The dependence of pressure with altitude

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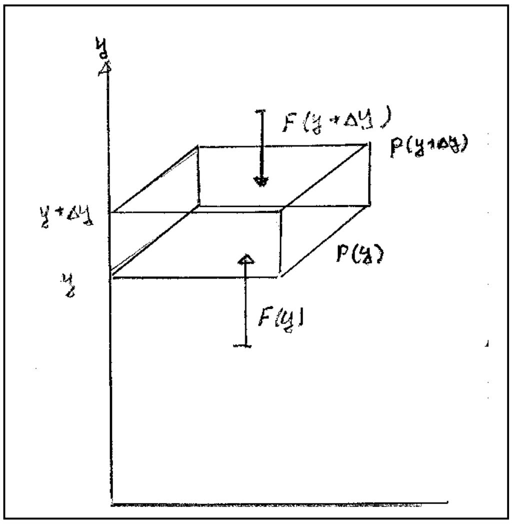
1. The dependence of pressure with altitude



We consider a rectangular horizontal section of the atmosphere. The area of the two end faces are A. The box is situated in the height y over the ground. The height of the box is Δy. The pressure on the upper and the lower side are $p(y+\Delta y)$ and $p(y)$ respectively.

The density of the air in the height y is $\rho(y)$.

The force on a flat piece with area A is $F = pA,$ where p is the pressure on the flat.

We then express that the difference in the force on the upper and the lower side is equal to the gravitational force on the air in the box, assuming that the air in the box is at rest. (g is the gravity acceleration)

$$p(y)A - p(y+\Delta y)A = m_{air}g = \rho(y)V_{air}g = \rho(y)A\Delta yg$$

So $$p(y)A - p(y+\Delta y)A = \rho(y)A\Delta yg$$

Dividing by $A\Delta y$: $\dfrac{p(y+\Delta y)-p(y)}{\Delta y} = -\rho(y)g$, and replacing $\dfrac{p(y+\Delta y)-p(y)}{\Delta y}$ by $\dfrac{dp}{dy}$ we find:

(1.1) $$\frac{dp}{dy} = -\rho(y)g$$

To solve this differential equation we need to know another relation between $\rho(y)$ *and* $p(y)$. This can however be obtained by:

1. The equation of state for ideal gasses: $PV = n_M RT$ (n_M is the number of moles)
2. Definition of the mole mass M: $m = n_M M \Leftrightarrow n_M = \dfrac{m}{M}$
3. The definition of density; $\rho = \dfrac{m}{V} \Leftrightarrow m = \rho V$

Insertion of the last two equations in (1) the equation of state gives:

(1.2) $$PV = n_M RT = \frac{m}{M}RT = \frac{\rho V}{M}RT \quad \Rightarrow \quad \rho = \frac{M}{RT}P$$

This expression for the density is then used in (1.1).

(1.3) $$\frac{dp}{dy} = -\frac{Mg}{RT}p$$

It is well known that the temperature decreases roughly by one centigrade for every 200 meters increase i altitude over the ground but Initially, we shall assume that the temperature is constant up through the atmosphere.

The differential equation (1.3) has the well known solution:

(1.4) $$p(y) = p_0 e^{-\frac{Mg}{RT}y}$$

Using the known values: M_{air} =29 *g/mol*, g = 9.82 m/s^2, R= 8.31 J/(*mol K*) and T = 273 K, we find:

(1.5) $$p(y) = p_0 e^{-1.2610^{-4}y}$$

Where y should be measured in meters
This results in a pressure drop of 1.3% per 100 m and a drop of 12% per 1000 m.

Next we shall look at the solution to the differential equation, where we take into account that the temperature drops linearly 1 ^0C per 200 m increase in altitude.
We put the temperature on the ground at 20 ^0C = 293 K. The temperature in the altitude y then becomes $T = T(y) = 293 - y/200$. Then the differential equation becomes:

(1.6) $$\frac{dp}{dy} = -\frac{Mg}{R(293 - \frac{y}{200})} p$$

The equation is solved in the usual way by separating the variables and integrating

$$\int_{p_0}^{p} \frac{dp}{p} = -\frac{Mg}{R} \int_0^y \frac{1}{293 - \dfrac{y}{200}} dy$$

$$\int_{p_0}^{p} \frac{dp}{p} = -\frac{Mg}{293R} \int_0^y \frac{1}{1 - \beta y} dy \qquad where \qquad \beta = \frac{1}{293 \cdot 200}$$

(1.6)
$$\ln(\frac{p}{p_0}) = \frac{Mg}{293R\beta} \ln(1 - \beta y) \qquad \Rightarrow$$

$$p = p_0 (1 - \beta y)^{\frac{Mg}{293R\beta}}$$

Although it looks rather different from (1.5), it turns out that it only causes a deviation from (1.5) of about 0.1 – 0.2 %.