# Linear motion of a particle in liquids and gasses

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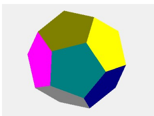

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## 3. Linear motion of a particle in liquids and gasses

When you analyze a mechanical system in order to determine the equation of motion, one is often referred to assume, that the system is without friction or dissipative forces.
This is usually only a realistic description to a certain degree, and sometimes completely unrealistic, but the differential equation, which describes the dynamics of the system, can only be solved in some cases, when the dissipative forces (coming from friction or viscosity) do not depend on the velocity. The latter is the case where to solid materials move relatively to each other.

The aim of this section is to draw attention some simple examples where the friction (the viscosity) is velocity dependent, either linearly or as the square of the velocity.

### 3.1 A ball sinking in a liquid

We shall first consider a body (characteristically a ball), which sinks in a liquid (water) under the influence of gravity.
If the speed is not too big (and for minor bodies it is not) we have a so called *laminar* streaming, and in that case, we can assume that the resistance to the movement is proportional to the speed of the sinking body.
If the speed of movement becomes larger, the resistance transforms into *turbulent* flow, where the resistance to the movement is more conspicuous, but empirically it is assumed to be proportional to the square of the speed. Turbulence is best described by the appearance of vortices in the liquid or in the air.

Incidentally turbulence is still one of partly unsolved problem hydrodynamics, since the Navier-Stokes equations (Newton's second law for hydrodynamics) do not allow a transition from laminar streaming to turbulent streaming, although both phenomena appear as solution to the equations.

A theoretical expression for the viscous force on a ball in a *laminar* flow is first given by Stoke, and is called Stokes law. If $r$ = radius of the ball, $v$= the speed, $\eta$= viscosity of the fluid, then:

(3.1) $$F_{visc} = 6\pi\eta rv$$

In the following examples, we shall abbreviate the constants $6\pi\eta r$ to one, and we then write the proportionality between the force and the speed as: $F_{visc} = \alpha \cdot v$. This formula is actually independent of the shape of the falling body, as long as the flow is laminar.

For a motion along the $x$ axis, we have the well known concepts.

Velocity: $v = \dfrac{dx}{dt}$, acceleration: $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$, and Newton's 2. law: $F_{res} = ma$

A body falling in a liquid is influenced by the following forces:

1. Gravity: $F_T = mg$.
2. The buoyancy $F_{up} = \rho_v Vg$

The buoyancy is equal to the gravity of the displaced liquid, where $\rho_v$ is the density of the fluid and $V = \frac{m}{\rho}$ is the volume of a body with mass *m* and density $\rho$.

3. The viscous force: $F_{visc} = \alpha v$.

The resulting force on the body is therefore:

$$F_T - F_{up} = mg - \rho_v Vg = \rho Vg - \rho_v Vg = (\rho - \rho_v)Vg = m_v g$$

Where $m_v\, g$ is the gravity of the body reduced by the buoyancy:The equation of motion is hereafter:

(3.2) $$F_{res} = ma \quad \Leftrightarrow \quad m\frac{dv}{dt} = m_v g - \alpha v \quad \Leftrightarrow \quad \frac{dv}{dt} + \frac{\alpha}{m}v = \frac{m_v}{m}g$$

To obtain more simplicity we put $g_v = \frac{m_v}{m}g$

The equation is solved by the same method, as we did in (2.4), by multiplication with $e^{\frac{\alpha}{m}\cdot t}$ and rearranging.

$$e^{\frac{\alpha}{m}\cdot t}\frac{dv}{dt} + \frac{\alpha}{m}e^{\frac{\alpha}{m}\cdot t}v = e^{\frac{\alpha}{m}\cdot t}g_v \quad \Leftrightarrow$$

(3.3) $$\frac{d(ve^{\frac{\alpha}{m}\cdot t})}{dt} = e^{\frac{\alpha}{m}\cdot t}g_v \quad \Leftrightarrow$$

$$ve^{\frac{\alpha}{m}\cdot t} = \frac{m}{\alpha}e^{\frac{\alpha}{m}\cdot t}g_v + c \quad \Leftrightarrow$$

$$v = \frac{mg_v}{\alpha} + ce^{-\frac{\alpha}{m}\cdot t} \quad \Leftrightarrow$$

(3.4) $$v = \frac{m_v g}{\alpha} + ce^{-\frac{\alpha}{m}\cdot t}$$

Adding the initial condition *v(0)=0*, we find $c = -\frac{m_v g}{\alpha}$, which inserted in the solution (3.4) gives:

(3.5) $$v = \frac{m_v g}{\alpha}(1 - e^{-\frac{\alpha}{m}\cdot t})$$

We can see that the velocity approaches asymptotically to $v_\infty = \frac{m_v g}{\alpha}$.

The half life of the velocity can be found in the traditional manner:

$$t_{\frac{1}{2}} = \frac{\ln 2}{k} \quad og \quad k = \frac{\alpha}{m} \quad \Rightarrow \quad t_{\frac{1}{2}} = \frac{m\ln 2}{\alpha}.$$

For the majority of motions in liquids, the final velocity is obtained rather quickly.

The equation (3.5) can of course be integrated to give the distance *x*.

(3.6) $$x = x_0 + \frac{m_V g}{\alpha}\left(t + \frac{m}{\alpha}\left(e^{-\frac{\alpha}{m}\cdot t} - 1\right)\right)$$

If the body has a initial velocity $v_0$ opposite to gravity, we must change sign on the *mg* term in (3.4) and $c = v_0 + \frac{m_V g}{\alpha}$ . In this case we find the solution:

(3.7) $$v = v_0 e^{-\frac{\alpha}{m}\cdot t} + \frac{m_V g}{\alpha}\left(e^{-\frac{\alpha}{m}\cdot t} - 1\right)$$

But again the velocity again approaches asymptotically to $v_\infty = -\frac{m_V g}{\alpha}$ .