Surface tension with applications

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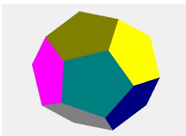


Ole Witt-Hansen 2021 (2009)

Contents

This paper is a translation of an article I wrote in 2009 for a physics journal. Most of the articles I have written are translated to English and displayed in my home page, but recently I found out that I had overlooked this article. I decided to enrol it in my homepage, since it is not so easy to find theoretical papers on surface tension, mostly because the description requires rather advanced mathematics.

1. Surface tension

We all know that surface tension in water is due to attraction of the molecules, caused by their dipole moment, and that the surface tension disappears if the water is added a little soap.
But what do we know apart from that?
Several years ago my son was going to graduate in the ninth grade. He asked casually, if I could tell him something about surface tension. Well apart from the notion that surface tension keeps water drops and soap bubbles together, and is the reason that some insects can skate on water, I knew actually nothing about the physics behind surface tension.
Neither, I have any recollection of a treatment of surface tension in high school or university.
On the other hand the physics of surface tension involves some of the most powerful tools in advanced mathematics, namely the calculus of variations.
It is not been so easy to find materials dealing with the physics of surface tension, but in Addison Wesley: University Physics from 1992 there is an excellent chapter dealing with the subject.
This is, however, not the case in the 2002 edition, where the trend in educational physics, was more or less, to discard the theoretical (mathematical) of physics in favour of a more phenomenological description serviced by computer programs and direct related to problems in daily experience. The derivations and proofs almost disappeared in the teaching of mathematics and physics.

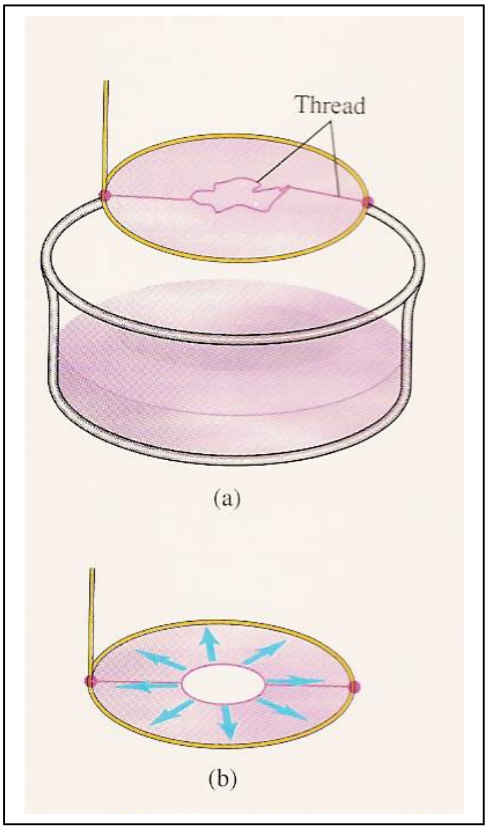
In the 1992 edition it is established that the (surface tension) force per unit length (on a mathematical cut) is a material constant γ. The force on a strip Δl is thus:

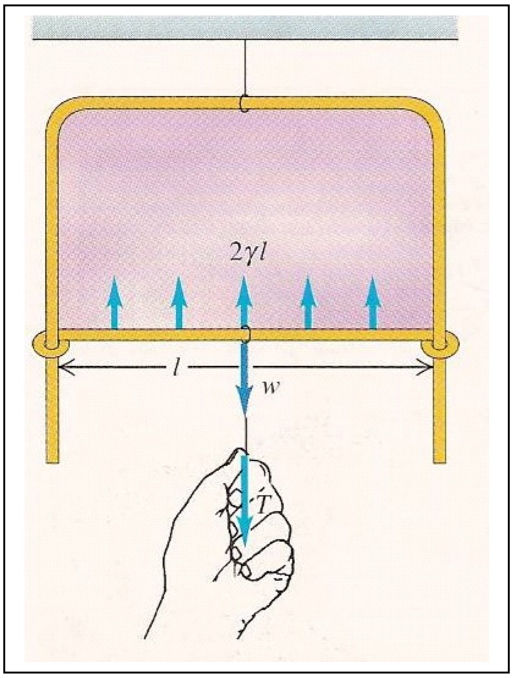
$$F = \gamma\Delta l$$

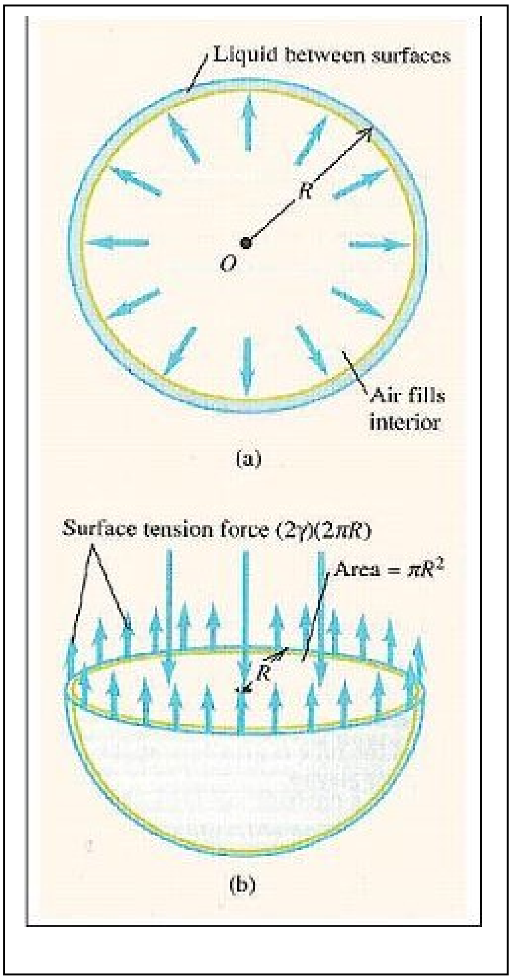
γ is a material constant, which is most often measured in *dyn/cm*, where 1 dyn = 1g 1 cm/s^2 is the traditional cgs unit for force.,For example, we have: $\gamma_{water} = 76\ dyn/cm$.

