# Kinematics and Dynamics of deformable bodies

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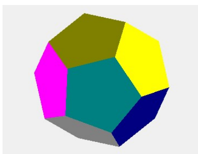

Ole Witt-Hansen (2021)

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## 1. Kinematics of deformable bodies

Our aim is first to show that the most general motion of a small element of a deformable body can be represented by as the sum of a *translation*, a *rotation* and an *extension* or *contraction* in three mutual orthogonal directions.
We shall therefore consider an infinitesimal displacement $d\vec{s} = (du, dv, dw)$ of an element, where the element has the position (*x, y, z*) in a Cartesian coordinate system.
We then have the three differentials

.

$$\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \\ dv &= \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz \\ dw &= \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz \end{aligned} \tag{1.1}$$

We ignore the translation, since it is the same for all points, and its differential is zero.
Then we rewrite these expressions, so that each new term will represent a rotation or an extension/contraction. We do this by an addition and subtraction of some of the partial derivatives.

$$du = \left[0 + \tfrac{1}{2}(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x})dy + \tfrac{1}{2}(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})dz\right] + \left[\frac{\partial u}{\partial x}dx + \tfrac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})dy + \tfrac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})dz\right] \tag{1.2}$$

You may convince yourself that this expression is actually equal to

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

Similarly for the *dv* and *dw*.

$$dv = \left[\tfrac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dx + 0 + \tfrac{1}{2}(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y})dz\right] + \left[\frac{\partial v}{\partial y}dy + \tfrac{1}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})dx + \tfrac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})dz\right] \tag{1.3}$$

$$dw = \left[\tfrac{1}{2}(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z})dx + \tfrac{1}{2}(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})dy + 0\right] + \left[\frac{\partial w}{\partial z}dz + \tfrac{1}{2}(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})dx + \tfrac{1}{2}(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z})dy\right] \tag{1.4}$$

The vector $d\vec{s} = (du, dv, dz)$ can therefore be split up into 2 vectors $d\vec{s} = d\vec{s}_1 + d\vec{s}_2$, indicated by the square parenthesis.
Regarding $d\vec{s}_1$, then we remind you of the formula for the rotation operator. If $\vec{s} = (u, v, w)$ then

$$\vec{\nabla} \times \vec{s} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{1.5}$$

If we introduce the vector:

(1.5) $$d\vec{\phi} = (d\phi_x, d\phi_y, d\phi_z) = \tfrac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}, \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}, \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

Then we can show (although it is a bit tricky), that the rotational part $d\vec{s}_1$ can be written as:

(1.6) $$d\vec{s}_1 = d\vec{\varphi} \times d\vec{r}$$

Where the cross product of two vectors $\vec{a} = (a_x, a_y, a_z)$ and $\vec{b} = (b_x, b_y, b_z)$ is given by:

$$\vec{a} \times \vec{b} = (a_z b_y - a_y b_z, a_x b_z - a_z b_x, a_y b_x - a_x b_y)$$

In our case:

(1.7) $$d\vec{\varphi} \times d\vec{r} = (d\varphi_z dy - d\varphi_y dz\ , d\varphi_x dz - d\varphi_z dx\ , d\varphi_y dx - d\varphi_x dy)\ ,$$

which we evaluate to get:

(1.8)

$$d\vec{\varphi} \times d\vec{r} = \tfrac{1}{2}\left( (\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x})dy - (\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z})dz\ ,\ (\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y})dz - (\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x})dx\ ,\ (\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z})dx - (\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y})dy \right)$$

This still not entirely equal to the two terms in $d\vec{s}_1$, (1.2) – (1.4) but if we change sign in the second terms in each coordinate, (by converting the minus sign outside the parenthesis to a plus sign), we find: (1.9)

$$d\vec{\varphi} \times d\vec{r} = \left( \tfrac{1}{2}(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x})dy + \tfrac{1}{2}(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})dz\ , \tfrac{1}{2}(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y})dz + \tfrac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dx\ ,\ \tfrac{1}{2}(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z})dx + \tfrac{1}{2}(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})dy \right)$$

This we compare to the first bracket in *du, dv, dw*, denoted with an index 1.

(1.10)
$$\begin{aligned}
du_1 &= \left[ \tfrac{1}{2}(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x})dy + \tfrac{1}{2}(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})dz \right] \\
dv_1 &= \left[ \tfrac{1}{2}(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y})dz + \tfrac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dx \right] \\
dw_1 &= \left[ \tfrac{1}{2}(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z})dx + \tfrac{1}{2}(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})dy \right]
\end{aligned}$$

So we have the equation $d\vec{s}_1 = \vec{\varphi} \times d\vec{r}$ . We have by definition the two equations:

(1.11) $$d\vec{s} = \vec{v}dt \qquad \text{and} \qquad d\vec{\varphi} = \vec{\omega}dt$$

Where $\vec{v}$ is the velocity, and $\vec{\omega}$ is the vortex vector, the significance of which was first recognized by Helmholtz. If we for simplicity write $(\dot{u}, \dot{v}, \dot{w})$ for $(du/dt, dv/dt, dw/dt)$, then the coordinates to $\vec{\omega}$ can be written:

