Elasticity

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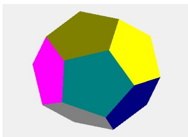


Ole Witt-Hansen   2019

Contents

1. Hooke's law

Elasticity of materials is the property that they regain their original shape, when the forces causing their deformations are removed. We find the property of elasticity in almost all solid materials.
For a homogenous bar, and for small deformation Hooke's law apply, which gives the deformation Δl as a result of a force *F*. Δl is simply proportional to *F*.
For traditional reasons the proportionality $\Delta l \propto F$ is written the other way round.

(1.1) $$F = k\Delta l$$

The lengthening Δl is proportional to the length of the material. This is obvious, since if we replace *l* with 2*l*. then the same force is acting on the two lengths *l*, the lengthening will be $2\Delta l$. Also the lengthening Δl must be inversely proportional to the thickness of bar. If we separate the bar with the double thickness, affected by the force *F,* into two identical bars, , each part will be affected with the force ½*F,* and therefore the lengthening will be $\frac{1}{2}\Delta l$. So the lengthening Δl is proportional to the length of the bar, and inversely proportional to the cross section. Thus we have:

$$\Delta l = k\frac{l}{A}F$$

Traditionally Hooke's law is written:

(1.2) $$F = EA\frac{\Delta l}{l}$$

Where *E* is called is called Young's modulus, which only depends only on the material in question.
The force per unit area is called the *stress*, and the stretch per unit length is called the *strain*.
Equation (1.2) may then be rewritten as.

(1.3) $$\frac{F}{A} = E\frac{\Delta l}{l}$$

Stress = (Young's modulus) x (strain)

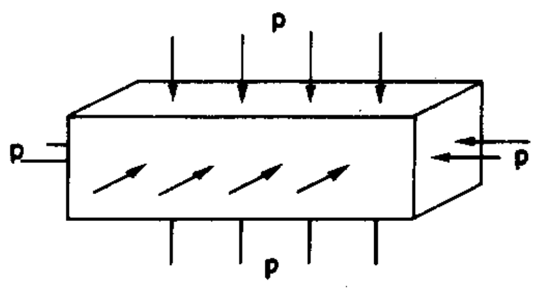
For a rectangular bar, there is a derived effect to Hooke's law. When you stretch the bar in one direction it contracts at right angles to the stretch. The contraction is proportional to the width of the bar, and also to $\Delta l / l$. The sidewise contraction is in the same proportion for both width and height, and it is usually written.

(1.4) $$\frac{\Delta w}{w} = \frac{\Delta h}{h} = -\sigma\frac{\Delta l}{l}$$

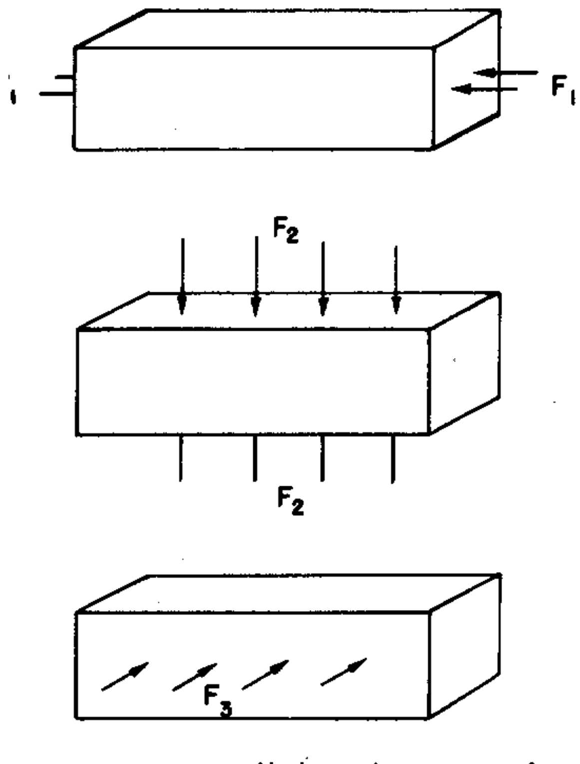
The constant σ is another constant of the material, and it is called *Poisson's ratio*.