If we are dealing with two parallel separated surfaces, as it is the case for a soap bubble, there is the same surface tension on both sides so the force perpendicular to a “cut” becomes $F = 2\gamma\Delta l$.

We have sought to illustrate this basic physical law, in the figure below. A very thin thread is floating in a soap sphere. When the inner surface bursts, the thread is stretched, caused by the surface tension.
In the second figure a soap surface is created in a rectangle, where one side can be displaced by a (small) force. The surface tension constant may in principle be determined by measuring the force F dividing it with the double length of the side.







The work that must be carried out to move the moveable rod a distance Δs is $\Delta W = F\Delta s = 2\gamma l\Delta s$.
From this follows that $\Delta W = 2\gamma\Delta A$, where $\Delta A = l\Delta s$ is the increment of the area of the rectangle.
Since the change of the potential energy $\Delta E_{pot} = \Delta W$, it follows immediately that the potential energy of a surface due to surface tension is proportional to the area of that surface.
It is a formality to show that this also applies to curved surfaces. You may just integrate $\mathrm{d}W = 2\gamma\mathrm{d}A$ over the surface.
However, that the potential energy due to surface tension is proportional to the area of the surface, delivers a simple explanation of many well known phenomena of daily life.
In fact it gives the explanation of the shape of water drops, soap bubbles and soap surfaces. They have namely in common that they have equilibrium at the least potential energy, that is, the least surface.
Equilibrium of a physical system having the least potential energy is in fact one of the most general first principle in physics.
The simple connection between force and surface tension may be applied to calculate the (over)pressure inside a soap bubble.

If we look at the force from the surface tension in a soap bubble then it is equal to the force from the gas along a meridian plane on each of the two shells. The force on each of the two shells is, however, the same as the force on the meridian plane, since the projection of the shells is equal to the cross section of the bubble.
The force from the surface tension along a meridian plane is. (Consult the figure above.)

$$F_S = 2\gamma(2\pi R)$$

The force on the meridian plane from the gas pressure P is:

$$F_G = P(\pi R^2)$$

We may then set up the equation: $F_S = F_G$ to established the expression for the gas pressure in the bubble.

$$2\gamma(2\pi R) = P(\pi R^2)$$

From the equation follows

$$P = \frac{4\gamma}{R}$$

From a table of physical constants we find γ_{soap}=25 *dyn/cm*, so for a bubble which has a radius of 1 *cm*, we find that the gas exerts a pressure of 100 dyn/cm^2 = 1 mN/ cm^2. (Since 1 dyn = 10^5 N).

2. Calculus of variation

The calculus of variation is one of the most powerful tools in mathematics and delivers the most fundamental first principles in theoretical physics and differential geometry.

The calculus of variation is based on the Lagrange formalism as a method of seeking minimum for a functional. The Lagrange formalism leads to differential equations that may or may not have a analytic solution. More specific we are seeking extremes (max or min) for the functional:

$$I(y) = \int_a^b F(y', y, x)dx$$

So our aim is to determine a differential equation, which gives the variation $\delta I = 0$, which leads to the Euler-Lagrange equations.
(The derivation of the Euler-Lagrange equations is shown in detail in:
www.olewitthansen.dk/mathematics/calculus_of_variation)

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

If $F(y', y, x)$ does not explicitly depend of x, one can show that this equation may be integrated to a first order differential equation.

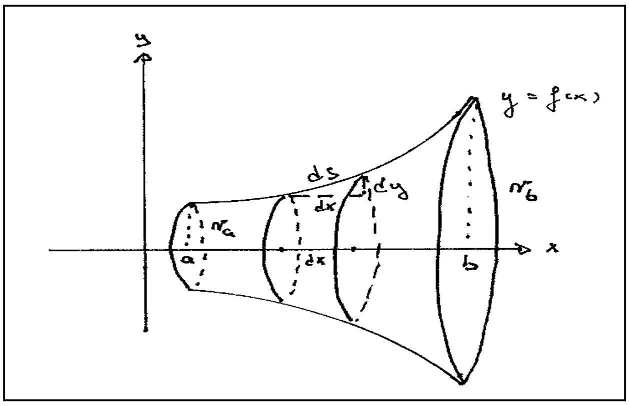
$$y'\frac{\partial F}{\partial y'} - F = Const$$

As it is has been demonstrated by physicists and mathematicians in the last century, the equation of motion for all developed physical theories, including Analytic mechanics, the Maxwell equations and the General theory of Relativity may be derived from "the principle of least action".
If a Lagrange function $L = T - U$ (The kinetic energy minus the potential energy), is written with the generalized coordinates q_i: $L(\dot{q}_i, q_i, t)$, Then the equation of motions are given by the Euler-Lagrange equations.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

2.1 Soap surfaces

It is a general experience, and a consequence of the Euler-Lagrange equations, that a system reaches its equilibrium when the potential energy is at minimum. Since the potential energy due to surface tension is proportional to the surface area, it follows that a surface, (delimited by some border conditions), have an equilibrium, when the surface is at minimum.
So the shape of a soap surface is clearly a variational problem, which has an analytic solution for some surfaces having rotational symmetry.
The shape of a rotational symmetric surface is solved by looking at the intersection with a plane through the axis of rotation, as it is shown in the figure below.



In problems of this kind the solution is always restricted by some *border conditions.*
In the example shown in the figure to the left, the border conditions are

$f(a) = r_a$ and $f(b) = r_b$,

where $y = f(x)$ is the intersecting curve, we want to determine.
To establish an expression for the surface area, we shall look at an (infinitesimal) strip with thickness *dx,* at *x,* where the circumference of this strip is

$$2\pi f(x) = 2\pi y.$$

For the infinitesimal distance *ds* on the surface along the *x*-axis, we have:

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (\frac{dy}{dx})^2}\,dx = \sqrt{1 + f'(x)^2}\,dx = \sqrt{1 + y'^2}\,dx$$

The contribution to the area from the infinitesimal strip is therefore:

$$dS = 2\pi y ds = 2\pi f(x)\sqrt{1 + f'(x)^2}\,dx \quad \text{or} \quad dS = 2\pi y\sqrt{1 + y'^2}\,dx$$

The area of the surface is then found by integration

$$S = 2\pi \int_a^b y\sqrt{1 + y'^2}\,dx$$

The integrand $F(y', y'x) = y\sqrt{1+y'^2}$ does not depend explicitly on x in this case, so we may therefore apply the simplified version of the Euler-Lagrange equations.

$$y'\frac{\partial F}{\partial y'} - F = C$$

C is a constant that must be determined by the border conditions. By inserting in the Euler-Lagrange equation, and performing the differentiation, we have:

$$y'\frac{2yy'}{2\sqrt{1+y'^2}} - y\sqrt{1+y'^2} = C$$

Multiplying the equation by $\sqrt{1+y'^2}$ and reducing, we obtain:

$$yy'^2 - y(1+y'^2) = C\sqrt{1+y'^2} \qquad \Leftrightarrow \qquad y = -C\sqrt{1+y'^2}$$

To get rid of the square root in $y = -C\sqrt{1+y'^2}$, we put: $y' = \sinh(\frac{x-x_0}{y_0})$.

