The heat pump
and
the laws of thermodynamics

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1. The heat pump, applied as an experimental device in high school

As a censor in the Danish 9-12 year high school, I was once introduced to an experiment with a heat pump. It was a liquid-liquid heat pump, which 30 years ago was popular (but no longer) extracting heat from the ground. Today most, if not all, heating pumps are air-air heating pumps.

From a theoretical point of view, the result of the high school experiment required only the first law of thermodynamics, since the heat Q_2, delivered to the second vessel should equal the heat Q_1 taken from the other vessel plus the work W done by the heating pump.

$$Q_2 = W + Q_1 \quad \text{, where the efficiency of the pump is calculated as } \eta = Q_2/W = 1 + Q_1/W$$

If we ignore the loss of energy in the machinery, then the efficiency is always greater than 1.

The purpose of the experiment was to determine the efficiency from the temperature curves from the water in the two vessels, since the heat content can be calculated from the caloric equations:
If m_1 and m_2 are the masses of water in the two vessels, and c is the heat capacity of water, we have:

$$Q_1 = -\, m_1 c\Delta T_1 \qquad \text{and} \qquad Q_2 = m_2 c\Delta T_2$$

The temperatures were collected in a computer and displayed on a screen.
They seemed to be linear, but the responsible teacher said that they were not.
I sought for a theoretical description of the temperature curves, but I soon realized that it requires an application of a differential formulation of the second law of thermodynamics, which clearly was beyond the theoretical level in third year of the Danish high school.

2. Analyse of the heat pump from the first and second law of thermodynamics.

We shall therefore proceed to write the first and second law in a differential form.

First law (conservartion of energy): $dQ_2 = dW + dQ_1$
Second law (The entropy): $dS = dS_1 + dS_2$, where $dS_1 = \dfrac{dQ_1}{T_1}$ *and* $dS_2 = \dfrac{dQ_2}{T_2}$
In the differential formulation, the processes are assumed to be reversible. (Which is the condition that we may calculate anything at all).

Reversibility means that: $dS_{total} = 0$, and we therefore have:

$$dS = 0 \quad \Leftrightarrow \quad dS_1 + dS_2 = 0 \quad \Leftrightarrow \quad \frac{dQ_1}{T_1} + \frac{dQ_2}{T_2} = 0$$

We shall first assume that $m_1 = m_2 = m$.
If we here insert the two expression $dQ_1 = c\, m_1 dT_1$ and $dQ_2 = c\, m_2 dT_2$, and subsequently divide by $c{\cdot}m$, we have the equation:

$$\frac{dQ_1}{T_1} + \frac{dQ_2}{T_2} = 0 \quad \Leftrightarrow \quad \frac{dT_1}{T_1} + \frac{dT_2}{T_2} = 0 \quad \Leftrightarrow \quad \ln T_1 + \ln T_1 = k_1 \quad \Leftrightarrow \quad T_1 T_2 = k$$

As a consequence of conservation of entropy, we arrive at the simple result that T_1 and T_2 are inversely proportional to each other.

To apply the first law, we must have at relation between the differentials dT_1 and dT_2.
However from the equation: $T_1T_2 = k$, it follows:

$$T_1 = \frac{k}{T_2} \Rightarrow dT_1 = -\frac{k}{T_2^{\,2}}dT_2 \quad \text{, and correspondingly} \quad T_2 = \frac{k}{T_1} \Rightarrow dT_2 = -\frac{k}{T_1^{\,2}}dT_1$$

We then insert either expression in the first law on differential form,

$$dQ_2 = dW + dQ_1 \quad \Leftrightarrow \quad cm \cdot dT_2 = Pdt - cm \cdot dT_1$$

The minus sign, because dT_1 is negative, and $dW = Pdt$, where P is the power of the heat pump.

$$cm \cdot dT_2 - cm\frac{k}{T_2^{\,2}}dT_2 = Pdt \Rightarrow (1 - \frac{k}{T_2^{\,2}})dT_2 = \frac{P}{cm}dt$$

$$cm \cdot dT_1 - cm\frac{k}{T_1^{\,2}}dT_1 = Pdt \Rightarrow (1 - \frac{k}{T_1^{\,2}})dT_1 = \frac{P}{cm}dt$$

There is apparently a complete symmetry between the two equations, and $k = T_1T_2 = T_0^2$, where T_0 is the common initial temperature.

If T_2 is increased, then T_1 will be diminished accordingly, as a consequence of the equation $k = T_1T_2 = T_0^2$.

Therefore the two factors in the parenthesis above have opposite sign, so T_1 will decrease, while T_2 will grow. Both equations can easily be integrated.