The two constants *E* and σ specify completely the elastic properties of a homogenous elastic material.

2. Uniform strain



A bar under uniform hydrostatic pressure.



Hydrostatic pressure is the superposition of three longitudinal compressions.

Since the principle of superposition applies to forces, so does it also to the equations of elasticity.
We shall first deal with a rectangular block under uniform hydrostatic pressure. (The figure to the left). If we assume that the block is submerged in water in a pressure tank, and if we discard the differences in pressure due to gravity, then the inward pressure, (the stress) is the same in all direction, and the force on each side is proportional to the area. This follows from de definition of pressure: $p = F/A$.
First we shall work out the change in length, which is the sum of changes caused by the three independent forces, illustrated in the figure to the left. First the stress caused by F_1 in the figure. According to (1.3)

$$p = \frac{F}{A} = E\frac{\Delta l}{l}, \text{ so}$$

(2.1)
$$\frac{\Delta l_1}{l} = -\frac{p}{E}$$

Next, we look at the stress caused by the force F_2. If we push on the two sides of the block with pressure p , the compressional strain is as before p/E, but this time we want the lengthwise strength. We can get that from the sidewise strain multiplied by $-\sigma$. The sidewise strain is:

$$\frac{\Delta w}{w} = -\frac{p}{E}$$

so

(2.2)
$$\frac{\Delta l_2}{l} = +\sigma\frac{p}{E}$$

Since there are no differences between the sides (2) and (3), we can combine the results from the three type of sides to give:

(2.3)
$$\frac{\Delta l}{l} = \frac{\Delta l_1}{l} + \frac{\Delta l_2}{l} + \frac{\Delta l_3}{l} = -\frac{p}{E} + \sigma\frac{p}{E} + \sigma\frac{p}{E} \Rightarrow$$
$$\frac{\Delta l}{l} = -\frac{p}{E}(1-2\sigma)$$

The problem is of course symmetrical in all three directions, therefore we must have:

(2.4)
$$\frac{\Delta w}{w} = \frac{\Delta h}{h} = -\frac{p}{E}(1-2\sigma)$$

The relative change in *volume* under hydrostatic pressure may also be calculated from $V = lwh$, so

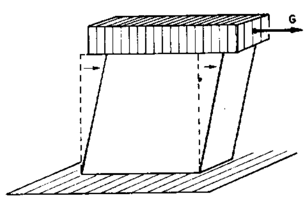
$$\Delta V = \frac{\partial V}{\partial l}\Delta l + \frac{\partial V}{\partial w}\Delta w + \frac{\partial V}{\partial h}\Delta h = wh\Delta l + lh\Delta w + lw\Delta h$$

(2.4)
$$\frac{\Delta V}{V} = \frac{\Delta l}{l} + \frac{\Delta w}{w} + \frac{\Delta h}{h} \quad \Rightarrow \quad \frac{\Delta V}{V} = -3\frac{p}{E}(1-2\sigma)$$

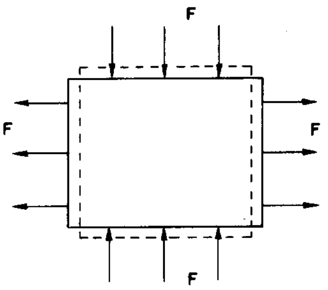
Often $\Delta V / V$ is denoted the volume strain, and the relation (2.4) is then written in a slightly different way:

(2.5)
$$p = -K\frac{\Delta V}{V} \qquad \text{where} \qquad K = 3\frac{E}{(1-2\sigma)}$$

3. Shear strain



A cube in uniform shear.