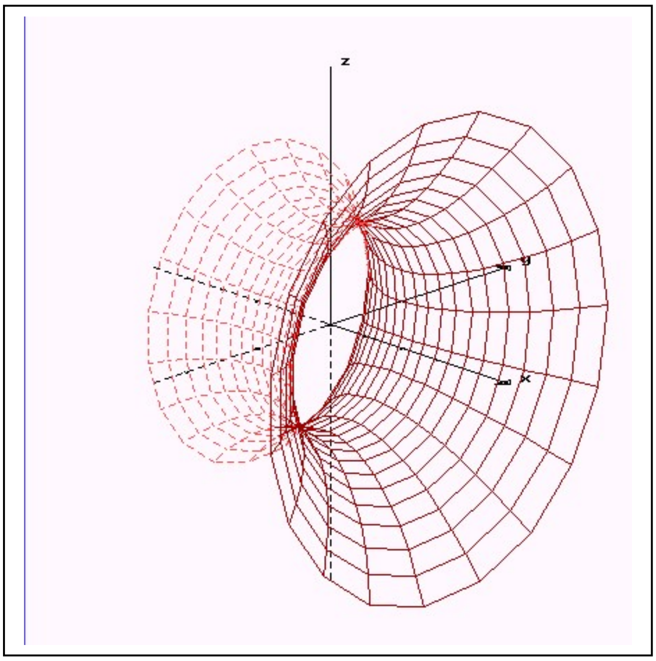
Since $\cosh^2 x + \sinh^2 x = 1$ and $\cosh' x = \sinh x$ follows that a solution is:

$$y = y_0 \cosh(\frac{x-x_0}{y_0}) + k \ .$$

Inserting in the equation, we find that y is a solution to the Euler-Lagrange equation, only if k=0 and $C = y_0$, So:

$$y = y_0 \cosh(\frac{x-x_0}{y_0})$$

Drawing the solution with a graphics computer program, it may look like the figure below.



2.2 Soap bubbles and wine barrels

If one wants to determine the shape of a soap bubble (we actually know that it is a sphere), then it is only a little more difficult than the calculation above.
The functional, which determine de least surface is actually the same, namely

$$O(y) = 2\pi\int_a^b y\sqrt{1+y'^2}\,dx\,,$$

But we must add a *side condition*, which determines that the volume must be held constant.

The volume of a body, which is the result of a rotation around the x-axis of a function $y = f(x)$ is given by the expression:

$$V(y) = \pi\int_a^b y^2 dx$$

In the calculus of variation it is proven that if you want to determine the extremes (max or min) , submitted to a side condition, that is, finding max or min for the functional

$$\int_a^b F(y',y,x)dx\,, \quad \text{having the side condition:} \quad \int_a^b G(y',y,x)dx = C\,,$$

So it can be accomplished by inventing a so called Lagrange multiplier λ, and instead determing the max or min for the functional:

$$\int_a^b (F(y',y,x) + \lambda G(y',y,x))dx\,,$$

where the constant λ is to be determined from the border conditions.

(For details consult e.g.: www.olewitthansen.dk/mathematics/calculus_of_variation).

In the case of a soap bubble, we must determine the min for the functional
.

$$O(y) = \int_a^b (2\pi y\sqrt{1+y'^2} + \lambda\pi y^2)dx$$

Since the integrand does not explicitly depend on x, we can apply the first order expression of the Euler-Lagrange equations.

$$y'\frac{\partial F}{\partial y'} - F = C\,,$$

When the differentiation after y' is performed it gives:

$$2y'\frac{2yy'}{2\sqrt{1+y'^2}} - 2y\sqrt{1+y'^2} - \lambda y^2 = C$$

Multiplying the equation by $\sqrt{1+y'^2}$ and reducing, we obtain:

$$2y + \sqrt{1+y'^2}\,(C + \lambda y^2) = 0$$

It is not really sensible to try to solve this differential equation analytically.
Instead, (since we know that the solution is a sphere), we shall investigate, if we can determine C and λ, such that the equation has the solution of the form: $y = \sqrt{r^2 - x^2}$, which is the equation for semi-circle. From $y = \sqrt{r^2 - x^2}$ it follows:

$$y' = \frac{-x}{\sqrt{r^2 - x^2}} \quad \Rightarrow \quad 1 + y'^2 = \frac{r^2}{r^2 - x^2} \quad \Rightarrow \quad \sqrt{1 + y'^2} = \frac{r}{\sqrt{r^2 - x^2}}$$

When inserting in the Euler-Lagrange equation, it gives:

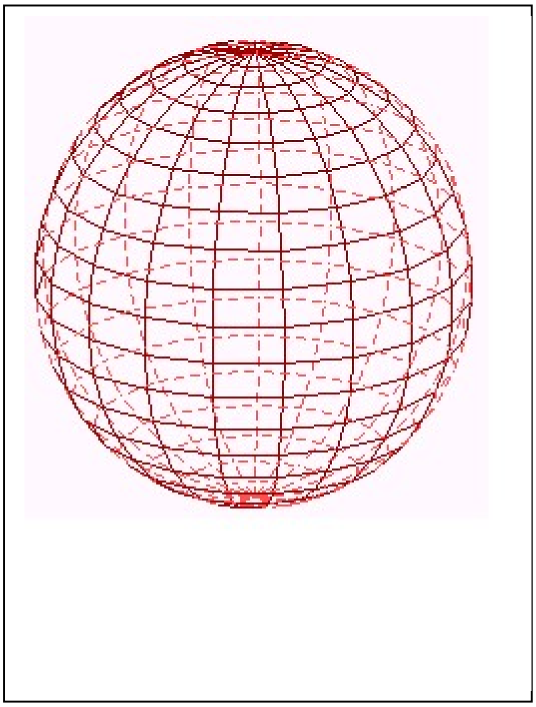
$$2\sqrt{r^2 - x^2} + \lambda(r^2 - x^2 + C)\frac{r}{\sqrt{r^2 - x^2}} = 0 \quad \Leftrightarrow$$

