Filling a tub from
a hot water tank
While heating the tank

This is an article from my home page: www.olewitthansen.dk

Ole Witt-Hansen 2023

Contents

1. Stating of the problem.

Many houses have a private tank of hot water from which they tab water to a shower or into a tub. Cold water is filled in the tank, as hot water is tapped, so that the volume of hot water tank is constant. When the temperature of the water in the hot water tank decreases a heater turns automatically on to restore the temperature. The hot water tank is diminished so that comply with max 4 showers, but it is not equipped to fill a large tub.
If you are to give birth in a tub, you will need to have a 400 l tub at 38^0C.
This article deals with, how can be done with a water tank of 60 l, and which can be heated to max 60^0.
Let us assume that the hot water tank holds M kg of water. And that it has the initial temperature T_{60} degrees Celsius.

Let us further assume that there is tapped $\mu = \frac{dm}{dt}$ water from the hot water tank per second.
The same amount of cold water with temperature T_0 is added to the hot water tank.
The heat capacity of water is c =4186 J/kgK, and P is the power of the heating apparatus.

We shall first determine how the temperature of the hot water container will change in time, as long as water is tapped from the hot water container.

We do this by establishing a caloric equation, expressing how the energy taken from the hot water tank is distributed to the outlet.

The tub can hold 400 l. The capacity of the heating tank is 60 litres. The heater delivers (max) 15 kW.
The initial temperature in the 60 l heating tank is 60^0 C.
The inlet cold water temperature to the hot water tank is 6.5^0
If we assume that the tub is filled by 0.2 l/s.
So the time to fill the tub is: $\frac{400}{0.2} s = 2000\, s = 33.3 \min$.

The final temperature in the 400 l tub is supposed to be 38^0 C, when the tub is filled.

To raise the temperature of 400 kg water from 6.5^0 to 38 deg requires:

$$\Delta E = mc\Delta T = 400 \cdot 4186 \cdot (38.5 - 6.5) = 56.092.4 kJ$$

If the heater delivers 15 kW it will take.

$$\Delta t = \frac{\Delta E}{P} = \frac{56{,}092.4 kJ}{15 kJ} = 3739.5\, s = 62.3 \min = 1h \;\; 2.3 \min$$, to provide heating the watert to the tub.

Thus it takes about the double time to heat the water than to fill the tub!

The water outlet from the tank equals to the inlet of cold water to the heating tank, which we assume to be 0.20 l/s.

The temperature in the hot water tank will, however, decrease, when the outlet is 0.20 l/s, where the time used to raise 0.20 l from 6.5^0 to 60^0 is given by the equation:

$$\Delta m(60-6.5)4186 = 15kW\Delta t \quad \Rightarrow \quad \Delta t = \frac{0.2(53.5)4186}{15kW} = 3.0\ s$$

Which is 3 times longer than the outlet of 0.20 l/s.

After about 1000 sec, the temperature in the hot water tank will be 24.5^0 C , the time *t* it takes to restore the temperature 24.5^0 C to 60^0 is given by the equation:

$$mc(\Delta T) = 15kW \cdot t \quad \Rightarrow \quad t = \frac{60 \cdot 4186 \cdot 35.5}{15KW} = 594s = 9.9\,\text{min}$$

To find the temperature of water in the hot water tank after a long time, where it is stabilized, we establish the equation:

$$c\frac{\Delta m}{\Delta t}(T-6.5) = 15kW \quad \Leftrightarrow \quad (T-6.5) = \frac{15kW}{0.2 \cdot 4186} = 17.9^0 \quad \Rightarrow \quad T = 24.5^0$$

So another method is certainly required, if we want 400 litres to be heated from 6.5 ^{0} C to 38^0C

We may calculate the *power* required to be delivered to heat the water in the tub in 33.3 min, which is the time used to fill the tub

$$P = \frac{\Delta E}{2000} = \frac{56092kJ}{2000} = 28.046kW$$. Which is about the double time needed to fill the tub.

So with the heating equipment available we will have to chose another approach..