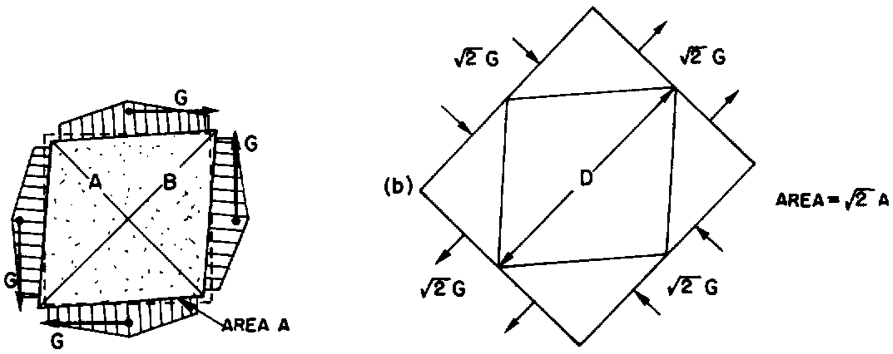


A cube with compressing forces on top and bottom and equal stretching forces on two sides.

Next we shall see what happens, when we put a "shear" stress on a bulk of matter. By a shear strain, we mean the kind of distortion show in the figure to the left. For simplicity, let us look at the strains in a cube of material subjected to the forces, as shown in the bottom figure to the left. Again we shall break it up in two parts: The vertical pushes and the horizontal pulls. Calling the area of the cube face for A, we have for the change in the horizontal length.

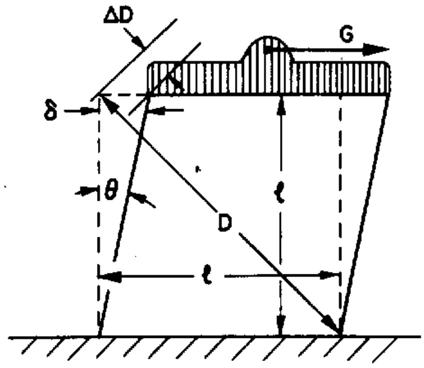
(3.1)
$$\frac{\Delta l}{l} = \frac{1}{E}\frac{F}{A} + \sigma\frac{1}{E}\frac{F}{A} = \frac{1+\sigma}{E}\frac{F}{A}$$

The change in the vertical height is just the negative of this. Now suppose we have the same cube and subject to the same shearing forces, as shown in the figure below to the left. Note that all the forces must be equal to keep the cube in equilibrium. The cube is then said to be in a state of pure shear.



The two pairs of shear forces in (a) produce the same stress as the compressing and stretching forces of (b).

The sum of the two vectors $\vec{G}$ is a vector along the diagonal having the length $\sqrt{2}G$
The area over which this force acts is $\sqrt{2}A$. Therefore the tensile stress normal to the plane is *G/A*. Similarly, if we examine a plane 45^0 the other way – the diagonal *B* in the figure – we se that there is a compressional stress normal to this plane of *–G/A*.
From this we see that the stress in a pure shear is equivalent to a combination of tension and compression stresses of equal strength at right angle to each other, and at 45^0 to the original faces of the cube. The internal stresses and strains are the same as we would find in a larger block of material with the forces shown in the figure above to the right. But this problem, we have already solved. The change in length of the diagonal is given by (3.1).



The shear strain θ is 2 ΔD/D.

(3.2) $$\frac{\Delta D}{D} = \frac{1+\sigma}{E}\frac{G}{A}$$

One diagonal is shortened, the other is elongated.
It is often convenient to express a shear strain in terms of the angle by which the cube is twisted, that is, the angle θ in the figure to the left. From the geometry of the figure, you can see that the horizontal shift δ of the top edge is equal to $\sqrt{2}\Delta D$, so

