The periods of passage for Mercury and Venus

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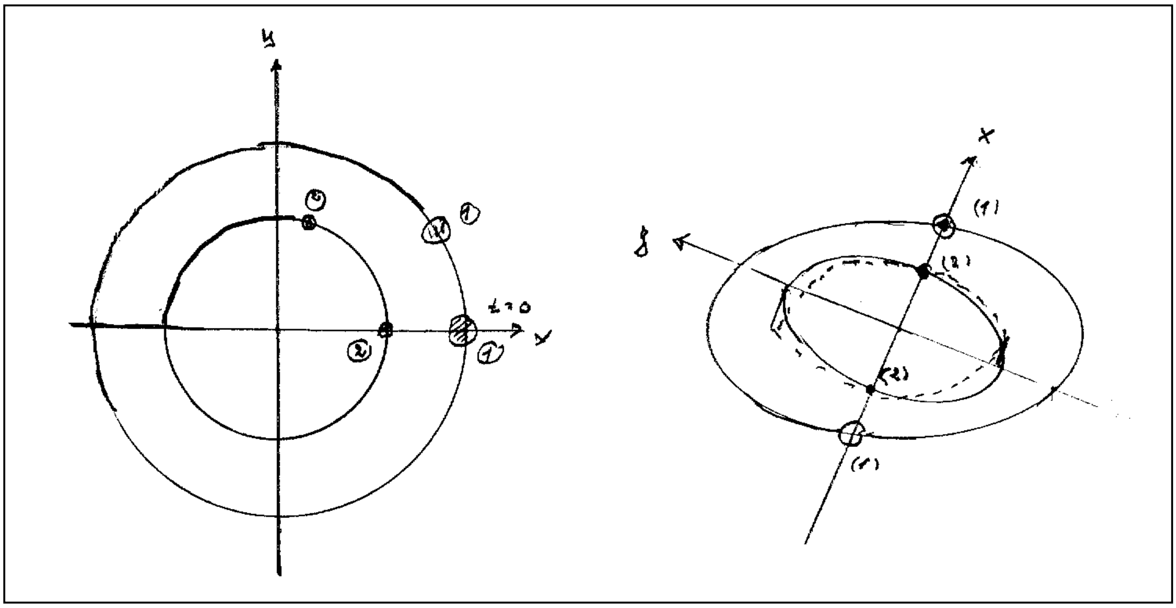


A passage of Mercury or Venus means that the planet together with the Earth lie on a straight line with passes through near the centre of the sun.
As we shall demonstrate theoretically below, the passage will happen with the period $0{,}371 \cdot T_{\text{Earth}}$ for Mercury, and with the period $1{,}597 \cdot T_{\text{Earth}}$, for Venus as we have sketched in the figure above.

However, the orbital plane of Mercury has an inclination of 7.0^0 with respect to the orbital plane of the Earth, and the orbital plane of Mercury has an inclination of 3.4^0 with respect to the orbital plane of the Earth. This makes the period much longer, since we should only observe a true passage when the planet and the earth are both positioned on the same side of the intersection line of the two planes.
The angular diameter of the sun as seen from the Earth is $0{,}53^0$, so the Earth and the planet does not necessarily need to be exactly on the intersection line, but otherwise the passage will then be seen on the upper or the lower part of the sun disc.
It requires, however, some calculations (as demonstrated below) to establish how far from the intersection line, the line of sight between the two planets may differ.

On November the 19th 2019 Denmark experienced a true passage of Mercury. Our aim is to predict the next passage of Mercury from the orbital periods of the Earth and Mercury.
But first we shall demonstrate the method, if the earth and the planet rotate in the same orbital plane.

2. The time between passages, when the planet and the Earth are in the same plane

If the planets share the same orbital plane as the earth, it is relatively simple to calculate the time elapsed between two passages. We shall assume that the planets perform a uniform circular motion. This is actually a restriction, since the planet in a elliptic motion move with slightly different speed in the orbital motion. If the eccentricity is small, we may discard from this.
If r_1 and r_2 are the radii of the circular motion, T_1 and T_2 are the orbital periods corresponding to the angular velocities: $\omega_1 = \dfrac{2\pi}{T_1}$ and $\omega_2 = \dfrac{2\pi}{T_2}$, then we may establish the parametric equation.

