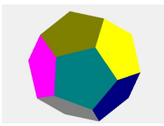
Can you score on a corner kick?

A theoretical approach

This is an article from my home page: www.olewitthansen.dk



Ole Witt-Hansen 2013

Contents

1. Is it possible to hit the goal from a corner kick?

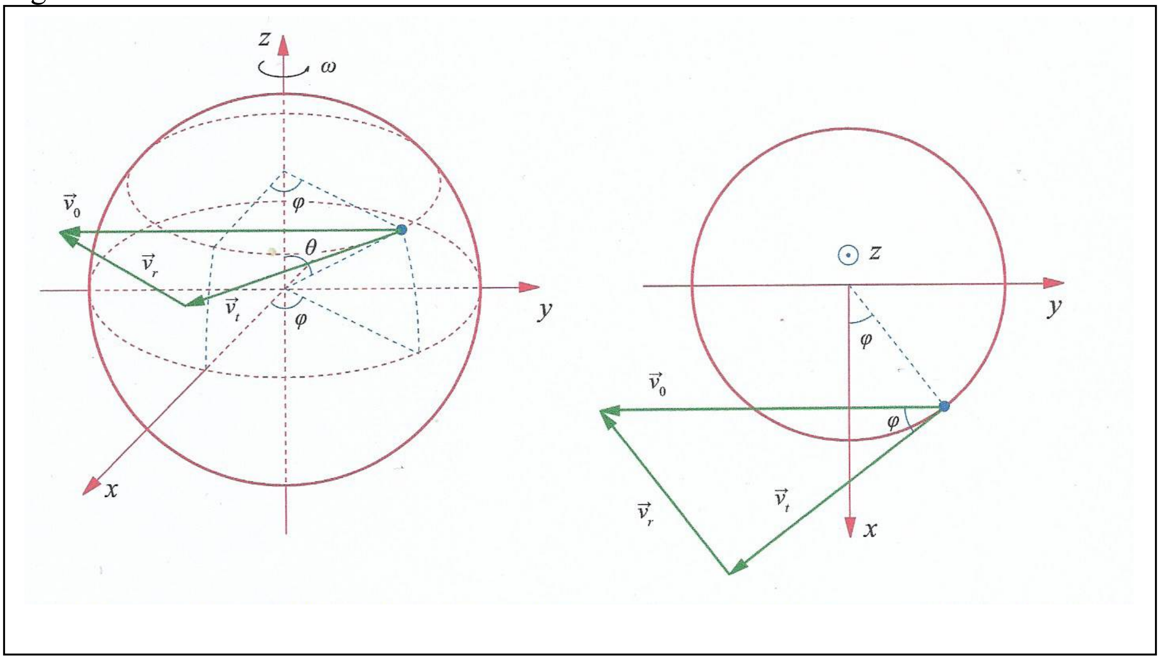
There are several videos on YouTube, demonstrating that it can actually be done (although it is extremely technical to perform).
(Whether you find that a theoretically calculation is equally technical depends on who you are). However, if we use some empirical formulas for the air resistance on a solid surface, it is possible to establish three coupled differential equations that can be solved numerically, and which demonstrates that it is (also theoretically) possible to score a goal from a corner kick.

1.1 Establishing the geometry of the problem

The figure below shows a (European) football with radius *r*, moving in the *y*-direction, which means that the air resistance is coming from the opposite (-*y*) direction.
A point on the football is designated by the polar coordinates (r, θ, φ). The spatial geometry may visually be hard to appreciate, so there is made a cut parallel to the $x - y$ plane, where the velocity of the wind $\vec{v}_0$ is dissolved into a tangential and a radial direction.
v_r designates the radial component of the wind , and v_t its tangential component.
From the figure to the right, we see that:

$$v_r = v_0 \sin\varphi \qquad v_t = v_0 \cos\varphi$$

Figure 1.1



If the ball rotates around the z – axis with angular frequency ω, then a point (r, θ, φ) will have a velocity in the *y* - direction, which is the sum of the balls velocity in the *y* - direction plus the *y* - component of the balls rotational velocity.
The speed in the uniform circular motion is given by: $v = \omega r$, and the speed in the circular motion, (corresponding to the polar angle θ) is therefore: $v_\theta = \omega r sin\theta$ and the *y*-component becomes:

(1.1) $$v_{\theta y} = \omega \cdot r \cdot sin\theta \cdot cos\varphi.$$

The velocity of a point on the football in the y-direction is thus: $v_0 + \omega r \sin\theta\cos\varphi$, and the radial and the tangential component are therefore:

(1.2) $$v_r = (v_0 + \omega r \sin\theta\cos\varphi)\sin\varphi \qquad v_t = (v_0 + \omega r \sin\theta\cos\varphi)\cos\varphi$$

We are interested in finding two forces, namely the force that acts opposite to the direction of motion of the ball, and the force in the x-direction, which acts perpendicular to the direction of motion of the ball.
To do so, we must calculate the components in the x- and y-direction of the radial velocity.
The radial velocities changes the speed and the direction of the ball, whereas the tangential velocities may possibly change the angular velocity.
In the following we shall, however, ignore the tangential velocities, since it is a minor effect.

From the figure above, you may convince yourself that the x- and y-components of the radial velocities are:

(1.3) $$v_{rx} = v_r\cos\varphi \qquad v_{ry} = v_r\sin\varphi$$

So we have:

(1.4) $$v_{rx} = (v_0 + \omega r \sin\theta\cos\varphi)\sin\varphi\cos\varphi \qquad v_{ry} = (v_0 + \omega r \sin\theta\cos\varphi)\sin\varphi\sin\varphi$$

For the air resistance, which acts opposite to the direction of motion, we shall apply the semi-empirical expression:

(1.5) $$F = -\tfrac{1}{2}c_w\rho A v^2 = -\gamma v^2$$

ρ is the density of the air. c_w is the so called form factor, A is the cross section of the ball, and v is the velocity of the body in the direction of the collective motion.

The forces F_x that act in the x-direction, which correspond to the angles φ and $\pi - \varphi$ are opposite directed, with respect to v_0, as it also appears from the expression for v_{rx}, but since we are going to square the velocities, we must evaluate F_x as:

(1.6) $$F_x = F_x(\varphi) - F_x(\pi - \varphi)$$

Having found the force F_x, acting in the point (r,θ,φ), it should then be multiplied by the area element dA of a sphere: $dA = r^2\sin\theta d\theta d\varphi$ and afterwards integrated over the semi-sphere: $\theta \in [0,\pi]$ *and* $\varphi \in [0,\tfrac{1}{2}\pi]$.

(1.7) $$dF = -\tfrac{1}{2}c_w\rho v^2 dA = -\gamma_1 v^2 dA$$

(1.8) $$\begin{aligned} dF_x(\varphi) - dF_x(\pi-\varphi) = -\gamma_1((v_0 + \omega r\sin\theta\cos\varphi)^2 \sin^2\varphi\cos^2\varphi\, r^2\sin\theta\, d\theta d\varphi - \\ (v_0 + \omega r\sin\theta\cos(\pi-\varphi))^2\sin^2\varphi\cos^2\varphi\, r^2\sin\theta\, d\theta d\varphi) \end{aligned}$$

The two terms are identical apart from a change of sign in the second term since $\cos(\pi-\varphi)=-\cos\varphi$. The two terms will therefore cancel each other, apart from two times the double product from the quadratic terms. After a minor reduction, we find:

(1.9) $$dF_x(\varphi)-dF_x(\pi-\varphi)=-\gamma_1 4v_0\omega\, r^3\sin^2\theta\cos^3\varphi\sin^2\varphi\, d\theta d\varphi$$

In the following, we shall make use of evaluating integrals of the type.