(3.3) $$\theta = \frac{\delta}{l} = 2\frac{\Delta D}{D}$$

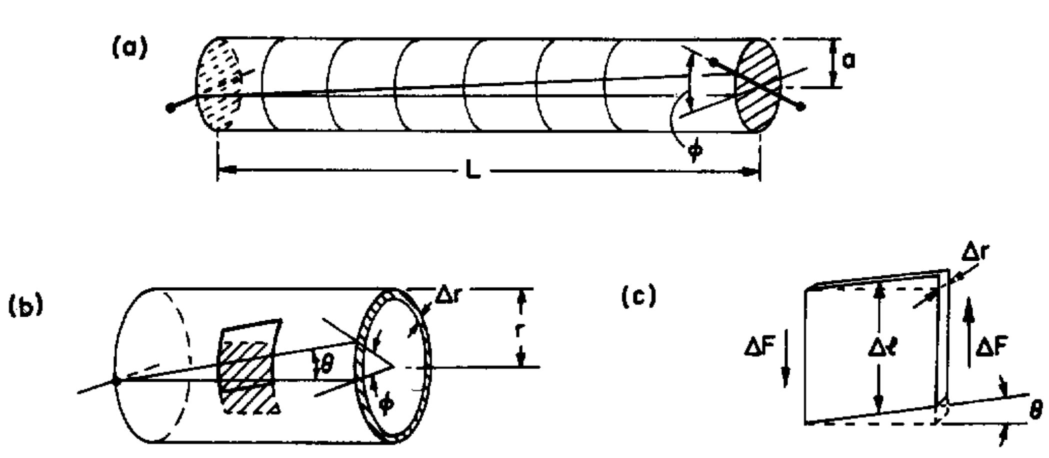
The *shear stress* η is defined as the tangential force on a face divided by the area. $\eta = G/A$.
Using the previous results (3,2) and (3,3), we get:

(3.4) $$\theta = 2\frac{1+\sigma}{E}\eta$$

This may also be written in the form "*stress = constant times strain*": $\eta = \mu\theta$ .The coefficient μ is usually called the *shear modulus*: It can be expressed by the other elastic constants as

(3.5) $$\mu = \frac{E}{2(1+\sigma)}$$

4. The torsion bar



(a) A cylindrical bar in torsion. (b) A cylindrical shell in torsion. (c) Each small piece of the shell is in shear.

We shall now turn to an example which is more complicated because different parts of the material are stressed by different amounts. We shall first consider a rod that is twisted. From experiments with the torsion pendulum we know that the torque on a twisted rod is proportional to the torsion angle, where the constant of proportionality depends on the material and the design of the rod or thread. The question is: In what way? We shall now try to answer this question which mainly is a geometrical one.
The figure above shows a cylindrical rod of length *L*, and radius *a*, with one end twisted by an angle ϕ with respect to the other. If we want to relate the strains to what we already know, we can think of the rod of being made up of many cylindrical shells and work out separately what happens to each shell.
We begin by looking at a thin short cylinder of radius *r* and thickness Δr, as shown in the figure above below to the left. If we look at a piece of this cylinder that was originally a small square, we can see that it has been distorted in to a parallelogram. Each such piece is in shear, and the shear angle is

(4.1) $$\theta = \frac{r\phi}{L}$$

The shear stress η in the material is therefore from (3.4).

(4.2) $$\eta = \mu\theta = \mu\frac{r\phi}{L}$$

The shear stress is the tangential force ΔF on the end of the square divided by the area $\Delta l \Delta r$ of the end. (See the figure above to the right).

(4.3) $$\eta = \frac{\Delta F}{\Delta l \Delta r}$$

The force ΔF on such a square contributes a torque $\Delta\tau$ around the axis of the rod equal to

(4.4) $$\Delta\tau = r\Delta F = r\eta\Delta l\Delta r$$

The total torque is the sum of such torques around a complete circumference of the cylinder.
So putting the pieces together, such that the $\Delta l's$ add up to the circumference $2\pi r$, we find that the total torque for a hollow cylinder is

$$r\eta(2\theta\pi r)\Delta r$$

Or using (4.3)

$$\tau = 2\pi\mu\frac{r^3\Delta r\phi}{L}$$

We thus find that the rotational "stiffness" τ/ϕ of a hollow cylinder is proportional to the radius to the third power, to the thickness Δr, and inversely proportional to the length *L*.
We now imagine a circular rod made up of a series of concentric tubes each twisted by the same angle ϕ. The total torque is then found by integrating over the radius *r*.