(1.12) $$\omega_x = \tfrac{1}{2}\left(\frac{\partial \dot{w}}{\partial y} - \frac{\partial \dot{v}}{\partial z}\right) \qquad \omega_y = \tfrac{1}{2}\left(\frac{\partial \dot{u}}{\partial z} - \frac{\partial \dot{w}}{\partial x}\right) \qquad \omega_z = \tfrac{1}{2}\left(\frac{\partial \dot{v}}{\partial x} - \frac{\partial \dot{u}}{\partial y}\right)$$

If we remove $d\vec{s}_1$ (the rotation) from (1.2) – (1.4), we are left with the vector $d\vec{s}_2$ =($du_2, dv_2, dw_2$), having the terms shown below. Since these equations contain no translation or rotation of the element, they must represent the deformation, that is, the compression and/or distortion of the element. The coordinates of $d\vec{s}_2$ are:

(1.13) $$\begin{aligned} du_2 &= \frac{\partial u}{\partial x}dx + \tfrac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})dy + \tfrac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})dz \\ dv_2 &= \tfrac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})dx + \frac{\partial v}{\partial y}dy + \tfrac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})dz \\ dw_2 &= \tfrac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})dx + \tfrac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})dy + \frac{\partial w}{\partial z}dz \end{aligned}$$

These nine coefficients to *dx, dy, dz* form a tensor.
The terms $\frac{\partial u}{\partial x}dx$, $\frac{\partial v}{\partial y}dy$, $\frac{\partial w}{\partial z}dz$, represent a compression/extension per unit length in three orthogonal directions. The other elements represent distortions of the *xy, yz* or *xz* plane of the material element.
For the tensor elements, we write them as $\varepsilon_{i\,j}$, where *i, j = x, y, z*, so that

(1.14) $$\begin{aligned} du_2 &= \varepsilon_{xx}dx + \varepsilon_{xy}dy + \varepsilon_{xz}dz \\ dv_2 &= \varepsilon_{yx}dx + \varepsilon_{yy}dy + \varepsilon_{yz}dz \\ dw_2 &= \varepsilon_{zx}dx + \varepsilon_{zy}dy + \varepsilon_{zz}dz \end{aligned} \quad \text{where} \quad \varepsilon = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}$$

If we compare with the equations above, we find:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}dx \ , \ \varepsilon_{xy} = \varepsilon_{yx} = \tfrac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \ , \ \varepsilon_{xz} = \varepsilon_{zx} = \tfrac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})$$

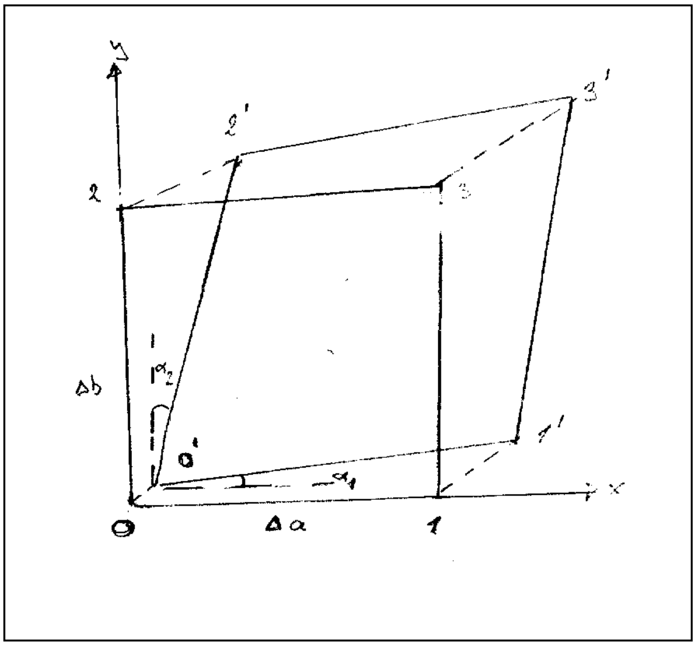
(1.15) $$\varepsilon_{yx} = \tfrac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \ , \ \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{yz} = \varepsilon_{yz} = \tfrac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})$$

$$\varepsilon_{zx} = \tfrac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \ , \ \varepsilon_{zy} = \tfrac{1}{2}(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}) \ , \ \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$\varepsilon$ is a 3x3 tensor called the *strain-tensor*, since they represent the compression and the distortions per unit length, due to some external force. Also they can be used to calculate the frequencies in the *x, y, z* directions when the element is acted on by a force, and then released, making free oscillations.
We shall then analyze the geometrical interpretation of the elements of the tensor $\varepsilon$.

We shall consider a $x\,y$ laminar of a rectangular box, which is so small that the deformations over the box are the same. We shall then look at a rectangle which is the $x\,y$ side of the box, having sides $\Delta a$ *and* $\Delta b$, both are considered infinitesimal small. See the figure below.
The laminar is 0132, and it is distorted into a parallelogram 0'1'3'2'.
We consider thus an infinitesimal distortion $(\Delta u, \Delta v, \Delta w)$.