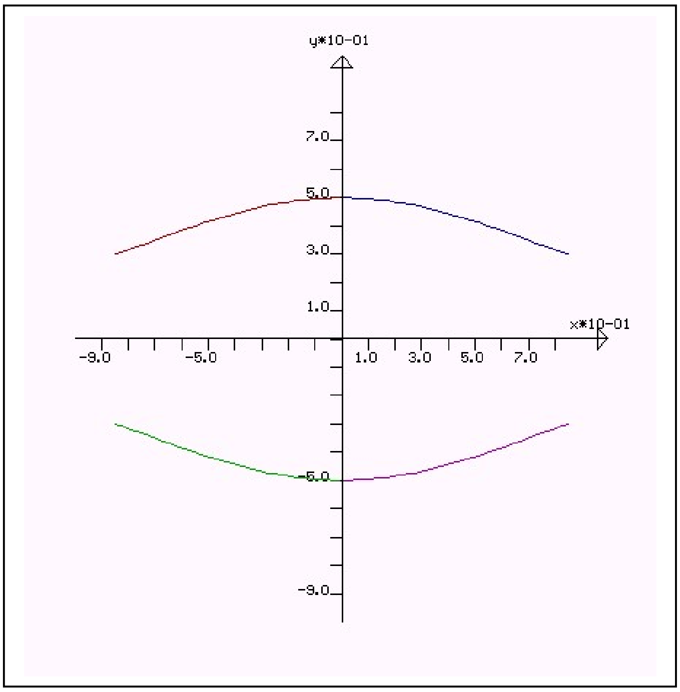
$$2(r^2 - x^2) + \lambda r(r^2 - x^2 + C) = 0$$

We can see that the last equation has solution for all x, only if : $C = 0$ and $\lambda = -2/r$.
The sphere is, however, the only solution if $y(r) = y(-r) = 0$. See the figure below.

On the other hand $C \neq 0$, means that the surface is not closed, but has two flat ends.
This could for example be the case if $f(a) = r_a$ and $f(b) = r_b$. If $r_a = r_b$, then the intersecting curve y = $f(x)$, might be the intersecting curve of a wine barrel.

The shapes of circular wine barrels with two flat ends, are containers having the least surface for a given volume has been known for more than a thousand years from carpenters having no knowledge the calculus of variations or computer simulations.





3. On the shape of water drops. Dew drops on a plane underlay

Above, we have demonstrated mathematically that a drop in a free fall has a spherical shape. This is, however, not the case for a dewdrop, which either is pended from a twig or lies on a leaf. In these cases both gravity and surface tension contributes to the potential energy
The shape of a supported dewdrop delivers therefore an obvious challenge to the calculus of variation, namely to determine the shape, so that the potential energy is at minimum.
If the water drop is on a horizontal underlay, it is not really possible to establish an expression which can be used in the calculus of variation. The reason is that the border condition with the underlay is not fixed.
However, you may establish some simplified assumptions. Since a free drop is a sphere, it is natural to assume that the drop lying on a flat underlay can be described by an ellipse which is rotated along a vertical axis.

3.1 The geometrics of an ellipsoid

To determine the potential energy due to gravity and due to surface tension it is necessary to know an expression for the surface and the volume of an ellipsoid, that is, when a function $y = f(x)$ is rotated 360^0 around the y-axis. The formulas are the same (with the exchange of x with y) as for a rotation around the x-axis.

The volume dV of a disc having radius x and thickness dy is:

$$dV = \pi x^2 dy = \pi x^2 dy/dx \cdot dx = \pi x^2 f'(x)dx = \pi x^2 y' dx$$

The surface of this disc is:

$$dS = 2\pi x ds = 2\pi x\sqrt{dx^2 + dy^2} = 2\pi x\sqrt{1 + (\tfrac{dx}{dy})^2}\,dy = 2\pi x\sqrt{1 + x'^2}\,dy$$

An ellipse with the two semi-axis a and b has the equation: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

The volume is most easily evaluated by integration.

$$V = \pi\int_{-b}^{b} x^2 dy = \pi a^2 \int_{-b}^{b}(1 - \frac{y^2}{b^2})dy = \pi a^2\left[y - \tfrac{1}{3}\frac{y^3}{b^2}\right]_{-b}^{b} = \tfrac{4}{3}\pi a^2 b$$

This is a well known formula, and it is seen that this gives the correct formula for the volume of a sphere, when $a = b = r$.
The formula for the surface of an ellipse when rotated around the minor axis is more involved. I have not been able to find the formula in a mathematical handbook, and the calculation is also somewhat more complicated. As is the case of the volume, the integration is most easily performed by integration along the y-axis.

$$S = 2\pi\int_{-b}^{b} x\sqrt{1 + x'^2}\,dy$$

From the equation of the ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Leftrightarrow x^2 = a^2(1 - \dfrac{y^2}{b^2})$

We find by implicit differentiation after y: $2xx' = -2y\dfrac{a^2}{b^2}$ and subsequently:

$$x' = -\frac{a^2}{b^2}\frac{y}{x} \quad \Rightarrow \quad 1 + x'^2 = \frac{x^2 b^4 + a^4 y^2}{x^2 b^4}$$

