The motion of a projectile

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4. The motion of a projectile

We next consider the trajectory of a projectile, fired at an angle θ, and the initial speed v_0.
First we shall review the trajectory for motion of a particle in the gravitational field near the surface of the earth. The projectile is thought to move in the x –y plane only influenced by gravity, so the equation of motion is:

4.1 The motion of a projectile without drag

(4.1) $\vec{F}_{res} = m\vec{g}$, where $\vec{g} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$ and $\vec{v}_0 = \begin{pmatrix} v_0 \cos\theta \\ v_0 \sin\theta \end{pmatrix}$ (The initial velocity)

The motion is with constant acceleration, and the solution is:

(4.2) $\vec{v} = \vec{a}t + \vec{v}_0$ and $\vec{r} = \frac{1}{2}\vec{a}t^2 + \vec{v}_0 t + \vec{r}_0$

If we put $\vec{r}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ it is found by straightforward insertion:

(4.3) $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_0 \cos\theta \\ v_0 \sin\theta - gt \end{pmatrix}$ and $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v_0 \cos\theta\, t \\ v_0 \sin\theta\, t - \frac{1}{2}gt^2 \end{pmatrix}$

The max height of the trajectory can be found by putting $v_y = 0 \Leftrightarrow t = \dfrac{v_0 \sin\theta}{g}$ and inserting in y, gives: $y_{\max} = \dfrac{(v_0 \sin\theta)^2}{2g}$

The width (in the x direction) of the trajectory, can be found by setting

$$y = 0 \quad \Leftrightarrow \quad v_0 \sin\theta\, t - \tfrac{1}{2}gt^2 = 0 \quad \Leftrightarrow \quad t = 0 \quad \vee \quad t = \frac{2v_0 \sin\theta}{g}$$

The maximum width (length of the throw), is then determined by inserting the second value for t into the expression for $x(t)$. The result it can be reduced to:

(4.4) $$x_{\max} = \frac{v_0^{\,2} \sin 2\theta}{g}$$

The longest throw is obtained when $\sin 2\theta = 1 \Leftrightarrow \theta = 45^0$, as is well known from elementary physics.
The trajectory is a parabola, by the way, since eliminating t en the expression for x and y leads to:

$$y = x\tan\theta - \frac{\frac{1}{2}g}{(v_0 \cos\theta)^2}x^2$$

4.2 The motion of a projectile with drag (air resistance)

We shall now consider the same motion as above, in the gravitational field of the earth, but this time paying to the inevitable resistance caused by the air.
Here we shall however only be concerned with *laminar flow*, with is the same as saying that the drag force is proportional to and directed opposite to the velocity.
For motion in the air, this is hardly applicable if the speed exceeds about 5.0 *m/s*, where the flow becomes turbulent, and the equations of motion do not have an analytic solution.

With turbulent flow, the drag force can empirically be represented by $F_{drag} = v^{\beta}$, where $1 < \beta \le 2$.
But we shall preliminary only be concerned with *laminar* flow.
When the motion takes place in gasses, we can safely discard the buoyancy. So in that case:

(4.5) $$F_{drag} = \alpha\,|\vec{v}\,| \quad and \quad \vec{F}_{drag} = -\alpha\vec{v}\,.$$

The equation of motion becomes:

(4.6) $$\frac{d\vec{v}}{dt} = \vec{g} - \frac{\alpha}{m}\vec{v} \quad \Leftrightarrow$$
$$\frac{dv_x}{dt} = -\frac{\alpha}{m}v_x \quad \wedge \quad \frac{dv_y}{dt} = -g - \frac{\alpha}{m}v_y$$

The differential equation (4.1) separates with respect to the *x*-direction and the y-direction.
But we have already solved such an equation for a linear motion in (3.2) to (3.5).
If the initial velocity is $\vec{v} = (v_0\cos\theta, v_0\sin\theta)$, we can simply copy the solution from (3.5).

(4.7) $$v_x = v_0\cos\theta \cdot e^{-\frac{\alpha}{m}\cdot t} \quad and \quad v_y = v_0\sin\theta \cdot e^{-\frac{\alpha}{m}\cdot t} - \frac{mg}{\alpha}(1 - e^{-\frac{\alpha}{m}\cdot t})$$

If $\frac{\alpha}{m}\cdot t$ << 1, that is, if the resistance of the air is small, we may apply the approximation

$e^x \approx 1 + x$ to the factor $e^{-\frac{\alpha}{m}\cdot t}$ to obtain.

$$v_x = v_0\cos\theta \cdot (1 - \frac{\alpha}{m}\cdot t) \quad and \quad v_y = v_0\sin\theta \cdot (1 - \frac{\alpha}{m}\cdot t) - \frac{mg}{\alpha}(1 - (1 - \frac{\alpha}{m}\cdot t)\,)$$

Dropping all terms proportional to α, we retrieve the formulas (4.3) for the motion without drag.

(4.8) $$v_x = v_0\cos\theta \quad and \quad v_y = v_0\sin\theta - g\cdot t$$

We may also find the position $x(t), y(t))$, by integrating (4.7) with respect to *t*.
Choosing $(x_0, y_0) = (0,0)$, we get:

$$x = v_0\cos\theta\int_0^t e^{-\frac{\alpha}{m}\cdot t}\,dt \quad and \quad y = v_0\sin\theta\int_0^t e^{-\frac{\alpha}{m}\cdot t}\,dt - \frac{mg}{\alpha}\int_0^t (1 - e^{-\frac{\alpha}{m}\cdot t})dt$$

(4.9) $$x = v_0 \cos\theta \cdot \frac{m}{\alpha}(1 - e^{-\frac{\alpha}{m}\cdot t}) \quad and \quad y = v_0 \sin\theta \cdot \frac{m}{\alpha}(1 - e^{-\frac{\alpha}{m}\cdot t}) - \frac{mg}{\alpha}(t - \frac{m}{\alpha}\cdot(1 - e^{-\frac{\alpha}{m}\cdot t}))$$

Again if $\frac{\alpha}{m} t << 1$, we may apply the approximation $e^x \approx 1 + x + \frac{1}{2}x^2$ with $x = -\frac{\alpha}{m}\cdot t$.
Dropping all terms, proportional to α, we retrieve the former expressions (4.3), derived without resistance.

(4.10) $$x = v_0 \cos\theta \cdot t \quad og \quad y = v_0 \sin\theta \cdot t - ½ g \cdot t^2$$

Neither (4.8) nor (4.9) are particular transparent, when determining the maximum height or the width of the throw. It is actually possible to determine y_{max}, but we cannot determine x_{max}, since the equation $y = 0$ is transcendent.
In a later section we shall however look at numerical solutions to differential equations.

As mentioned, the equation of motion for a projectile can not be solved when assuming turbulent flow i.e. when the drag force is proportional to v^2, $F_{drag} = \alpha\, v^2$. Below is shown a numerical solution with: α = 0 (no air resistance) , α = 0.0001, α = 0.0005, α = 0.001.