2. Calculating the temperature in the hot water tank, where the Inlet temperature of water $T_{6.5} = 6.5^0C$

To do so we establish a caloric equation expressing that the energy of the remaining energy of th e water in the tub $c(M-dm)T$ plus the inlet of cold $+cdmT_{6.5}$, plus the power delivered by the heater Pdt equals the final energy in the tank.

$$c(M-dm)T + cdmT_{6.5} + Pdt = cM(T+dT) \quad \Leftrightarrow$$

$$-cdmT + cdmT_{6.5} + Pdt = cMdT \quad \Leftrightarrow$$

$$c\frac{dm}{dt}(T_{6.5}-T) + P = cM\frac{dT}{dt} \quad \Leftrightarrow$$

$$\frac{MdT}{-\frac{dm}{dt}T + \frac{dm}{dt}T_{6.5} + \frac{P}{c}} = dt$$

If we put $\mu = \frac{dm}{dt}$ we get:

$$\frac{MdT}{-\mu T + \mu T_{6.5} + \frac{P}{c}} = dt \quad \Leftrightarrow \quad \frac{\frac{M}{\mu} dT}{-T + T_{6.5} + \frac{P}{\mu c}} = dt \quad \Leftrightarrow \quad \frac{-\frac{M}{\mu} dT}{T - T_{6.5} - \frac{P}{\mu c}} = dt$$

This is a differential equation of the form: $\frac{kdy}{ay+b} = dx$ and it has the solution: $\frac{k}{a}\ln(ay+b) = x + c$,

So our differential equation becomes

$$\frac{-\frac{M}{\mu} dT}{T - T_{6.5} - \frac{P}{\mu c}} = dt$$

And it has the solution: $-\frac{M}{\mu}\ln(T - T_{6.5} - \frac{P}{\mu c}) = t + K \quad \Leftrightarrow$

$$\ln(T - T_{6.5} - \frac{P}{\mu c}) = -\frac{\mu}{M}t + K \quad \Leftrightarrow \quad = (T - T_{6.5} - \frac{P}{\mu c}) = \exp^{K} \exp\left(-\frac{\mu}{M}t\right) \quad \Leftrightarrow$$

$$T = T_{6.5} + \frac{P}{\mu c} + \exp^{K} \exp\left(-\frac{\mu}{M}t\right) \quad \Rightarrow$$

Let us assume that the initial temperature in the hot water tank is $T = T_{60}$ then we have:

$$T_{60} = T_{6.5} + \frac{P}{\mu c} + \exp^{K} \quad \Rightarrow \quad \exp^{K} = T_{60} - T_{6.5} - \frac{P}{c\mu}$$

So the solution becomes:

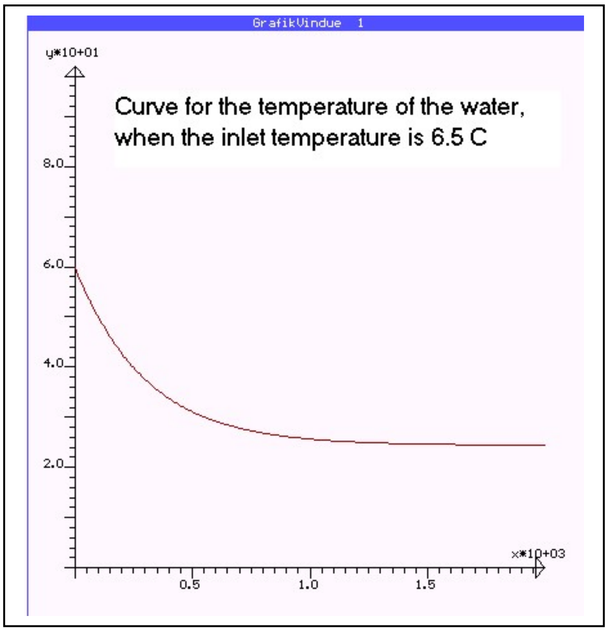
$$T = T_{6.5} + \frac{P}{c\mu} + (T_{60} - T_{6.5} - \frac{P}{c\mu})\exp\left(-\frac{\mu}{M}t\right)$$

4. Temperature curve for the water (that is let out of the tank and into the tub), when the inlet water to the hot water tank is 6.5 C

Inserting the values:

$\mu = 0.2\, l/s: \; P = 15\, kW \;\; T_{6.5} = 6.5, \; and \;\; T_{60} = 60 \; we \; find \qquad T = 24.4 + 35.6\exp\left(-\frac{1}{300}t\right)$

The graph for the temperature curve of the hot water tank is shown below.



We can see when the heater runs continuously from the start then the temperature will be about 24.5^0 after 2000 sec.
However, if the heater starts from a lower temperature where the temperature is less than 60^0, then the final temperature will probably be lower.