Inserting $x^2 = a^2(1 - \frac{y^2}{b^2})$ in the numerator, we get:

$$1 + x'^2 = \frac{x^2 b^4 + a^4 y^2}{x^2 b^4} = \frac{a^2(1 - \frac{y^2}{b^2})b^4 + a^4 y^2}{x^2 b^4} = \frac{a^2 b^4 - a^2 b^2 y^2 + a^4 y^2}{x^2 b^4}$$

$$1 + x'^2 = \frac{a^2 b^4 - a^2 b^2 y^2 + a^4 y^2}{x^2 b^4} = \frac{a^2}{x^2}\frac{b^4 - b^2 y^2 + a^2 y^2}{b^4} = \frac{a^2}{x^2}\left(1 - \frac{a^2(1 - \frac{b^2}{a^2})y^2}{b^4}\right)$$

Introducing the eccentricity *e*, where $e^2 = 1 - \frac{b^2}{a^2}$, we deduce, after some rearranging of the integrand:

$$S = 2\pi \int_{-b}^{b} x\sqrt{1 + x'^2}\,dy = 2\pi a \int_{-b}^{b} \sqrt{1 + \frac{a^2 e^2}{b^4} y^2}\,dy$$

Finally, if we put $k = \frac{ae}{b^2}$ then the integral is reduced to:

$$S = 2\pi a \int_{-b}^{b} \sqrt{1 + \frac{a^2 e^2}{b^4} y^2}\,dy = 2\pi a \int_{-b}^{b} \sqrt{1 + (ky)^2}\,dy$$

We have then reduced the integral to finding an integral of $\sqrt{1 + x^2}$

We make the substitution $x = \sinh t$ => $1+x^2 = \cosh^2 t$ and $dx = \cosh t\, dt$, and then

$$\int \sqrt{1 + x^2}\,dx = \int \sqrt{1 + \sinh^2 t}\cosh t\,dt = \int \cosh^2 t\,dt = \tfrac{1}{2}\int (\cosh 2t + 1)\,dt = \tfrac{1}{2}(\tfrac{1}{2}\sinh 2t + t)$$

$$\tfrac{1}{2}(\tfrac{1}{2}\sinh 2t + t) = \tfrac{1}{2}(\sinh t \cosh t + t) = \tfrac{1}{2}(\sinh t\sqrt{1 + \sinh^2 t} + t) = \tfrac{1}{2}(x\sqrt{1 + x^2} + \sinh^{-1} x)$$

Applying this on the surface integral $S = 2\pi a \int_{-b}^{b} \sqrt{1 + (ky)^2}\,dy$, we obtain after the substitution:

$x = ky = \frac{ae}{b^2} y \Rightarrow dy = \frac{b^2}{ae} dx$, with the new limits: $x = -\frac{ae}{b}$ *and* $x = \frac{ae}{b}$

Which after insertion of the limits and reduction can be transformed into:

$$\text{r}\, S = 2\pi ab\left(\frac{a}{b} + \frac{\sinh^{-1}(\frac{ae}{b})}{\frac{ae}{b}}\right)$$

For small x we have $\sinh^{-1} x \approx x - \frac{1}{6}x^3 \;\Rightarrow\; \dfrac{\sinh^{-1} x}{x} \approx 1 - \frac{1}{6}x^2$ *so that* $\dfrac{\sinh^{-1} x}{x} \to 1$ *for* $x \to 0$

We notice that for $a = b = r$ and $e \to 0$ $S = 4\pi r^2$, which is the surface of a sphere.

Since the elliptical shape in advance is an approximation, we shall in the first approximation substitute the ratio $\dfrac{\sinh^{-1}(\frac{ae}{b})}{\frac{ae}{b}}$ with 1. Hereby we obtain a relatively simple expression for the surface area. $S = 2\pi ab\left(\frac{a}{b} + 1\right) = 2\pi(a^2 + ab)$

Hereafter, we may directly present an exact and an approximate expression for the potential energy from the surface tension.

$$E_{pot}(S) = \gamma S = 2\pi\gamma ab\left(\frac{a}{b} + \frac{\sinh^{-1}(\frac{ae}{b})}{\frac{ae}{b}}\right) \approx 2\pi\gamma(a^2 + ab) \;\; for \;\; \frac{ae}{b} << 1$$

We define r from the equation $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi a^2 b \;\Rightarrow\; a^2 b = r^3$, where r is the radius in a sphere having the same volume as the volume of the ellipsoid. Then we make the following rewriting.

$$\frac{ae}{b} = \frac{a\sqrt{1 - \frac{b^2}{a^2}}}{b} = \sqrt{\left(\frac{r}{b}\right)^3 - 1} \;\; , \;\; ab = \frac{r^3}{a} = r^{\frac{3}{2}}\sqrt{b} \;\text{ and }\; \frac{a}{b} = \left(\frac{r}{b}\right)^{\frac{3}{2}}$$

Then the expression for the surface tension potential energy E_{pot}(S) can then be reduced to:

$$E_{pot}(S) = \gamma S = 2\pi\gamma\left(\frac{r^3}{b} + \frac{r^{\frac{3}{2}}b^2}{\sqrt{r^3 - b^3}}\sinh^{-1}\left(\sqrt{\left(\tfrac{r}{b}\right)^3 - 1}\right)\right) \approx 2\pi\gamma\left(\frac{r^3}{b} + r^{\frac{3}{2}}\sqrt{b}\right)$$

For a body suspended in the gravity field, we know from mechanics, that the potential energy from gravity can be found as the whole mass of the body is situated in the CM (centre of mass) of the body. The CM for an ellipsoid is found in the centre of symmetry of the ellipsoid.
If the ellipsoid is supported on a plane surface at $y = 0$, then the CM is raised the amount b, and the potential energy from gravity is $E_{pot}(M) = mgb$. Since b most often is used as a constant, and not a variable, we shall use z instead of b. such that $E_{pot}(M) = mgz$ .We shall then determine the min of:

$$E_{pot}(z) = E_{pot}(S) + E_{pot}(M)$$

The *exact* expression for the sum of the two potential energies is:

$$E_{pot}(z) = \rho V g z + 2\pi\gamma\left(\frac{r^3}{z} + \frac{r^{\frac{3}{2}} z^2}{\sqrt{r^3 - z^3}}\sinh^{-1}\left(\sqrt{\left(\frac{r}{z}\right)^3 - 1}\right)\right) = \tfrac{4}{3}\rho\pi r^3 g z + 2\pi\gamma\left(\frac{r^3}{z} + r^{\frac{3}{2}}\sqrt{z}\,\frac{\sinh^{-1}\sqrt{\left(\frac{r}{z}\right)^3 - 1}}{\sqrt{\left(\frac{r}{z}\right)^3 - 1}}\right)$$