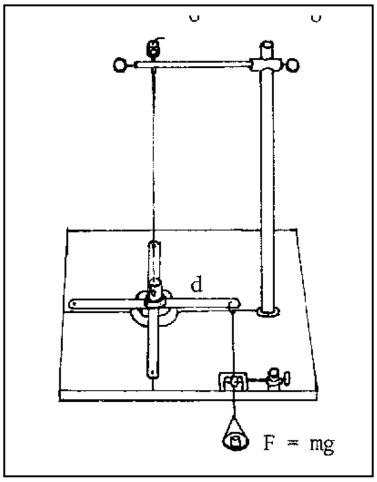
$$\tau = 2\pi\mu\frac{\phi}{L}\int_0^a r^3 dr$$

(4.4) $$\tau = \mu\frac{\pi a^4}{2L}\phi$$

For a rod in torsion the torque is proportional to the angle and to the fourth power of the diameter, and inversely proportional to the length. A rod which is twice as thick is sixteen times as stiff for torsion.

5. The torsion pendulum and torsion waves

We shall then briefly look into a matter of elastic dynamics called torsion waves, which in fact is a generalization of the torsion pendulum.



Until about forty years ago the torsion pendulum was one of many experiments that the students performed with a rapport in the 9-12 grade of the Danish high school. It was done with the apparatus shown to the left. (The physical notations are somewhat different from that of the US)
The torque H necessary to twist the metal wire an angle φ is proportional to φ. We write this as.

(5.1) $$H = D\varphi$$

D is denoted the torsion coefficient or the directional moment. It depends on the length, the thickness and of the materiel of the wire. In contrast to Hooke's law (5,1) also applies to larger angles, that is, over 360^0,at least for longer and thinner wires.
If a symmetric mass having moment of inertia I is hung in he wire, turned an angle φ ,and then released, it will be affected by a torque $H = -D\varphi$ from the wire. According to Newton's law for rotation the system will obey the equation

(5.2) $$H = I\frac{d^2\varphi}{dt^2} \quad \Leftrightarrow \quad -D\varphi = I\frac{d^2\varphi}{dt^2} \quad \Leftrightarrow \quad \frac{d^2\varphi}{dt^2} = -\frac{D}{I}\varphi$$

The well known solution to this differential equation is a harmonic oscillation, which may be described as:

(5.3) $$\varphi = \varphi_0\cos(\omega t + \alpha_0) \text{ , where } \omega = \sqrt{\frac{D}{I}} \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{I}{D}}$$

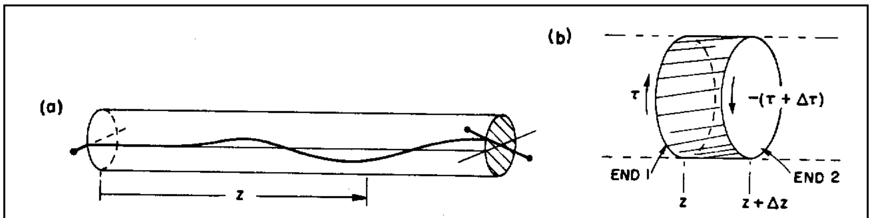
The experiment is not really meaningful, without knowledge of the torsion coefficient D, so D is performed in a static determination. As indicated in the figure, this is accomplished by delivering a torque from small masses in a cup, connected to the pendulum with a cord.
By measuring the masses and the angle, you may plot the results into a (m, φ) graph and determine D , from the slope.
Then one can set the torsion pendulum in oscillations, measure the period an compare with the formula (5,3).