(2.1) $\vec{r}_1 = r_1(\cos\omega_1 t, \sin\omega_1 t)$ and $\vec{r}_2 = r_2(\cos\omega_2 t, \sin\omega_2 t)$,

Let us assume that we have a passage at $t = 0$ on the positive side of x-axis, and we shall then find the occurrence of the next passage. That is, where the planets are on the same line through the centre of the sun. The condition is that the two phases have the same cosine at time t. Then their phases can only differ by an amount of 2 pi.

(2.2) $$\omega_2 t = \omega_1 t + p2\pi\ ,\ p = 0,1,2 \iff \frac{2\pi}{T_2}t = \frac{2\pi}{T_1}t + p2\pi \iff t = p\frac{T_2 T_1}{T_1 - T_2},\quad p = 0, 1, 2$$

Applying $T_1 = T_{\text{earth}} = 1$, $T_2 = T_{\text{Mercury}} = 0.241$, $T_2 = T_{Venus} = 0.615$, we find for the time elapse between two consecutive passages:

(2.3) $$t_{mercury} = \frac{1\cdot 0.241}{1-0.241} = 0.318\cdot T_{earth} \quad\text{and}\quad t_{venus} = \frac{1\cdot 0.615}{1-0.615} = 1.597\cdot T_{earth}$$

3. Times between passages for Mercury, when the orbital planes are different

We shall now turn to the case where a passage can only occur, when the two planets are on the same side of the intersection line between the two orbital planes of the planets.
Let us therefore assume that we have a passage at $t = 0$, and we want to calculate the time for the next passage. This requires that the cosines to the phases are both 1.
We know that $\cos(\omega t) = 1 \iff \omega t = p2\pi$, so we find:

(3.1) $$\omega_1 t = p_1 2\pi \quad and \quad \omega_2 t = p_2 2\pi \quad\Rightarrow\quad t = \frac{p_1 2\pi}{\omega_1} \ and\ t = \frac{p_2 2\pi}{\omega_2} \quad\Rightarrow$$
$$\frac{p_1 2\pi}{\omega_1} = \frac{p_2 2\pi}{\omega_2} \quad\Rightarrow\quad p_1 T_1 = p_2 T_2$$

We might have established the last equation, without the preceding calculations, since it simply expresses that, the time for p_1 rounds of the earth is equal to p_2 rounds for the planet.
First we use this equation on Mercury, when we put $T_1 = T_{\text{earth}} = 1$ and $T_2 = T_{\text{mercury}} = 0.241$.

(3.2) $$p_1 = 0.241 p_2$$

Since p_1 and p_2 must be integers, the equation has the obvious solution:

$$p_1 = 241 \text{ og } p_2 = 1000.$$

This is somewhat distressing, however, since 241 is a prime, so the equation cannot be shortened. The conclusion is therefore that the period for passing for Mercury is 241 years, and that there is no minor period. From the astronomical observations, one finds, however, that there are not less than 14 passages of Mercury in the years between 2000 and 2100.

07. maj 2003, 08. nov 2006 , 09. maj 2016, 11. nov 2019, 13. nov 2030, 7. nov 2039 , 07. maj 2049

What we see, according to the astronomical data, is that we have periods of 3, 10, 3, 13,10 years. The period $p_1 = 3$, does not fit the equation at all, since it gives $p_2 = 12.5$, but it indicates that the ”passage” is with the planets on the opposite side of the sun. The period $p_1 = 6$ gives $p_2 = 24.90$, The period $p_1 = 10$, gives $p_2 = 41.49$, (opposite passage?) The period $p_1 = 13$, gives $p_2 = 53.94$. Only $p_1 = 13$, might qualify as a real passage.

One may notice however that another reference, which apply the times for the orbital periods with 5 or 6 digits: $T_{\text{earth}} = 365{,}254$ days and $T_{\text{Mercury}} = 87{,}969$ days. The problem is, however, that these numbers are relatively primes, so they cannot be shortened to yield at shorter period.
Moreover the periods are not in accordance with the astronomical data, which have the periods: 6, 7, 13, 46, 217, where only the period 13 is almost in accordance with the astronomical data.