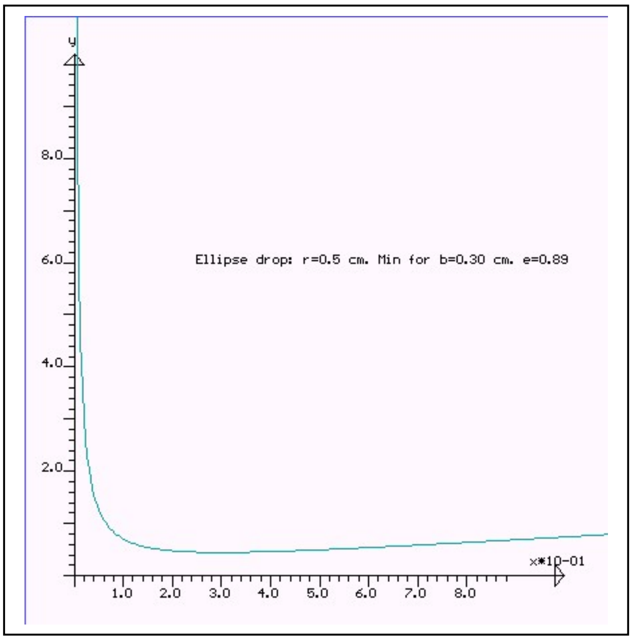
And an approximate expression becomes, if we put $k_1 = \frac{4}{3}\pi r^3 \rho g$, $k_2 = 2\pi\gamma r^3$, $k_3 = 2\pi\gamma r^{\frac{3}{2}}$ then the two expressions for $E_{pot}(z)$ are written:

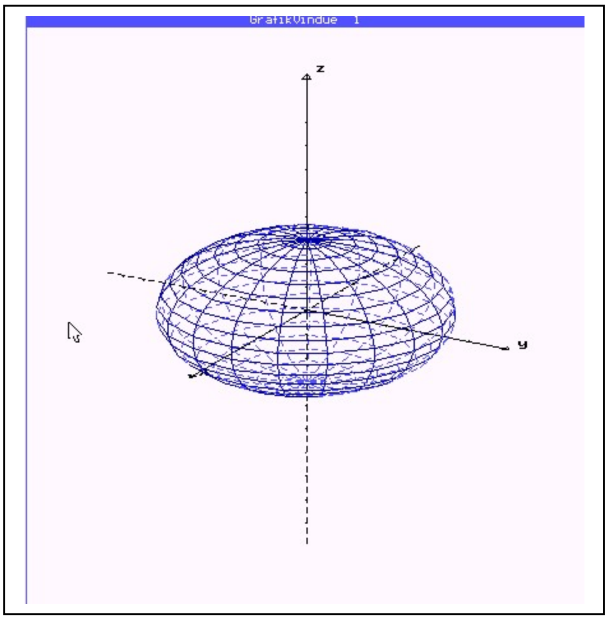
$$E_{pot}(z) \approx k_1 z + \frac{k_2}{z} + k_3\sqrt{z}) \quad \text{and} \quad E_{pot}(z) \approx k_1 z + \frac{k_2}{z} + k_3\sqrt{z}\,\frac{\sinh^{-1}\left(\sqrt{\left(\frac{r}{z}\right)^3 - 1}\right.}{\sqrt{\left(\frac{r}{z}\right)^3 - 1}}$$

It turns out that it is not possible to determine the minimum for the sum of the two expressions analytically. This stems from the fact, the border conditions are not fixed.
It is, however, possible to find the min for the approximate expressions, when $\sqrt{\left(\frac{r}{z}\right)^3 - 1}$ is small, that is, when the shape of the ellipsoid does not deviate substantially from a sphere.

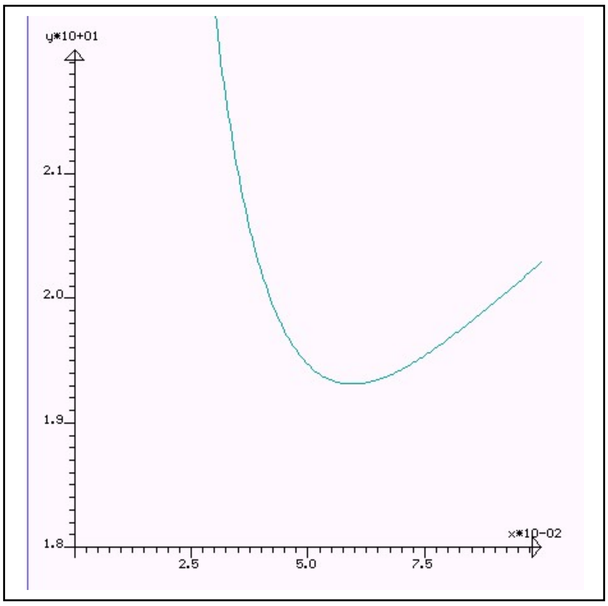
It is most practical to do the numerical calculations in cgs units, and in these units we have:
$\gamma = 76$ dyn/cm, g = 982 cm/s^2 and $\rho = 1.0$ g/cm^3. For a rain drop, it corresponds to $r = 0.5$ cm.

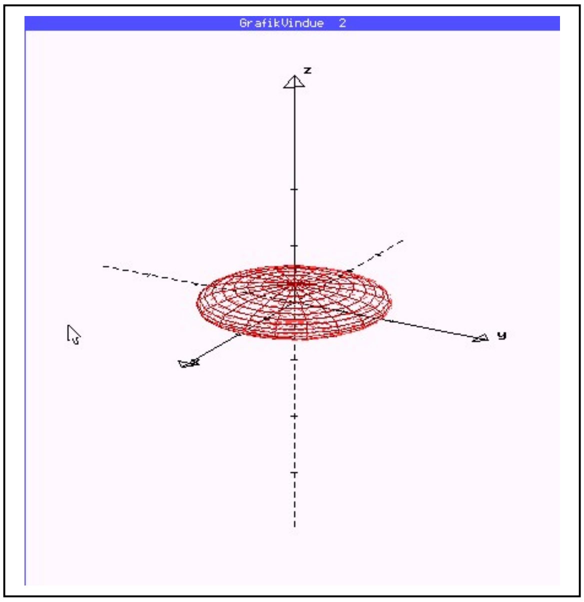
The resulting two curves for E_{pot} are shown below. The min is found by differentiation of the two expressions above, followed by solving $f'(x) = 0$ numerically. This gives the result ($b = 0.2988$) and $b = 0.2677$ corresponding to $a = 0.4395$ and $e = 0.79$. In the figure below to the right is shown the corresponding ellipsoid.