One may however also perform a dynamic determination of the torsion coefficient based on the formula (5,3).
Namely, you may place one or more circular discs with known moment of inertia I_1 and I_2 on top of the pendulum. Since the moments of inertia along the same axis are additive, we may write

(5.4) $$T_1 = 2\pi\sqrt{\frac{I_0+I_1}{D}} \quad \wedge \quad T_2 = 2\pi\sqrt{\frac{I_0+I_2}{D}} \quad \Rightarrow \quad D = 4\pi^2\frac{I_2-I_1}{T_2^2-T_1^2}$$

5.1 Torsion waves

We shall then look into torsional waves in a rod of finite thickness, as illustrated below.



(a) A torsional wave on a rod. (b) A volume element of the rod.

Let z be the distance to some point down the rod. For a static torsion the torque is the same everywhere along the rod and proportional to ϕ / L, the total torsion angle over the total length. What matters to the material is the local torsion strain, which is $\partial\phi / \partial z$, so when the torsion along the rod is not uniform, we should replace (4.4) by

(5.5) $$\tau(z) = \mu\frac{\pi a^4}{2}\frac{\partial\phi}{\partial z}$$

Let us then analyze an element of length Δz shown magnified in the figure above to the right. There is a torque $\tau(z)$ at "end 1" of the little hung of rod, and a torque $\tau(z+\Delta z)$ at "end 2".
The net torque $\Delta\tau$ acting is then $\Delta z = \tau(z+\Delta z) - \tau(z) = \frac{\partial\tau}{\partial z}\Delta z$.
Differentiating (5.5) we get:

$$\Delta\tau(z) = \mu\frac{\pi a^4}{2}\frac{\partial^2\phi}{\partial z^2}\Delta z$$

The effect of this net torque is to give an angular acceleration to the little slice of the rod. The mass of the slice is:

$$\Delta m = (\pi a^2 \Delta z)\rho$$

Where ρ is the density of the material. The moment of inertia of a circular disc is ½mv^2. Calling the moment of inertia of our little piece of rod for ΔI we have:

(5.6) $$\Delta I = \tfrac{\pi}{2}\rho a^4 \Delta z$$

Newton's law says that the torque is equal to the moment of inertia times the angular acceleration.

(5.7) $$\Delta\tau = \Delta I \frac{\partial^2 \phi}{\partial t^2}$$

Inserting the expressions we have obtained for $\Delta\tau$ and ΔI, we have:

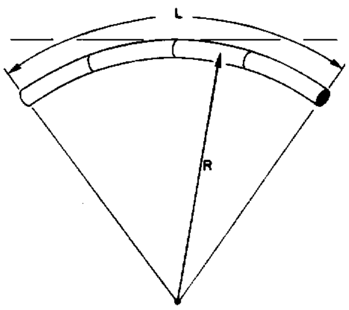
$$\mu \frac{\pi a^4}{2} \frac{\partial^2 \phi}{\partial z^2} \Delta z = \frac{\pi}{2} \rho a^4 \Delta z \frac{\partial^2 \phi}{\partial t^2}$$

(5.8) $$\frac{\partial^2 \phi}{\partial z^2} - \frac{\rho}{\mu} \frac{\partial^2 \phi}{\partial t^2} = 0$$

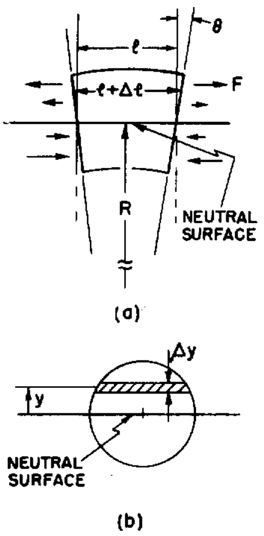
The last equation is recognized as the wave equation, corresponding to a speed of propagation

$$v = \sqrt{\frac{\mu}{\rho}}$$

6. The bent beam



A bent beam.



(a) Small segment of a bent beam. (b) Cross section of the beam.