3. Filling the tub with hot water from the heater

The differential equation below expresses that the increase in the water temperature in the tub comes from the hot water from the heater.
But the temperature of the hot water inlet temperature T_{in} depends on time as explained above.
μ is the volume inlet of hot water per sec.
The differential equation is the following, where T is the temperature of the water in the tub having mass M_T , and T_{in} is the temperature of the inlet water to the tub.

$$d(M_T T) = \mu dm T_{in}$$

$$M_T dT + T dM_T = \mu dm T_{in}$$

And with division by dt

$$M_T \frac{dT}{dt} + T\frac{dM_T}{dt} = \mu \frac{dm}{dt} T_{in} \qquad \text{with } M_T = \mu t \qquad \frac{dM_T}{dt} = \frac{dm}{dt} = \mu$$

And inserting the expression for the temperature for the hot water in the tank at time t, we get:

$$T_m = T_{6.5} + \frac{P}{c\mu} + (T_{60} - T_{6.5} - \frac{P}{c\mu})\exp\left(-\frac{\mu}{M}t\right)$$

of the hot water at time t: Then We get:

$$c\mu t\frac{dT}{dt} + c\mu T = c\mu\left(T_{6.5} + \frac{P}{c\mu} + (T_{60} - T_{6.5} - \frac{P}{c\mu})\exp\left(-\frac{\mu}{M}t\right)\right)$$

And by division by $c\mu t$.

$$\frac{dT}{dt} = -\frac{T}{t} + \frac{1}{t}\left(T_{6.5} + \frac{P}{c\mu} + (T_{60} - T_{6.5} - \frac{P}{c\mu})\exp\left(-\frac{\mu}{M}t\right)\right)$$

This differential equation can hardly be solved, so it is solved analytically.

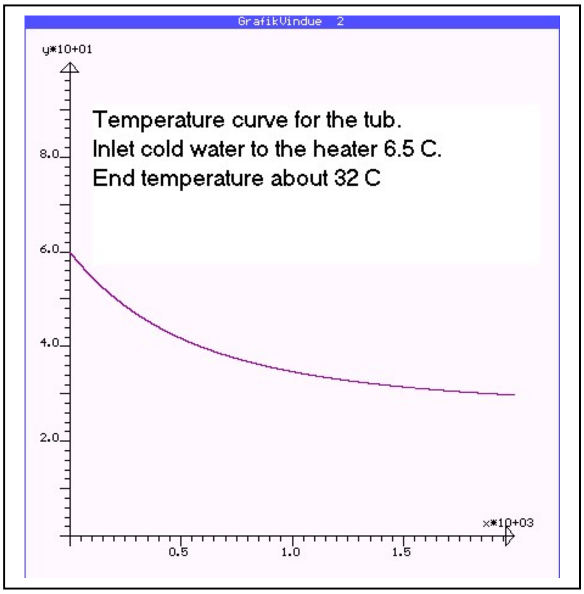
3a. Temperature curve for the water in the tub, 0 to 2000 sec, when it is filled with water from the hot water tank. Inlet of cold water to the heater is 6.5^0 C

$$\frac{dT}{dt} = -\frac{T}{t} + \frac{1}{t}\left(T_{6.5} + \frac{P}{c\mu} + (T_{60} - T_{6.5} - \frac{P}{c\mu})\exp\left(-\frac{\mu}{M}t\right)\right)$$

$$\frac{dT}{dt} = -\frac{T}{t} + \frac{1}{t}\left(6.5 + \frac{P}{c\mu} + (60 - 6.5 - \frac{P}{c\mu})\exp\left(-\frac{\mu}{M}t\right)\right)$$

$$\frac{dT}{dt} = -\frac{T}{t} + \frac{1}{t}\left(24.4 + 35.6)\exp\left(-\frac{\mu}{M}t\right)\right)$$

-



The temperature in the tub decreases as it is filled with water, because the temperature of hot water from the tank also decreases.The final temperature in the tub becomes about 32^0. This is still 6^0 below the wanted temperature 38^0.

If we want to compensate for this by adding 100^{0}C water, we should calculate the number of litres required, so that the total number of litres in the tub remain 400 l.

$$c(M-m)32 + cm100 = cM\cdot 38 \quad \Leftrightarrow \quad m(100-32) = M(38-32) \quad \Rightarrow$$

$$m = \frac{M\cdot 6}{68} = \frac{400\cdot 8}{68} = 23.7 \ \ litres$$

It is perhaps surprising that you should provide 23.7 litres boiling water to raise the temperature just 6^0 of a 400 l tub
In practice it is not realistic to do, within a reasonable time.

