Why can a box only rotate freely about two of its three principal axes

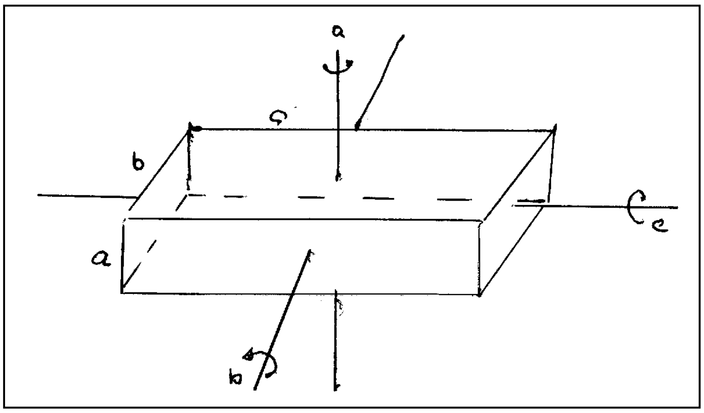
This is an article from my home page: www.olewitthansen.dk

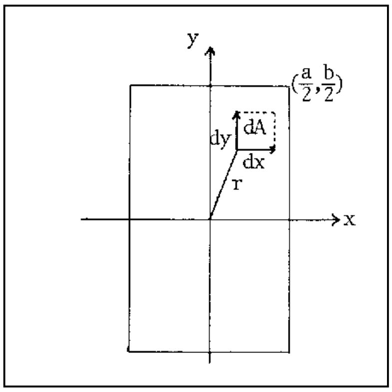
Contents

Ole Witt-Hansen 2009

1. Moment of inertia about the three axes of a box

A rectangular box has three symmetry axes. If the sides of the box are *a, b, c,* then we shall assume that $a < b < c$. The moment of inertia about any of the symmetry axes can be calculated as the moment of inertia of a rectangular plate with respect to an axis through its symmetry centre as shown in the figure to the right.





The procedure of evaluation the moment of inertia about the three axes *a, b, c*, is the same, so we settle for evaluation the moment of inertia about the axis *c*. The moment of inertia is however the same as that of a plate with sides *a* and *b*. For such a plate the moment of inertia with respect to an axis perpendicular to and through its symmetry centre can be evaluated by integration as:

$$I_c = \int r^2 dm$$

If the mass of the plate is *m,* and the area of the plate is $A = ab$, the density per unit area is $\rho = \frac{m}{A}$, so in this case: $dm = \frac{m}{A} dxdy$ as depicted in the figure to the right. $r^2 = x^2 + y^2$, and the moment of inertia is evaluated first by an integration over *x*, followed by an integration over *y*.

$$I_c = \int r^2 dm = \frac{m}{ab} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2 + y^2) dxdy$$

First we do the integral over *x*.

$$I_c = \frac{m}{ab} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2 + y^2) dxdy = \frac{m}{ab} \int_{-\frac{b}{2}}^{\frac{b}{2}} \left[\tfrac{1}{3} x^3 + y^2 x\right]_{-\frac{a}{2}}^{\frac{a}{2}} dy =$$

$$I_c = \frac{m}{ab} \int_{-\frac{b}{2}}^{\frac{b}{2}} (\tfrac{1}{12} a^3 + ay^2) dy = \frac{m}{ab} \left[\tfrac{1}{12} a^3 y + \tfrac{1}{3} ay^3\right]_{-\frac{b}{2}}^{\frac{b}{2}} = \frac{m}{12}(a^2 + b^2)$$

In the same manner we find: $I_b = \frac{m}{12}(a^2 + c^2)$ and $I_a = \frac{m}{12}(b^2 + c^2)$, so

$$I_c = \frac{m}{12}(a^2 + b^2) \ , \ I_b = \frac{m}{12}(a^2 + c^2) \ , \ I_a = \frac{m}{12}(b^2 + c^2)$$

Since we have assumed that $a < b < c$ it follows: $I_a > I_b > I_c$.

2. Rotation about the three axes

The experience is that any rectangular box may be put in stable rotation about the longest and the shortest axis, but if you try to make it rotate freely about the medium axis the rotational motion quickly becomes chaotic.
You may wonder why, but below we shall present a theoretical explanation, but it is not as entirely trivial as you may think.
For an arbitrary free rotation of a box the kinetic energy as well as the angular momentum are conserved.

$$E = ½ I_a \omega_a^2 + ½ I_b \omega_b^2 + ½ I_c \omega_c^2 \quad \text{and} \quad L^2 = I_a^2 \omega_a^2 + I_b^2 \omega_b^2 + I_c^2 \omega_c^2$$

From these two equations we can eliminate ω_a^2 by multiplying the first equation by $2I_a$ and subtract it from the second equation. Then we find:

$$L^2 - 2EI_a = I_b(I_b - I_a)\omega_b^2 + I_c(I_c - I_a)\omega_c^2$$

We obtain two similar expressions by eliminating ω_b^2 and ω_c^2 from the two equations

$$L^2 - 2EI_b = I_a(I_a - I_b)\omega_a^2 + I_c(I_c - I_b)\omega_c^2$$

$$L^2 - 2EI_c = I_a(I_a - I_c)\omega_a^2 + I_b(I_b - I_c)\omega_b^2$$

The three expression can be considered as quadratic forms in two of the angular velocities.
As we have assumed $I_a > I_b > I_c$, we can see that $I_b - I_a < 0$ and $I_c - I_a < 0$, in the first expression, so it can never be positive. Similarly for the last expression $I_a - I_c > 0$ and $I_b - I_c > 0$, so it can never be negative.

For the second expression however: $I_a - I_b > 0$ and $I_c - I_b < 0$, so the second expression can obtain both positive as well as negative values, and it can be zero even if ω_a and ω_c are non-zero.

3. Why can't the box rotate freely about the middle axis

However, if the initial state is a rotation about any of the three axis *a, b, c*, all three expressions above will be zero. For the two of them because the angular velocities are zero, and for the third, say the *a*-axis

$$E = ½ I_a \omega_a^2 \ \text{ and } \ L^2 = I_a^2 \omega_a^2 \quad \Rightarrow \quad 2EI_a - L^2 = 0$$

This leaves only the possibility $\omega_b = \omega_c = 0$ from the first equation.
Similarly when the initial rotation is about the *c* axis, the third equation leaves only the possibility $\omega_b = \omega_a = 0$. But for rotation about the *b* axis, even if

$$L^2 - 2EI_b = I_a(I_a - I_b)\omega_a^{\,2} + I_c(I_c - I_b)\omega_c^{\,2} = 0$$

The coefficients $I_a(I_a - I_b)$ and $I_c(I_c - I_b)$ have opposite signs, so there are an infinite number of possibilities for non zero values of ω_a and ω_c.
Conservation of energy and angular momentum forbids any rotation about the two other axes, when the rotation is initially about the major and minor axis.

But this is not the case when rotating about the middle axis!

This may serve for an explanation, why the rotation about the major and the minor axes are stable, but an attempt to rotate about the middle axis becomes chaotic.