Again we stress that from a Newtonian geometrical calculation, there should be only one least period. So again we conclude that the discrepancy is probably caused from the mutual interaction of the inner planets, and it is not possible to predict the periods of passages from Newtonian geometrical calculations, but rather from accurate numerical computer calculations, far from the scope of this article.
The non integral values of p_2 12.5 and 41.49, might correspond to a passage of Mercury but on the opposite side of the sun. This is also in accordance with the graphical representation presented in the end of this article. The value $p_2 = 53.94$, might be a true passage, and from the graphical representation it seems to be.

4. The times between passages for Venus, when the orbital planes are different

If we apply the Newtonian geometric calculation for Venus, the results are equally stressing.
If we put $T_1 = T_{\text{earth}} = 1$ and$T_2 = T_{\text{venus}} = 0.615$, it leads to the equation.

(4.1) $$p_1 T_1 = p_2 T_2 , \quad \text{and for Venus} \quad p_1 = 0.615 p_2 ,$$

The equation expresses that the time for p_1 rounds for the earth $p_1 T_1$ should be exactly equal to the the time for p_2 rounds for Venus $p_2 T_2$. The equation has the obvious integral number solution:

$$p_1 = 615 \quad og \quad p_2 = 1000 ,$$

which may be shortened to: $p_1 = 123 \quad and \quad p_2 = 200$.
The period for passages of Venus should therefore be 123 years, and nothing else.
However, as it is seen from the astronomical table below, the actual periods for passages occur with the periods $p_1 = 8, 5, 7, 8, 122, 8, 5$ years.

8 juni	2004	05.13	08.20	11.26	6 h 03 min	Visible from Denmark
6 juni	2012	22.09	01.29	04.49	6 h 40 min	Most late passage, apart from 2019
11 december	2117	23.58	02.48	05.38	5 h 40 min	Not visible from Europe
8 december	2125	13.15	16.01	18.48	5 h 33 min	Partly visible from Western Europe .
11 juni	2247	08.42	11.33	14.25	5 h 43 min	Visible from Europe
9 juni	2255	01.08	04.38	08.08	7 h 00 min	Partly visible from Europe
13 december	2360	22.32	01.44	04.56	6 h 24 min	

We shall then calculate the values for p_2 corresponding to: p_1 = 8, 5, 7, 8, 122, resulting in:
p_2 = 13,00, 8.13, 11.38, 198,31.
As it was the case for Mercury our Newtonian geometric calculation does not qualify at all, only $p_1 = 8$ gives a value of p_2 very near to an integer value.

5. Passage when the planets are in opposite positions with the sun

Now the two planets may lie on the intersection line between the two orbital planes, but opposite to the sun, although it is not a visible passage. We shall therefore make a calculation of the possible occurrences of such a configuration of the planets, on the condition, that we have a true passage (planets are on same side of the sun) at $t = 0$, and the earth is again in the same position but the planet is at an angle pi. This gives the two equations:

$$\omega_1 t = p_1 2\pi \quad and \quad \omega_2 t = \pi + p_2 2\pi \quad \Rightarrow$$

$$t = \frac{2p_1\pi}{\omega_1} \quad and \quad t = \frac{(2p_2+1)\pi}{\omega_2} \quad \Rightarrow$$