(1.10) $$\int\sin^n x\cos^m x dx$$

If *n* is even and *m* is odd (or vice versa) then it is relatively simple to evaluate the integrals by using the formula $\cos^2 x$ = 1 - $\sin^2 x$, (which is the same as: $\sin^2 x$ = 1 - $\cos^2 x$), and followed by a elementary substitution.
If both *m* and *n* are even, then the integral may be evaluated by a successive application of the same formulas and subsequently reducing the power of the trigonometric functions using the formulas:

(1.11) $$\cos^2 x=\frac{1+\cos 2x}{2}\quad and\quad \sin^2 x=\frac{1-\cos 2x}{2}$$

If both *n* and *m* are odd, one may apply the formula:

(1.12) $$\sin 2x=2\sin x\cos x$$

$$F_x=-\gamma_1 4v_0\omega r^3\int_0^{\frac{\pi}{2}}\int_0^{\pi}\sin^2\theta\cos^3\varphi\sin^2\varphi d\theta d\varphi$$

$$\int_0^{\pi}\sin^2\theta d\theta=\int_0^{\pi}\frac{1-\cos 2\theta}{2}d\theta=\left[\frac{\theta}{2}-\tfrac{1}{4}\sin 2\theta\right]_0^{\pi}=\frac{\pi}{2}$$

(1.13)
$$\int_0^{\frac{\pi}{2}}\cos^3\varphi\sin^2\varphi d\varphi=\int_0^{\frac{\pi}{2}}(1-\sin^2\varphi)\cos\varphi\sin^2\varphi d\varphi\ =$$
$$\int_0^{\frac{\pi}{2}}(\sin^2-\sin^4\varphi)d\sin\varphi=\left[\tfrac{1}{3}\sin^3\theta-\tfrac{1}{5}\sin^5\theta\right]_0^{\frac{\pi}{2}}=\frac{1}{3}-\frac{1}{5}=\frac{2}{15}$$

The final expression for F_x then becomes:

(1.13) $$F_x=-\frac{1}{2}c_w\rho\frac{4}{15}\pi v_0\omega\, r^3.$$

Notice that there are no transverse forces to the direction of motion, if ω = 0. (Of course)

The expression for the velocity in the y-direction is stated above: $v_{ry} = (v_0 + \omega r \sin\theta\cos\varphi)\sin^2\varphi$

The expression for the y- component of the radial force is: $F_y = -½ c_w \rho A v_{ry}{}^2 = -\gamma v^2$

This expressionshould be integrated over the semi-sphere.

(1.14) $$F_y = -\gamma_1 \int_0^\pi\int_0^\pi v_{ry}{}^2 r^2 \sin\theta d\theta d\varphi = -\gamma_1 \int_0^\pi\int_0^\pi ((v_0 + \omega r\sin\theta\cos\varphi)\sin^2\varphi)^2 r^2 \sin\theta d\theta d\varphi$$

The term that has cos φ as a factor will disappear in evaluating the integral, since cos φ is odd in the interval from 0 to pi, while all the other factors are even.
When the integrand is evaluated, we are left with.

(1.15) $$F_y = -\gamma_1 r^2 \int_0^\pi\int_0^\pi (v_0{}^2 \sin^4\varphi\sin\theta + \omega^2 r^2 \sin^4\varphi \sin^3\theta\cos^2\varphi) d\theta d\varphi$$

In almost in the same manner as we did above we find for the 4 integrals.

(1.16) $$\int_0^\pi \sin^4\varphi d\varphi = \frac{3\pi}{8} \qquad \int_0^\pi \sin^3\theta d\theta = \frac{4}{3} \qquad \int_0^\pi \sin\theta d\theta = 2 \qquad \int_0^\pi \sin^4\varphi\cos^2\varphi d\varphi = \frac{\pi}{16}$$

What is demonstrated here doing it by hand can of course be checked using a CAS, (but I was taught mathematics before the appearance of mathematical computers)

2. The equations of motion

We are then able to write the expression for F_y, the force opposite the direction of motion.

(2.1) $$F_y = -\frac{1}{2} c_w \rho r^2 \left(\frac{3\pi}{4} v_0{}^2 + \frac{\pi}{12}\omega^2 r^2\right)$$

Actually we have assumed that the football moves only in the y-direction. But this is not strictly correct, if you kick a ball in the air from the corner. Since the air resistance is always opposite to the velocity then the correct expression is:

$$\vec{F} = -½ c_w \rho A v^2 \frac{\vec{v}}{v} = -½ c_w \rho A v \vec{v}$$

Although the velocity might not be entirely perpendicular to the axis of rotation, we shall keep the expression for F_x, since anything else would be mathematically reckless, and the correction will possibly have a minimal effect on the results.