Now we shall look into bending of a rod or a beam, which is another matter concerning elasticity. (See the figure to the left).
We shall first analyze which forces act on a cross section of the bar.
To ease the mathematics, we shall assume a circular cross section of the bar, but this is no real limitation to the general results.
However, our results will be correct only when the radius of the bend is much larger than the thickness of the beam.
If you bend a bar, it will follow a curve, like the one shown in the figure to the left. Since the bar is curved it will be stretched at the upside of the bar and compressed at the inside. Therefore we must have a "middle" of the bar, where it is neither stretched nor compressed. This is called the *neutral surface*.
For a pure bending the, a thin slice of the bar is shown to the left (a). The material below the neutral surface has a compressional strain, which geometrically is proportional to the distance from the neutral surface. So the longitudinal stretch Δl is proportional to the height y. Geometrically, we must have:

$$\frac{\Delta l}{l} = \frac{y}{R}$$

So the stress – the force per unit area – in a small strip at y is also proportional to the distance from the neutral surface.

(6.1) $$\frac{\Delta F}{\Delta A} = E \frac{y}{R}$$

We shall then analyze the forces that produce such a strain. The forces acting on the little segment are drawn in figure (a). The forces above and below the neutral surface have opposite direction and together they make a bending moment of force *M,* that is, a torque about the neutral line.
We can then compute the total moment by integration the force times the distance from the neutral surface for one of the faces of the segment.

$$M = \int y dF$$

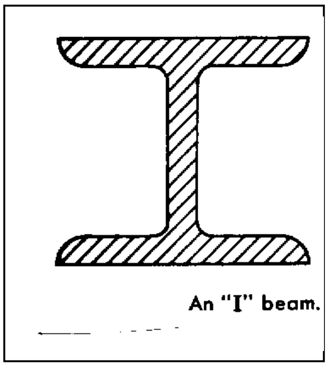
We have from (6.1) $dF = Ey/R\ dA$, so

(6.2) $$M = \frac{E}{R}\int y^2 dA$$

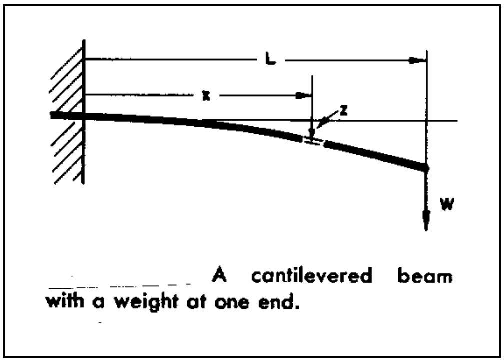
We shall denote the integral of $y^2 dA$ as "the moment of inertia" of the geometric cross section about a horizontal axis through its "centre of mass" as *I*.

(6.3) $$M = \frac{EI}{R} \quad \text{where} \quad I = \int y^2 dA$$

Equation (6,3) is the relation between the bending moment *M* and the curvature $1/R$ of the beam.



The stiffness of the beam is proportional to Young's module *E* and to the moment of inertia *I*.
The latter means that if you want to have a stiff beam, with a given amount of, say Aluminium then you should put as much as you can far away from the neutral surface to create a large moment of inertia.
This cannot be carried to extremes of course, since otherwise the beam will buckle or twist and do not regain its original shape.
This you already know from experience, since the way one produces an *H* shaped steel beams is quite familiar. (See the figure to the left).



As another example of the beam equation (6.3), we shall find the deflection of a beam which is fixed in one end, with a force ***F*** (in the figure *W*) acting in the free end.
We are interested in what the shape of the beam may be.
The mathematical formula for the curvature in a point of $z = z(x)$ is:

(6.4) $$\frac{1}{R} = \frac{d^2z/dx^2}{(1+(dz/dx)^2)^{\frac{3}{2}}}$$

Since we are only interested in small bending,
We shall neglect the square of the differential quotient in the denominator compared to 1, and put

(6.5) $$\frac{1}{R} = \frac{d^2z}{dx^2}$$

We shall also need the bending moment *M*. It is a function of *x*, because it is equal to the torque of any neutral axis of any cross section. We shall neglect the weight of the beam and take only the downward force *F*, at the end of the beam. Then the bending moment at *x* becomes:

(6.5) $$M(x) = F\cdot(L - x)$$

This is the torque about the point at x, exerted by the force F equal to the torque that the beam must support at x. We then have.