(5.1) $$p_1 T_1 = (p_2 + \tfrac{1}{2})T_2$$

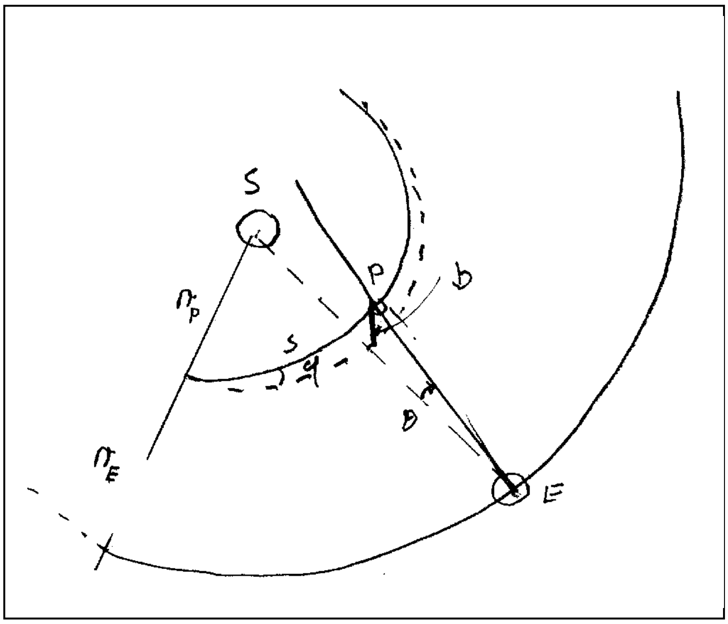
If we put $T_1 = T_{earth} = 1$ and $T_2 = T_{mercury} = 0.241$, We find in the same way as before.

$p_1 T_1 = 0.241(p_2 + \tfrac{1}{2})T_2$, which has the solution

$p_1 = 241 \ and \ (p_2 + \tfrac{1}{2}) = 1000$

It is immediately seen, that the equation can have no integral solution p_2 .
The geometrical model it leads to the conclusion that if the planets have a passage on the same side of the sun, then it is impossible to have a passage on opposite sides of the sun.
The result for Venus is the same, since the right side of the equation: $(p_2 + \tfrac{1}{2}) = 1000$ is always an integral number, while the left side is not.
But again, this result probably does not comply with the actual astronomical observations.

6. If we consider the correction due to the angular diameter of the sun.



In the preceding sections, we have seen that a simple Newtonian geometric model cannot reproduce the astronomical data for the passages of Mercury and Venus, based on the assumption that earth and the planet are on a line.
Now the sun has an angular diameter of 0.53^0, so if the earth and a planet lie on a line that intersects the sun, but not in its centre, one may observe a partial passage of the planet in the upper or lower part of the sun.
We shall now look into whether such an adaptation can change the previous conclusions for the periods for a passage of the planets.

As shown in the figure above is sketched the sun *S*, the earth *E*, and a Planet *P*. The radius in the planets orbit is r_P. The radius in the planets orbit is r_E. The angle between the two orbital planes is φ. In the position of the planet the vertical distance to the orbital plan of the earth is *b*. Finally is θ the angular extension of the sun as seen from the earth. Finally *s* is the arc, which the planet has moved from the intersection line until the time *t*. T_P is the orbital period for the planet, and T_E =1, is the orbital period for the earth.
We shall now estimate how far the planet can be from the intersection line, so that the line that connects the planet and the earth still hits the sun disc. We have from the figure:

(6.1) $$s = \frac{t}{T_P} 2\pi r_P \ , \ b = s\varphi \ , \ \theta = \frac{b}{r_E - r_P} \quad \text{which gives:} \quad \theta = \frac{t}{T_P} 2\pi r_P \frac{\varphi}{r_E - r_P},$$

And isolating the time *t*.

(6.2) $$t = \frac{\theta T_P}{2\pi r_P} \frac{r_E - r_P}{\varphi}$$

Using the data for Mercury:

$T_{mercury} =$ 0.241, $r_{mercury}$ = 0.387, $\varphi_{mercury}$ = 7.0^0 = 0.122, r_{earth} =1, and ½θ = 0.265^0 = 0.004 rad, we get:

$$t_{mercury} = 0.00514\ T_E$$

Using the data for Venus:

T_{venus} =0,615, r_{earth} =1, $r_{venus} =$ 0,723, φ_{venus} = 3,39^0 = 0.00463 and ½θ = 0.265^0 = 0,004 rad, we get

$$t_{venus} = 0.0324\ T_E$$

To investigate if such a correction might bring the geometric model more in accordance with the astronomical observation, we scrutinize the values of p_1 and p_2, adding a small correction, as described above. We therefore modify the equations to:

$$\omega_1 t = p_1 2\pi \quad og \quad \omega_2 t = (p_2 + \delta)2\pi \quad \Rightarrow \ t = \frac{p_1 2\pi}{\omega_1} \quad og$$

$$t = \frac{(p_2 + \delta)2\pi}{\omega_2} \quad \Rightarrow \quad \frac{p_1 2\pi}{\omega_1} = \frac{(p_2 + \delta)2\pi}{\omega_2} \quad \Rightarrow$$