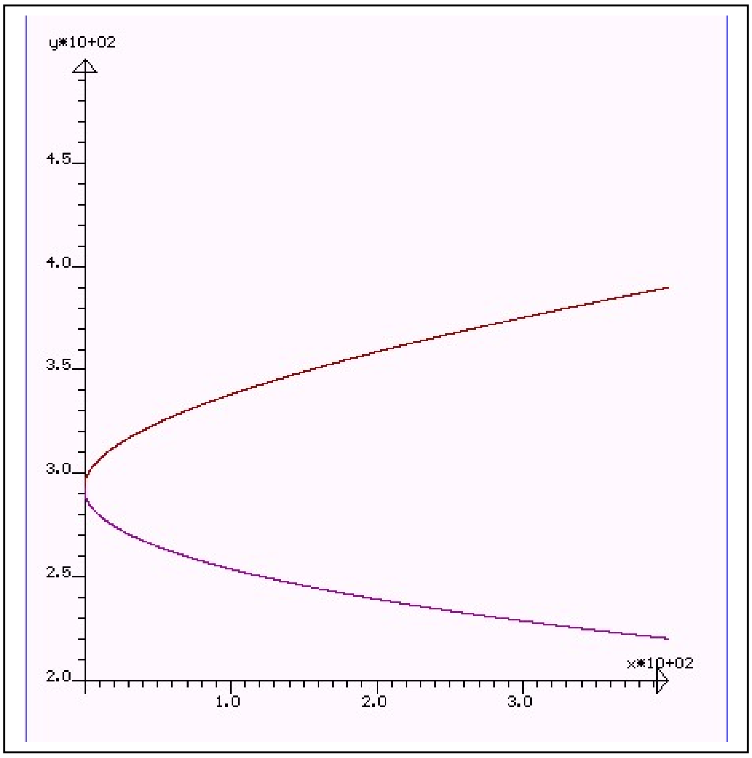
$$T_2 - T_{20} + \frac{k}{T_2} - \frac{k}{T_{20}} = \frac{P}{cm}t \qquad \text{og} \qquad T_1 - T_{10} + \frac{k}{T_1} - \frac{k}{T_{10}} = \frac{P}{cm}t$$

If we multiply the first equation by T_2T_{20} , then appears a quadratic equation.

$$T_2^{\,2} - (T_{20} + \frac{k}{T_{20}}k + \frac{P}{cm}t)T_2 + k = 0 \quad \Leftrightarrow \quad T_2^{\,2} - qT_2 + k = 0\,; \; where \;\; q = (T_{20} + \frac{k}{T_{20}}k + \frac{P}{cm}t)$$

A quite similar equation may be obtained for T_1. Even when the quadratic equation is easy to solve, the solution is not adequate for a graphic representation. Actually it is much better to make a numeric, graphic computer solution.

If we have two vessels both with 2 liters of water, and a heat pump having the power of 500 W, then it results in the following graphic representation.



Under these conditions, however, the temperature in the primary vessel quickly falls below freezing point 273 K, so the curves are not realistic. To get a better description, we must assume that the vessel that delivers the heat is much larger than the other. This will, however, modify the equations somewhat.

4. Solving the temperature equations with different masses in the two vessels.

The reformulations of the two equations is then:

$$dS = \frac{dQ_1}{T_1} + \frac{dQ_2}{T_2} = 0 \quad \text{and} \quad dQ_1 = c\, m_1 dT_1 \quad \text{and} \quad dQ_2 = c\, m_2 dT_2 \quad \Rightarrow$$

$$\frac{m_1 dT_1}{T_1} + \frac{m_2 dT_2}{T_2} = 0$$

Integrating we find: $m_1 \ln T_1 + m_2 \ln T_2 = k \quad \Leftrightarrow \quad \ln T_1 + \frac{m_1}{m_2} \ln T_2 = k$

Which leads to $T_1 T_2^{\beta} = k, \quad where \quad \beta = \frac{m_2}{m_1}$. This results in the equations.

$$T_1 = \frac{k}{T_2^{\beta}} \Rightarrow dT_1 = -\beta \frac{k}{T_2^{1+\beta}} dT_2 \qquad\qquad T_2^{\beta} = \frac{k}{T_1} \Rightarrow \beta T_2^{\beta-1} dT_2 = -\frac{k}{T_1^2} dT_1$$

$$dT_2 = -\frac{k}{\beta T_2^{\beta-1} T_1^2} dT_1 = -\frac{k}{\beta (T_2^{\beta} T_1) T_2^{-1} T_1} dT_1 = -\frac{k T_2}{\beta k T_1} dT_1 = -\frac{k (k T_1^{-1})^{\frac{1}{\beta}}}{\beta k T_1} dT_1$$

$$dT_2 = -\frac{(kT_1^{-1})^{\frac{1}{\beta}}}{\beta T_1} dT_1 = -\frac{k^{\frac{1}{\beta}}}{\beta T_1^{1+\frac{1}{\beta}}} dT_1 \qquad dT_2 = -\frac{\left(\frac{k}{T_1}\right)^{\frac{1}{\beta}}}{\beta T_1} dT_1$$

Then we may establish the differential equations for the two temperatures.

$$cm_2 \cdot dT_2 + cm_1 \cdot dT_1 = Pdt \quad =>$$

$$cm_2 \cdot dT_2 - cm_1 \frac{\beta k}{T_2^{\beta+1}} dT_2 = Pdt \quad \Rightarrow$$

$$(1 - \frac{k}{T_2^{\beta+1}})\beta dT_2 = \frac{P}{cm_1} dt$$

$$cm_1 \cdot dT_1 + cm_2 \cdot dT_2 = Pdt \quad =>$$

$$cm_1 \cdot dT_1 - cm_2 \frac{\left(\frac{k}{T_1}\right)^{\frac{1}{\beta}}}{\beta T_1} dT_1 = Pdt \quad \Rightarrow$$

$$(1 - (\frac{k}{T_1})^{\frac{1}{\beta}} \frac{1}{T_1}) dT_1 = \frac{P}{cm_1} dt$$

In this case an analytic solution is hardly available, so we stick to the numeric graphic computer solution. If we put $\beta = m_2/m_1 = 0.1$, then the following solution curves appear.