The acceleration is as usual found by dividing the forces by the mass of the football. Below is the acceleration written with the help of the three base vectors $(\vec{i}, \vec{j}, \vec{k})$

(2.2) $$\vec{a} = \frac{\vec{F}_x}{m} + \frac{\vec{F}_y}{m} + \frac{\vec{F}_z}{m} = -\frac{1}{2m} c_w \rho \frac{4}{15}\pi v \omega r^3 \vec{i} - \frac{1}{2m} c_w \rho r^2 \left(\frac{3\pi}{4} v \vec{v} + \frac{\pi}{12}\omega^2 r^2 \vec{j}\right) - g\vec{k}$$

Written out in coordinates after the *x, y* and *z*-axis, we have:

(2.3)
$$a_x = -\frac{1}{2m}c_w\rho r^2\frac{3\pi}{4}vv_x - \frac{1}{2m}c_w\rho\frac{4}{15}\pi v\omega r^3$$
$$a_y = -\frac{1}{2m}c_w\rho r^2(\frac{3\pi}{4}vv_y + \frac{\pi}{12}\omega^2 r^2)$$
$$a_z = -\frac{1}{2m}c_w\rho r^2\frac{3\pi}{4}vv_z - g$$

These three coupled second order differential equations, have no analytic solution (I believe) and even if you should succeed (but you won't) to find an analytic solution, it would probably be difficult to interpret.
But the equation may be solved numerically and the solutions plotted in a genuine 3D projection.

Inserting the numerically values for the constants: ρ_{air} =1.293 kg/m^3. The form factor c_w = 0.4. The radius of the football: r = 0.10 m. The mass of the football: m= 0.400 kg, we get the following numerical equations.

(2.4)
$$a_x = -1{,}52\cdot 10^{-2}vv_x - 5.40\cdot 10^{-4}v\omega$$
$$a_y = -1{,}52\cdot 10^{-2}vv_y + 1.68\cdot 10^{-5}\omega^2$$
$$a_z = -1{,}52\cdot 10^{-2}vv_z - 9{,}82$$

3. Graphic solution to the equations of motion

We have put the width of the football lane to 60 *m*.
The difficult task is now to determine (guess on) the speed of the ball, the angles θ, φ, which determine the direction, and the right screw (rotation) so that the football ends in the goal (if possible at all on the theoretical conditions?).
Certainly, it was not the first try that led to an acceptable result, but after experimenting a bit, the following values seem to accomplish the job.

Speed v_0 = 22.5 *m/s*, angular velocity: ω = 31.4 /s, Polar angle: θ = 60^0, Azimuth angle φ = 80^0 .

Using these values, it appears from the 3D figures that the football will have the right deflection in the air, and end up in the goal. But studying the solution more carefully, it turns out to be a deception, caused by the 3D projection. With these values the football lands just outside the goal. This is revealed if you look at the graphs for $x(t)$, $y(t)$ and $z(t)$ where can see, that although the ball is near the goal, it is not inside. (It does not cross the x-coordinate.)

So the graphs shown below, where the football actually enters the goal, are made with an unrealistic high rotation of the ball.

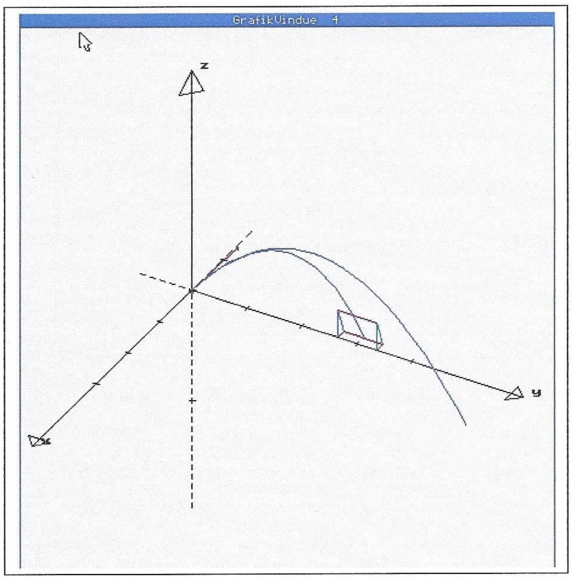
Whether it is actually possible to determine realistic values of speed, angles and air resistances to prove theoretically that you can score on a corner kick, I don't know, but perhaps it is not so important after all. There are many unknown factors, when dealing wit turbulent air drag.