For r = 0.10 cm, the approximate formula cannot be applied any longer, so we have used the exact formula.
The figure to the left shows the min for the potential energy for r = 1.0 cm. The figure to the right shows the very compressed ellipsoid for r = 1.0 cm. This is what could be expected, since gravity is proportional to the mass, whereas the surface tension is proportional to the surface.





3.2 Raindrops are falling…

As we have demonstrated above one cannot really apply the calculus of variation for a raindrop or a mercury drop on a plane underlay, and the reason is that we do not have a fixed boundary condition.

A different situation comes about, if the water drop is suspended on a leaf or a twig.
Here it is actually possible to reproduce the form of the hanging drop, by determining the minimum for the potential energy from gravity and from surface tension. We have two suppositions:

1. The potential energy from the surface tension is least for a sphere.
2. The potential energy from gravity: $E_{pot} = mgh$, is least if the liquid drop is smeared out in a large area as possible and situated as low as possible.

If the mass becomes too big, then the surface tension can no longer keep the liquid together, in accordance with the observation that you never see water drops on more than a few mm, likewise when tiny mercury drops are collected on a table, they do not form larger drops, but rather splashes out.

The potential energy of a liquid drop having surface S has, (as a consequence of the surface tension), the potential energy $E_{pot} = \gamma S$, where γ is a material constant.

To determine the shape of a hanging drop is an obvious problem for the calculus of variation, since it amounts to determine the minimum for the sum of potential energies.

$$E_{pot}(\text{Surface tension}) + E_{pot}(\text{Gravity}).$$

Unfortunately the resulting differential equations have no analytical solution, and we must satisfy ourselves with a computer generated simulation to the form of the drop.
However, one may determine the approximate maximal size of the drops from the analytic equation.
But to do so, we must first establish some formulas for the surface of a rotational body around the y-axis, and the gravitational potential energy of a rotational symmetric disc with thickness dy in the height y. First we find the expression for the differential surface.

$$dS = 2\pi x\sqrt{dx^2 + dy^2} = 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi x\sqrt{1+y'^2}\,dx$$

$$dE_{pot}(S) = \gamma dS = 2\pi\gamma x\sqrt{1+y'^2}\,dx$$

And then for the differential volume

$$dV = \pi x^2 dy = \pi x^2 \frac{dy}{dx}dx = \pi x^2 y'\,dx$$

$$dE_{pot}(V) = (\rho dV)gy = \rho g\pi x^2 yy'\,dx$$

Then the task is to find min for the collected potential energy.

$$2\pi\gamma\int_{-r}^{r} x\sqrt{1+y'^2}\,dx + \pi\rho g\int_{-r}^{r} x^2 yy'\,dx = \pi\int_{-r}^{r}(2\gamma x\sqrt{1+y'^2} + \rho g x^2 yy')dx$$

Due to the side condition:

$$\pi\int_{-r}^{r} x^2 y'\,dx = V_0$$

We shall drop the common factor π, and the Lagrange function $F(y',y,x)$ is hereafter, when using a Lagrange multiplier λ

$$F(y',y,x) = 2\gamma x\sqrt{1+y'^2} + \rho g x^2 yy' + \lambda x^2 y'$$

In this case, however, F does depend explicitly on x, so we are compelled to use the general form of the Euler – Lagrange equations, which is a second order differential equation in y.

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

First we evaluate the partial derivatives.

$$\frac{\partial F}{\partial y} = \rho g x^2 y'$$

$$\frac{\partial F}{\partial y'} = \frac{2\gamma x y'}{\sqrt{1+y'^2}} + \rho g x^2 y + \lambda x^2$$

And then we do the differentiation after x.

$$\frac{d}{dx}\frac{\partial F}{\partial y'} = 2\gamma\left(\frac{y'}{\sqrt{1+y'^2}} + x\left(\frac{y''}{\sqrt{1+y'^2}} - \frac{y'2y'y''}{2(1+y'^2)^{\frac{3}{2}}}\right)\right) + \rho g(2xy + x^2 y') + 2\lambda x$$

Inserting in the Eulers-Lagrange equation: $\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y}$, reducing and rearranging, we find:

$$\frac{2\gamma}{\sqrt{1+y'^2}}\left(y' + \frac{xy''}{1+y'^2}\right) + \rho g 2xy + 2\lambda x = 0$$

Finally isolating y'':

$$y'' = -(1+y'^2)\left(\sqrt{1+y'^2}\left(\frac{\rho g}{\gamma} y + \frac{\lambda}{\gamma}\right) + \frac{y'}{x}\right)$$

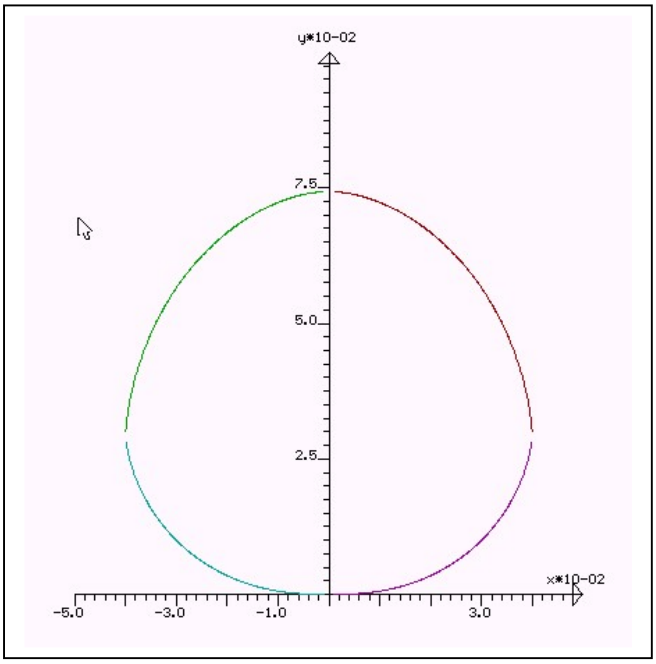
If we want a solution with a starting point in (0,0), which lies in the upper half plane, we must remove the minus sign. Otherwise the solution will move downwards in the negative half plane, giving a mirror symmetric solution. We are then looking for a solution to:

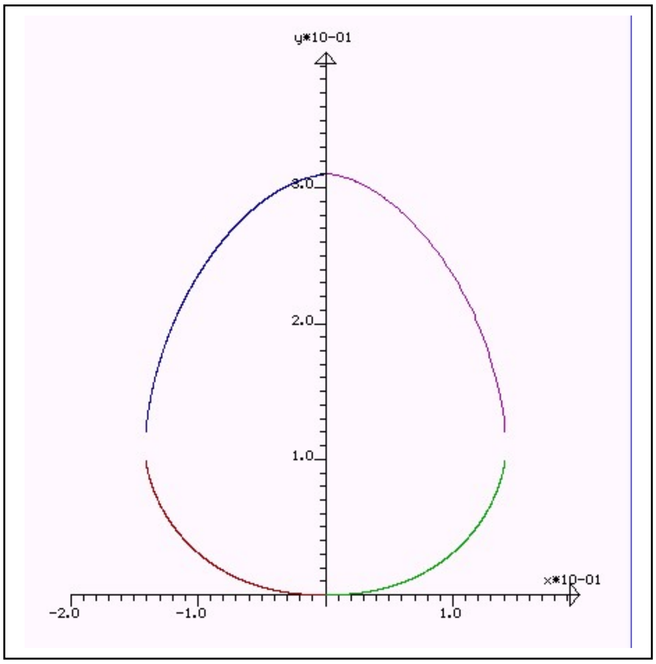
$$y'' = (1+y'^2)\left(\sqrt{1+y'^2}\left(\frac{\rho g}{\gamma} y + \frac{\lambda}{\gamma}\right) + \frac{y'}{x}\right)$$

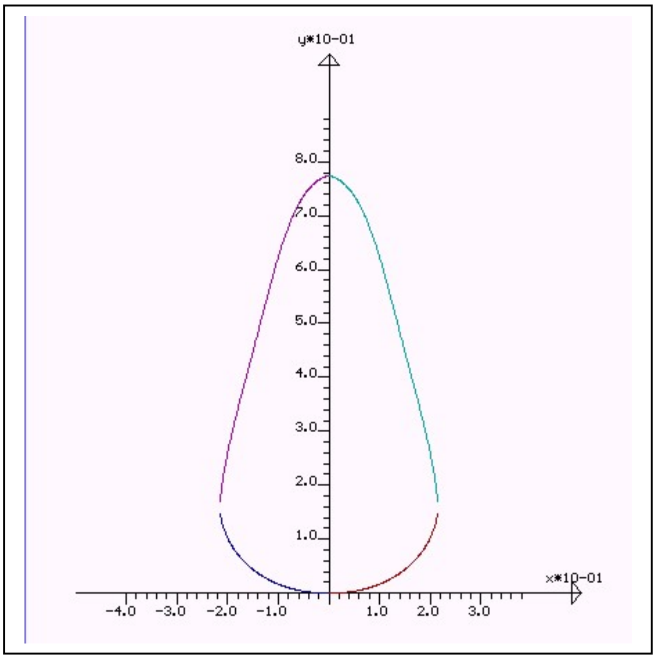
It is not possible (at least I can't) to find an analytic solution to the differential equation, so we are referred to a numeric computer graphic solution.

It is difficult to comment on the value of λ, so we are left to making numerical trials for different values.
Below are shown solutions for $\lambda = 50$, $\lambda = 100$ and $\lambda = 500$, where the largest drop belongs to $\lambda = 50$. The numeric solutions are, however, not without problems, since the intersecting curve has two vertical tangents, the expression is not defined for $x = 0$. One is therefore referred to divide the solution in four different pieces, being careful with the limits of integration.

Below is shown the cross section for three drops corresponding to the widths 0.35 mm, 1.5 mm and 2.3 mm. The largest drop has the length 8 mm. Finally is shown a photo of a hanging drop. The resemblance with the last computer generated drop is remarkable.
If you experiment with larger drops, you will find no solution for the upper part of the drop.
It is, however, possible to give a qualitative analytic reason for that. Physically it probably means that the drop gets too heavy and falls. We shall elaborate on that following the pictures.







From the differential equation it is possible to understand, why there is an upper limit for the size of the drop. If we denote the point, where the curve of the cross section goes from convex to concave with (*a, b*), then *y''* must be less than zero for *y* > *b*. This gives the inequality.

$$\sqrt{1+y'^2}\left(\frac{\rho g}{\gamma}b+\frac{\lambda}{\gamma}\right)+\frac{y'}{a}<0 \quad\Leftrightarrow\quad a\left(\frac{\rho g}{\gamma}b+\frac{\lambda}{\gamma}\right)<-\frac{y'}{\sqrt{1+y'^2}}$$

The function $f(y')=\dfrac{y'}{\sqrt{1+y'^2}}$ has values in the interval]-1, 1[so the condition can be written as:

$$a\left(\frac{\rho g}{\gamma}b+\frac{\lambda}{\gamma}\right)<1 \;\Leftrightarrow\; \lambda<\frac{\gamma}{a}-\rho g b$$

Inserting γ = 76 dyn/cm and ρg = 982 dyn/cm^3, we get the inequality:

$$\lambda<\frac{76}{a}-982b$$

Estimating b = 0.5a, and investigating when $\frac{76}{a}-491a>0$, we find that a <0.4 cm, which means that the width of the drop is at most 4 mm.
This is in good accordance with the results from the results from the numerical solution of the Euler-Lagrange differential equations.
Estimating b = 0.3 a, we arrive at a max width of the drop 0.5 cm.