(6.3) $$p_1 T_1 = (p_2 + \delta) T_2$$

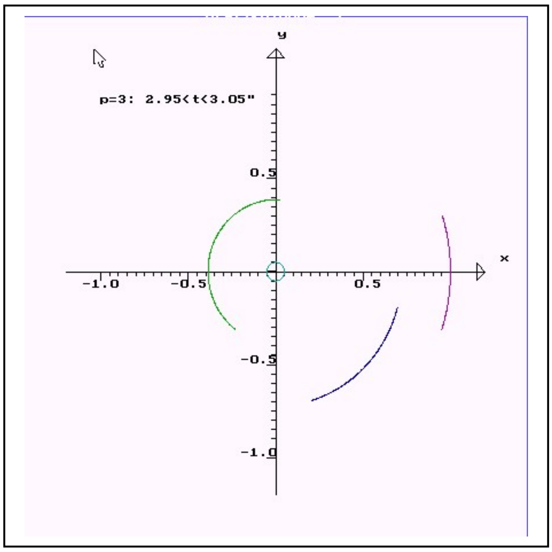
First we look at Mercury, where: $T_1 = T_{\text{earth}} = 1$, $T_2 = T_{\text{Mercury}}$ = 0,241 and $\delta_{Mercury}$ = 0,00514 , and $p_1 = (p_2 + \delta)0.241$. The periods are due to the values p_1 = 3, 10, 3, 13 and 10 years.
$p_1 = 3 \Rightarrow p_2$ =12.5, p_1 = 10, $\Rightarrow p_2$ = 41.49. The period p_1 = 13, $\Rightarrow p_2$ = 53.94.

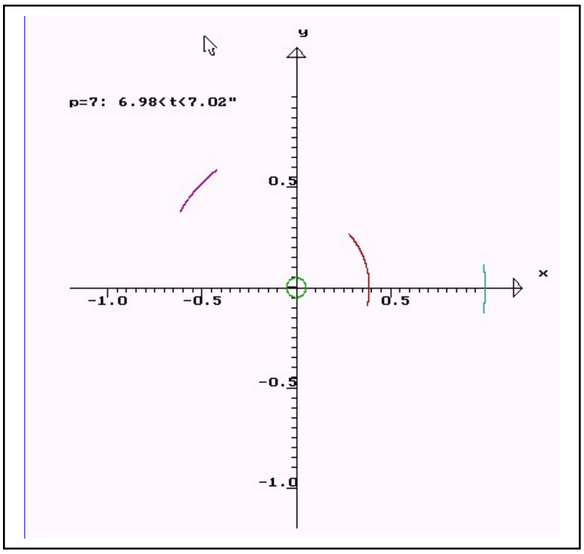
As it is obvious, then not even for the period p_1 = 13, p_2 = 53.94, the deviation from an integral number, can be explained from the correction taking into account the solar diameter.
The correction is far too small.

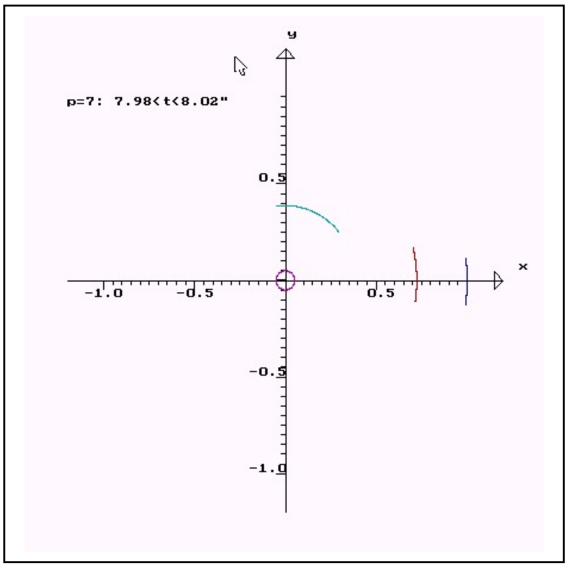
Regarding Venus the situation is almost the same: $T_1 = T_{\text{earth}} = 1$, $T_2 = T_{\text{Venus}} = 0.615$ and $\delta_{Venus} = 0{,}0324$, $p_1 = (p_2 + \delta)0.615$. According to the astronomical observation the periods for passages are: p_1 = 5, 7, 8, 122, which gives: p_2 = 8.13, 11.38, 13.00, 198.31.
Except for p_2 =13 none of these numbers can be transformed into an integral number by adding the correction. $\delta_{Venus} = 0.0324$ to p_2.

