The longest throw from a given height

Finding extremum using Lagrange multipliers

This is an article from my home page: www.olewitthansen.dk

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1. Obliquely throw in the gravitational field

It is well known from kinematics and Newton's laws that you will get the longest throw, with a throw angle of 45^0.
The result is however based on the assumption that the origin of the throw and the impact point is in the same horizontal level.

If the motion starts at (0,0) with the speed v_0 and the throw angle is α, then the equation of motion becomes. (See: Elementary Physics 1, "Kinematics" on my home page).

(1.1) $$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_0 \cos\alpha \\ v_0 \sin\alpha - gt \end{pmatrix} \quad and \quad \vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v_0 \cos\alpha \cdot t \\ v_0 \sin\alpha \cdot t - ½gt^2 \end{pmatrix}$$

The highest point of the trajectory is found by setting: $v_y = v_0 \sin\alpha - gt = 0$ solving for t and insert it in the expression for $y = v_0 \sin\alpha \cdot t - ½gt^2$ to give y_{max}

(1.2) $$y_{\max} = \frac{(v_0 \sin\alpha)^2}{2g}.$$

The throw width is found by solving the equation $y = 0$ with respect to t giving: $t = \dfrac{2v_0 \sin\alpha}{g}$

and insert the t found in the expression for x. A minor reduction gives the result:

(1.3) $$x_{\max} = \frac{v_0{}^2 \sin 2\alpha}{g}$$

From which it is obvious that the longest throw is, when sin2α = 1 ,that is, α= 45^0.

We shall then assume that the origin of the throw is at height h. It could be a throw with a spear, a hammer throw or a throw from a tower, and we want to determine the throwing angle that results in the longest horizontal throw.
At a glance one could expect a similar calculation as above, but it turns out that the mathematics becomes more intricate than that.

The equation of motion are now:

(1.4) $$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_0 \cos\alpha \\ v_0 \sin\alpha - gt \end{pmatrix} \quad and \quad \vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v_0 \cos\alpha \cdot t \\ h + v_0 \sin\alpha \cdot t - ½gt^2 \end{pmatrix}$$

Our task is then to determine max for: $x(\alpha) = v_0 \cos\alpha \cdot t$, where t is one of the solution to the equation:

$$y = 0 \quad \Leftrightarrow \quad h + v_0 \sin\alpha \cdot t - ½gt^2 = 0.$$

Even if it is possible to find the solution of the quadratic equation with respect to t and inserting it in $x(\alpha) = v_0 \cos\alpha\, t$, and even differentiate the expression, but you get blocked when you try to solve the equation: $x'(\alpha) = 0$.
A value where $f'(x) = 0$, is called an extremum

The solution to these kinds of problems is called *extremum with side condition*.
If the equation $f'(x) = 0$ cannot be solved directly, the problem may often be solved by the help of the so called *Lagrange multipliers*.

2. Introducing Lagrange multipliers

We shall illustrate the method for a function of two variables, but it may readily be generalized to several variables.

Let there be given a function: $y = f(x,y)$, where we wish to determine the extrema..

A point of extremum is found by solving the differential $dy = 0$.

(2.1) $$dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

If dx and dy are independent variables, we have the well known result:

(2.2) $$\frac{\partial f}{\partial x} = 0 \quad and \quad \frac{\partial f}{\partial y} = 0$$

But if the two variables are bound together in a side condition: $g(x,y) = c,$ and where, it is not possible (or just incomprehensible) to solve for x or y and insert in $f(x,y)$, then we may write the differential of $g(x,y)$, and hold it together with the differential for $f(x,y)$.

(2.3) $$\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0 \qquad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

If these two equations must remain valid for all dx and dy, then it follows that:

$$(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \quad \text{and} \quad (\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}) \qquad \text{must be proportional to each other.}$$

Since if you consider the two equations as a system of equations having the unknowns dx and dy, then the determinant of the system must be zero, (since the right side of the system of equations is zero). The determinant is:

$$\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial x}=0 \Leftrightarrow \frac{\frac{\partial f}{\partial x}}{\frac{\partial g}{\partial x}}=\frac{\frac{\partial f}{\partial y}}{\frac{\partial g}{\partial y}}=-\lambda \Leftrightarrow \frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}=0 \quad\wedge\quad \frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}=0 \quad\Leftrightarrow$$

(2.5) $$\frac{\partial}{\partial x}(f+\lambda g)=0 \quad\wedge\quad \frac{\partial}{\partial y}(f+\lambda g)=0$$

There is a tradition to put the constant of proportionality equal to - λ.

The last two equations are, however, the same as determining the extrema for $f(x,y)+\lambda g(x,y)$.

The constant λ is to be determined from the border conditions.

3. Finding the longest throw from a given height

We return to the oblique throw from a given height, and put:

$$x=f(t,\alpha)=v_0\cos\alpha\cdot t \quad and \quad y=g(t,\alpha)=h+v_0\sin\alpha\cdot t-½gt^2$$

We then seek max for $x(t,\alpha)$ under the side condition $y=0$ and form the function: $F=f+\lambda g$

(3.1) $$F(t,\alpha)=v_0\cos\alpha\cdot t+\lambda(h+v_0\sin\alpha\cdot t-½gt^2).$$

Calculating the partial derivatives, and putting them to zero:

(3.2) $$\begin{aligned}\frac{\partial F}{\partial t}=0 &\Leftrightarrow\quad v_0\cos\alpha+\lambda v_0\sin\alpha-\lambda gt=0\\ \frac{\partial F}{\partial\alpha}=0 &\Leftrightarrow -v_0\sin\alpha\cdot t+\lambda v_0\cos\alpha\cdot t=0\end{aligned}$$

From the last equation, we find: $\lambda=\tan\alpha$, which is the inserted in the first equation to determine *t.*

$$v_0\cos\alpha-gt\tan\alpha+\tan\alpha v_0\sin\alpha=0.$$

Multiplying by cosα*,* and solving for *t*, gives: $t=\frac{v_0}{g\sin\alpha}$, when inserted in the side condition to determine α gives.

$$-½g(\frac{v_0}{g\sin\alpha})^2+v_0\sin\alpha(\frac{v_0}{g\sin\alpha})+h=0$$

Which can be solved for sin*α* to give the angle, resulting in the largest throw:

(3.3) $$\sin^2 \alpha = \tfrac{1}{2}\frac{v_0^{\,2}}{v_0^{\,2} + gh} = \tfrac{1}{2}\frac{1}{1+\dfrac{gh}{v_0^{\,2}}}$$

If h=0, then $\sin^2 \alpha = \tfrac{1}{2}$, so the angle is 45^0, as it should be.

Examples

If h = 2.0 m and $v_0 = 20$ *m/s* = 72 *km/h*, we find the throwing angle for the longest throw to be $\alpha = 43.66^0$. A very delicate correction for a javelin thrower.

On the other hand if h = 20.0 *m* and $v_0 = 20$ *m/s* = 72 *km/h*, we find the throwing angle for the longest throw to be $\alpha = 35.39^0$.

Below are shown graphs for 4 throws, corresponding to $v_0 = 20$ *m/s*, with heights h = 0 *m*, 2.0 *m* (a javelin thrower), 5 *m* and 10 *m*.
What we see, is that the higher from which the object is thrown, the longer it comes, when having the same speed. (Which is certainly not surprising). It is more surprising that throwing the spear from the height of 2 meters, and having the right angle, it increases the throw width with about 8 meters.