(6.6) $$F(L-x)=\frac{EI}{R}=EI\frac{d^2z}{dx^2}$$

Which gives

(6.7) $$\frac{d^2z}{dx^2}=\frac{F}{EI}(L-x)$$

It may easily be integrated to

(6.8) $$z=\frac{F}{EI}\left(\frac{Lx^2}{2}-\frac{x^3}{6}\right)$$

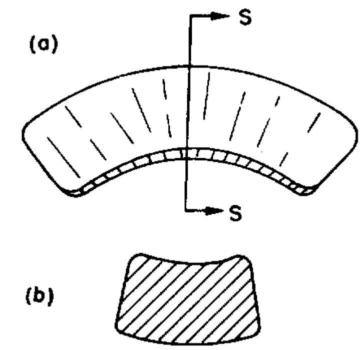
here we have used: $z(0) = 0$ and that dz/dx is also zero at $x = 0$. This is the shape of the bended beam. The displacement at the free end is:

$$z(L)=\frac{F}{EI}\frac{L^3}{3}$$

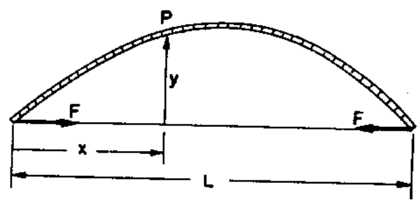
The polynomial (6,8) is by the way a spline polynomial that is used to give the smoothest approximation to a function, given by a series of equidistant points.
One spline polynomial covers 3 points, In the end points and middle point it has the same value as the function, whereas in left end points it also has the same slope as the preceding spline polynomial. Physically, we know that this is exactly the case for a bent beam.

7. Buckling

We shall turn to the concept of buckling of beams or rods.



(a) A bent eraser; (b) cross section.



A buckled beam.

Consider the situation sketched above to the right in which a rod that normally is straight, is held in its bent shape by two opposite forces that push on the ends of the rod. Our aim is then to find the shape of the buckled rod and the magnitude of the forces at the ends.
Let the deflection of the rod from a straight line be $y(x)$, where x is the distance from one end. The bending moment M at a point P in the figure is equal to the force multiplied by the moment arm, which also is the perpendicular distance y.

(7.1) $$M(x) = Fy$$

Using the equation (6,3), we have

$$\frac{EI}{R} = Fy$$

For small deflections, we can take $1/R = -d^2y/dx^2$ (the minus sign, because the curvature is downward), so we get:

(7.2) $$\frac{d^2y}{dx^2} = -\frac{F}{EI}y$$

But this is just the differential equation for the trigonometric functions.
So for small deflections the curve for such a bent beam is a sine curve. The "wavelength" of the sine wave is seen to be $2L$. So the general solution is

(7.3) $$y = c\sin \pi x/L$$

Differentiating twice, we have

(7.4) $$\frac{d^2y}{dx^2} = -\frac{\pi^2}{L^2}y$$

And when compared to (7.2), we can find the force.

(7.5) $$F = \pi^2\frac{EI}{L^2}$$

It is noteworthy that according to (7,5) the force is independent of the bending displacement y.
As a consequence, if the force is less than F, given in equation (7.5) there will be no bending at all.
But if the force is just slightly greater than this force, the material will suddenly bend a substantial amount, that is, for a force larger than the "critical" force π^2EI/L^2 (often called the Euler force), the beam will buckle.
For example if the loading of the columns that hold the second floor of a building exceed the Euler force the building will probably collapse.

References: The Feynmann Lectures on physics II (1965).
Arnold Sommerfeld: Mechanics of deformable bodies (64).