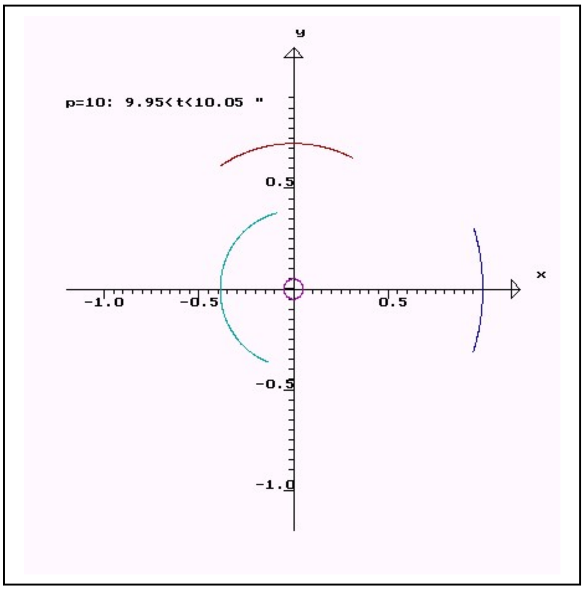
7. Computer graphical display of the alleged passages of Mercury and Venus

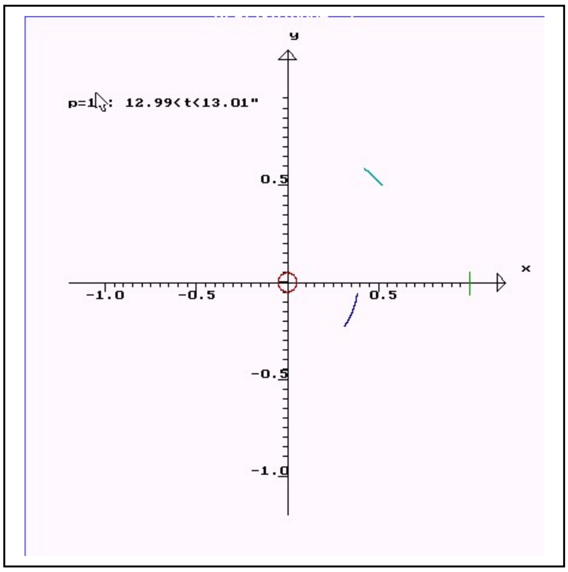
The graphic solution is displayed for p_1 = 3, 7, 8, 10 13, 123, 241, but it is only the values of p_1 = 8, 13, 123, 241, which result in a situation, which might resemble a passage of either Mercury or Venus. True enough, we get a real passage for 123, 241, since they are the values for the period obtained from the geometric model.
As we have stated already that from this model, there ought to be only one period.
For p = 3 and p = 10, it seems that the two planets are aligned, but on different sides of the sun. But this is not what we usually denote as a passage.
The rather large discrepancy between observation and a Newtonian geometrical model, I do not know, but the most likely reason is of course the mutual attraction between the earth, Venus and Mercury. But this requires at least the solution of 8 coupled second order differential equations.