To boil 1 litre water from 10^0 to 100^0 from a 2 kW el-kettle takes 3.5 min. So the transaction would take 3.5*23.7 =83 min = 1 h and 23 min. and the water in the tub, will probably cool off in such a period.
So to rise the temperature in the tub to 38^0 within an hour we must find another way.

Also the heater supplies 15 kW, which is 7.5 times that of an el-kettle. So initiating the hot water tank to 60^0 applying the heater (15 kW) is far less time consuming.

One litre of $100^{0\,C}$ water will raise the temperature of 399 litre only 0.175 ^{0}, as shown below:

$$(M-1)30+100 = M(30+\Delta T) \quad \Leftrightarrow \quad 30+\Delta T = \frac{(M-1)30+100}{M} = \frac{399\cdot 30+100}{400} = 30.175^0$$

$$\Delta T = 0.175^0$$

In accordance with $0.175 \cdot 23.7^0 = 6^0$ (when filled 1 litre at the time)

4. filling the tub, initiating the hot water tank to 60^0 after 1000 sec.

$$\frac{dT}{dt} = -\frac{T}{t} + \frac{1}{t}\left(T_{6.5} + \frac{P}{c\mu} + (T_{60} - T_{6.5} - \frac{P}{c\mu})\exp\left(-\frac{\mu}{M}t\right)\right)$$

$$\frac{dT}{dt} = -\frac{T}{t} + \frac{1}{t}\left(6.5 + \frac{P}{c\mu} + (60 - 6.5 - \frac{P}{c\mu})\exp\left(-\frac{\mu}{M}t\right)\right)$$

0 - 1000: $\frac{dT}{dt} = -\frac{T}{t} + \frac{1}{t}\left(24.4 + 35.6)\exp\left(-\frac{\mu}{M}t\right)\right)$

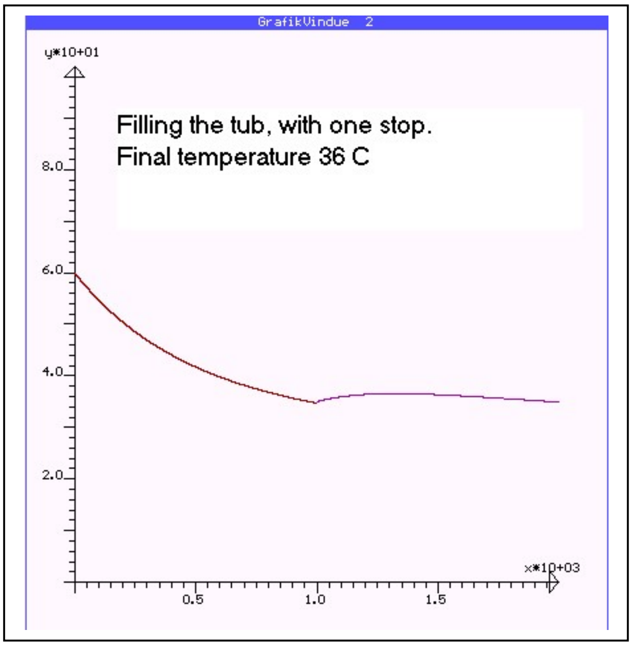
Temperature after 1000 sec 42^0. (So far so good)

Next step, hot water tank is initiated to 60^0. (10 to 15 min). Start temperature 42^0.

$$\frac{dT}{dt} = -\frac{T}{t+1000} + \frac{1}{t+1000}\left(24.4 + (30 - 6.5 - \frac{P}{c\mu})\exp\left(-\frac{\mu}{M}(t-1000)\right)\right)$$

$$\frac{dT}{dt} = -\frac{T}{t+500} + \frac{1}{t+500}\left(24.4 + 23.6)\exp\left(-\frac{\mu}{M}(t-500\right)\right)$$

The graph for the temperature in the tub is shown below. Final temperature is 36^0



We can see that the temperature stabilizes around 36^0 (instead of decreases)
The final temperature after 2000 sec. is about 36.
The increase in temperature is lower, because of the increase of water in the tub,

5a. filling the tub, initiating the heater twice to 60^0 after 1000 sec and 1500 sec

The graph for the temperature in the tub is shown below.