The calculations show, however, clearly that if you kick the ball from the left corner, and kick the football in the right side obtaining a positive rotation of the ball, and using the semi-empirical formulas for the air drag, then the trajectory will be deflected to the left towards the goal.
Also the formulas for turbulent air resistance are mostly empirical, and they depend rather heavily on the shape of the moving object.

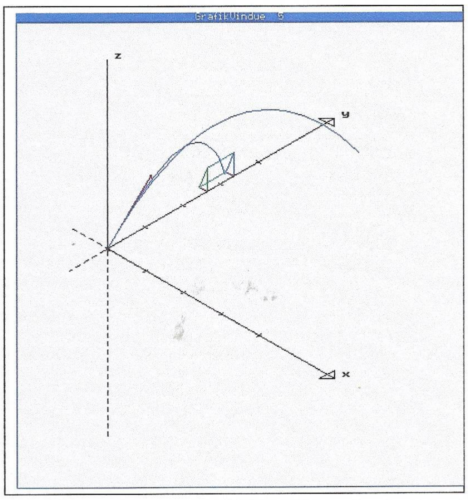
The first two of the three graphs show the trajectory of the same corner kick, but observed from different seats of the spectators.

In the third graph is plotted the graphs for $x(t)$, $y(t)$ and $z(t)$ are with and without air resistance.
It is seen that the ball actually crosses the x-axis and lands inside the goal.

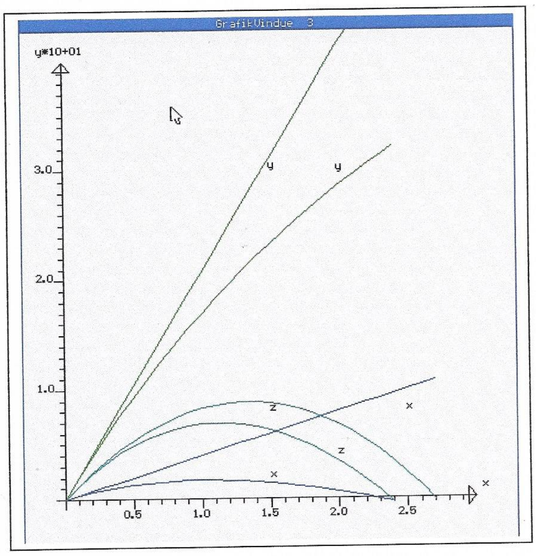
Kick from the origin. The two trajectories of the ball are with and without air resistance. The ball passes the x-axis.



Same kick, but seen from another perspective.



The two trajectories for $x(t)$, $y(t)$ and $z(t)$ are with and without air resistance. It is seen that the ball crosses the x-axis and lands inside the goal.



4. Postscript. The old days of scientific programming

The numerical solution of the 3 coupled second order differential equations and the 3D graphical representation of the solutions comes from a program that I developed in the early 90ties.
The program is a DOS program, written in Borland's Turbo 7.0. There were no windows, no menu's nothing that we later considered as granted. You had to do it all for your self, from lines, squares, circles, colours in the graphic system.
You could not print the graphics screen from the program, and the ability to make a screen dump ended with Windows 98. After Windows XP, dos programs can no longer run on a computer.
So the program is run on a Windows XP computer, and the screen dumps are done on a Windows 98 computer.
The obvious question is, why do I not use a modern program, and the answer is that even if I am aware of the existence of such programs, I have not found one which is able to the same. But there might of course.