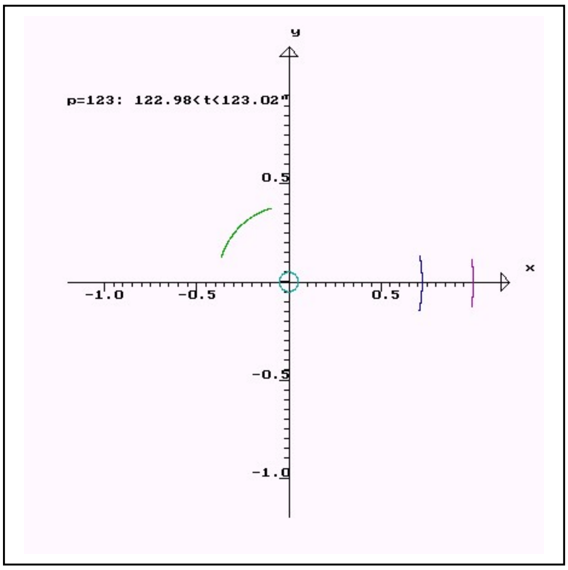


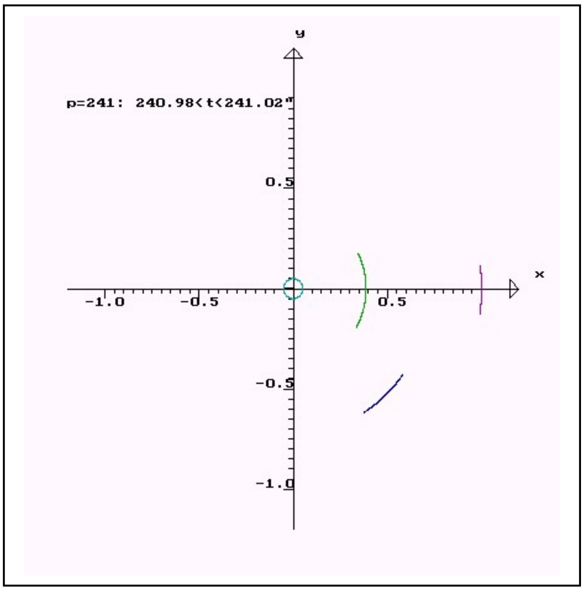












8. Computer simulation of the three planets from the analytic equations of motion.

From Newton's law of gravitation, one may establish 6 second order differential equations, if we assume that the planets all have a circular motion, and that they all circulate in the same orbital plane.
However, it is practical to use more appropriate units, so we shall use the planetary units (PU), instead of the SI – units.
Length: Mm, *time*: h, and *velocity*: Mm/h. With these units, we have:

$G = 6.67\ 10^{-11}$ SI = $8.644\ 10^{-22}$ PU,
M_{earth} = $5.98\ 10^{24}$ kg, M_{moon} = $7.35\ 10^{22}$ kg, M_{venus} = $4.875\ 10^{24}$ kg, M_{mecury} = $5.79\ 10^{4}$ kg

Radii in the circular orbits

$r_{earth} = 1.496\,10^5\ Mm,\ r_{venus} = 1.082\,10^5\ Mm,\ \ r_{mecury} = 5.79\,10^4\ Mm$

Periods in their orbital motion

$T_{earth} = 1\ y = 8.76\,10^3 h,\ T_{venus} = 0.615\,T_{earth} = 5.387\,10^3 h, T_{mecury} = 0.241 T_{earth} = 2.111\,10^3 h,$

Velocity in their orbital motion:

$v_{earth} = 1.073\,10^2 Mm/h,\ v_{venus} = 1.262\,10^2 Mm/h,\ \ v_{mecury} = 1.723\,10^2 Mm/h,$

Gravitational constant times mass of body (all in planetary units)

$GM_{sun} = 1.792\,10^9\ ,\ \ GM_{earth+moon} = 5.233\,10^3\ ,\ \ GM_{venus} = 4.213\,10^3\ ,\ GM_{mecury} = 284$

We can then use these constants to establish the differential equations for the motions of the three planets. Mercury = (1), Venus = (2), Earth = (3). All orbits are considered as having the fixed sun in their centre.