0 – 1000: $\frac{dT}{dt} = -\frac{T}{t} + \frac{1}{t}\left(T_{6.5} + \frac{P}{c\mu} + (T_{60} - T_{6.5} - \frac{P}{c\mu})\exp\left(-\frac{\mu}{M}t\right)\right)$

Which is the same as before.

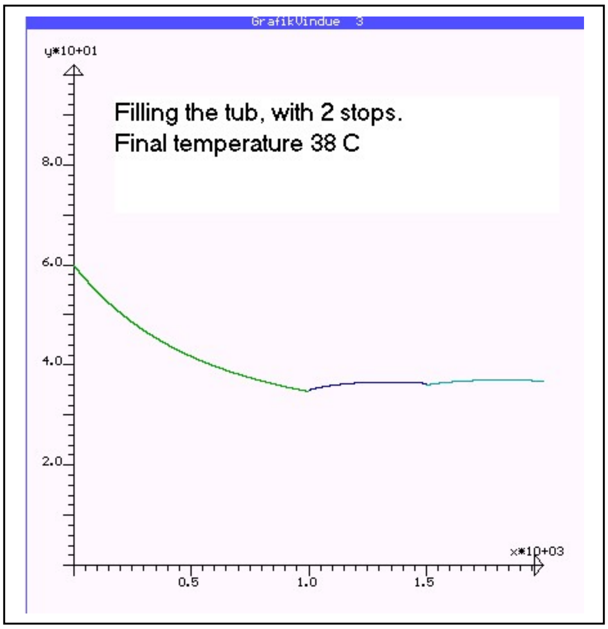
But if the heater is now initiated when the temperature in the tub is 36^0.

1000 – 1500, Where the he initial temperature is 38 ^{0}.

$$\frac{dT}{dt} = -\frac{T}{t+1000} + \frac{1}{t+1000}\left(6.5 + \frac{P}{c\mu} + (60 - 6.5 - \frac{P}{c\mu})\exp\left(-\frac{\mu}{M}(t-1000)\right)\right)$$

1500 – 2000: The initial temperatur is 38:

$$\frac{dT}{dt} = -\frac{T}{t+1500} + \frac{1}{t+1500}\left(24.4 + 35.6\exp\left(-\frac{\mu}{M}(t-1500\right)\right)$$

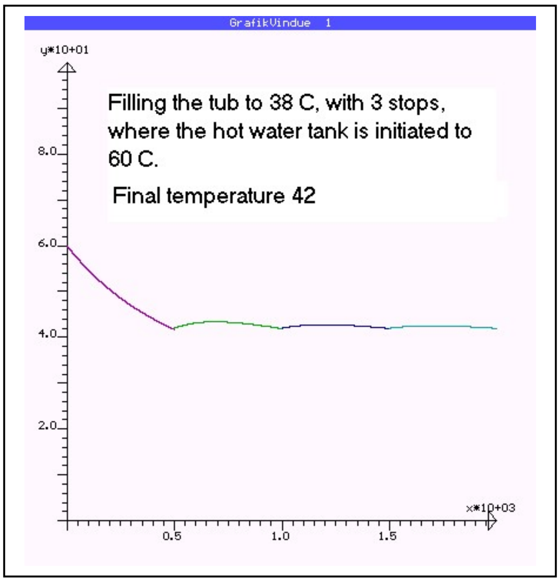


The temperature reaches about 38^0 (In theory) but we have neglected heat dissipation to the surroundings, which can neither be neglected nor estimated.

5a. filling the tub, initiating the heater to 60^0 3 times after 500 sec, 1000 sec and 1500 sec

The graph for the temperature in the tub is shown below.

$$\frac{dT}{dt} = -\frac{T}{t+1000} + \frac{1}{t+1000}\left(6.5 + \frac{P}{c\mu} + (60 - 6.5 - \frac{P}{c\mu})\exp\left(-\frac{\mu}{M}(t-1000)\right)\right)$$



The final temperature is read from the curve to about 42^0, but heat dissipation may alter this.

However, we don't know the exact power of the heater, and we have neglected any dissipation af heat to the surroundings, so my conclusion is that it cannot be accomplished.

The curve shows that the final temperature is about 44 C.
However it is with the assumption that the heater constant delivers 15 kW, and there is not heat dissipation, but this can probably not be uphold, and the temperature in the tub will allegedly be lower.