the deviation perpendicular to the motion on a lane of 40 *m*.
If the stone is sent away with an deviation in angle on 1^0, then the deviation at the en of the lane:

$$1.0^0 \frac{\pi}{180} 40m = 0.69\, m = 69\, cm\,.$$

I have therefore always wondered how they do it!

The results have been derived for curling stones, but they are equally applicable for billiard balls.