As we can see above the attraction from the sun is more than 500.000 stronger than the mutual attraction from the planets, when the distances are compatible, so we may not expect any visible deviation from a uniform circular motion. Nevertheless, it is a comforting fact that the motion of the planets comply with Newton's law of gravitation.

$$\ddot{x}_1 = -\frac{1.792\,10^9 x_1}{\sqrt{x_1^2 + y_1^2}^3} + \frac{5233(x_3 - x_1)}{\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}^3} + \frac{4213(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3}$$

$$\ddot{y}_1 = -\frac{1.792\,10^9 y_1}{\sqrt{x_1^2 + y_1^2}^3} + \frac{5233(y_3 - y_1)}{\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}^3} + \frac{4213(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3}$$

$$\ddot{x}_2 = -\frac{1.792\,10^9 x_2}{\sqrt{x_2^2 + y_2^2}^3} + \frac{5233(x_3 - x_2)}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}^3} + \frac{284(x_1 - x_2)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3}$$

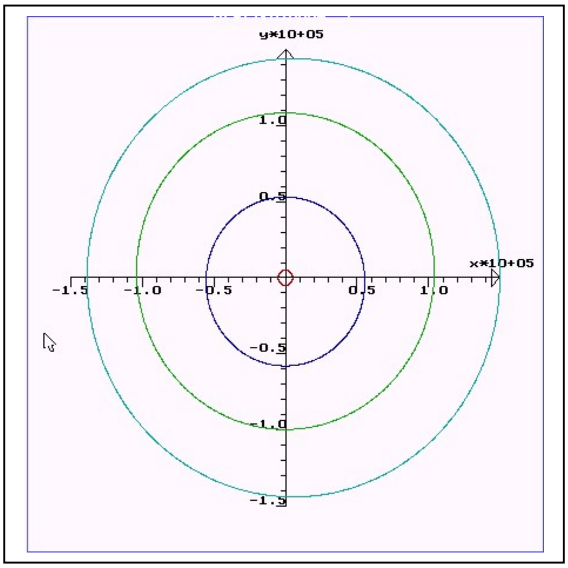
$$\ddot{y}_2 = -\frac{1.792\,10^9 y_2}{\sqrt{x_2^2 + y_2^2}^3} + \frac{5233(y_3 - y_2)}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}^3} + \frac{284(y_1 - y_2)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3}$$

$$\ddot{x}_3 = -\frac{1.792\,10^9 x_3}{\sqrt{x_3^2 + y_3^2}^3} + \frac{5233(x_2 - x_3)}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}^3} + \frac{284(x_1 - x_3)}{\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}^3}$$

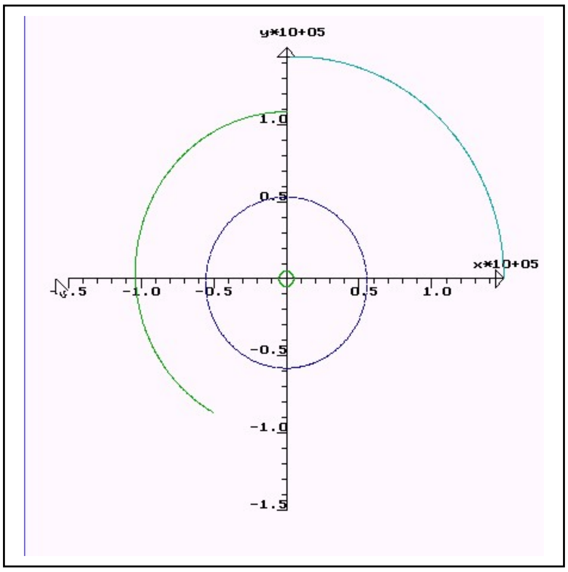
$$\ddot{y}_3 = -\frac{1.792\,10^9 y_3}{\sqrt{x_3^2 + y_3^2}^3} + \frac{5233(y_2 - y_3)}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}^3} + \frac{284(y_1 - y_3)}{\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}^3}$$

As is clear from the graphic computer solutions below, there are no sign whatsoever of perturbations in the orbits of the three planets. The solutions are merely to confirm, that the planets move in accordance to Newton's law of gravitation.

Time period of the Earths orbital motion



Time period of Mercury's orbital motion



Time period of the orbital motion of Venus