Figure (1.17)




$\Delta v$ is the amount that the rectangle is stretched along the $y$ - axis, and $\Delta u$ is the amount that the rectangle is stretched along the $x$ – axis. The figure is two dimensional, (in the $x-y$ plane), although it might look three dimensional from the figure, but both the rectangle and the parallelogram are in the same plane
From the figure one can see that the angle $\alpha_1$ can be calculated from $\Delta v$ the stretch in $y$ direction divided by the length of stretched 0'1' line, having the length $\Delta a + \Delta u$.
Notice that we consider $\Delta u << \Delta a$.

(1.18) $$\sin\alpha_1 = \alpha_1 = \frac{\Delta v}{\Delta a + \Delta u} \approx \frac{\Delta v}{\Delta a} \approx \frac{\partial v}{\partial x}$$

Likewise $\alpha_2$ can be calculated from the stretch $\Delta u$ in the $x$ – direction , divided by the length of the stretched 0'2' line, which is $\Delta b + \Delta v$

$$\sin\alpha_2 = \alpha_2 = \frac{\Delta u}{\Delta b + \Delta v} \approx \frac{\Delta u}{\Delta b} \approx \frac{\partial u}{\partial y}$$

We then find:

(1.19) $$\alpha_1 + \alpha_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \varepsilon_{xy}$$

It is of course possible to give a similar interpretation of the other components of the tensor $\varepsilon$
However the physical significance of the tensor elements is not so transparent.

We may write a linear form from the expression for $d\vec{s}_2$ by replacing $d\vec{s}_2$ with $\vec{s}_2$, $(du_2, dv_2, dw_2)$ with $(u_2, v_2, w_2)$ and $(dx, dy, dz)$ with $(x,y,z) = \vec{r}$

$$
\begin{aligned}
u_2 &= \varepsilon_{xx}x + \varepsilon_{xy}y + \varepsilon_{xz}z \\
v_2 &= \varepsilon_{yx}x + \varepsilon_{yy}y + \varepsilon_{yz}z \\
w_2 &= \varepsilon_{zx}x + \varepsilon_{zy}y + \varepsilon_{zz}z
\end{aligned} \tag{1.20}
$$

We then establish the bilinear form:

$$\vec{s}_2 \cdot \vec{r} = \varepsilon_{xx}x^2 + 2\varepsilon_{xy}xy + \varepsilon_{yy}y^2 + \dots$$

This may be an ellipsoid or a hyperboloid. We know, however, it has three principal axes, $(x_1, x_2, x_3)$ and by a coordinate transformation to these axes, it can be brought to the form:

$$f(x_1, x_2, x_3) = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 + \varepsilon_3 x_3^2 \tag{1.21}$$

The coefficients $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are called the principal extensions or contractions.

A small relative compression of the volume can in the system of the principal axes be written as:

$$\frac{\Delta V}{V} = (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - 1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \quad \text{(omitting higher order terms)}$$

Since $\varepsilon$ is considered to be a tensor, we know that the trace of the tensor is an invariant under coordinate transformations, so

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = Tr(\varepsilon) = \varepsilon_{xx} + \varepsilon_{xy} + \varepsilon_{zz} = \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial w}$$

If the relative compression is denoted $\Phi$, then we have already foreseen that

$$\Phi = \varepsilon_{xx} + \varepsilon_{xy} + \varepsilon_{zz} = \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial z} \tag{1.21}$$

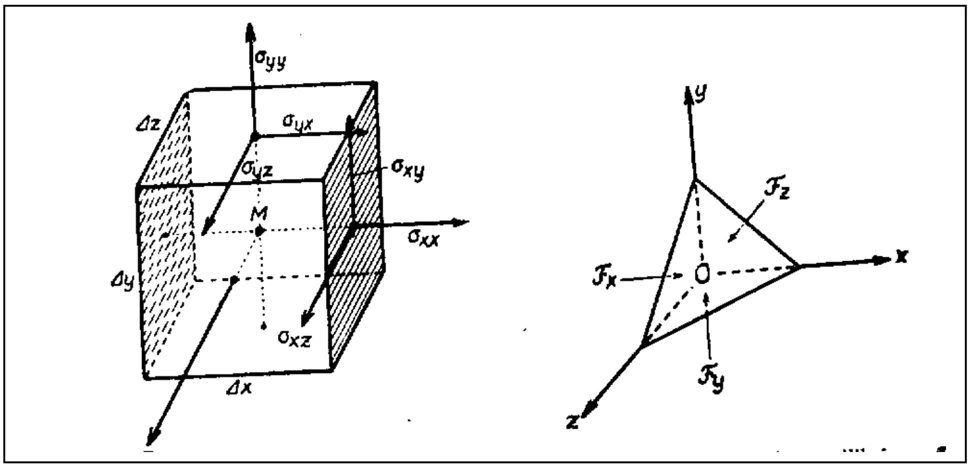
represents the cubic compression or dilatation

## 2. Dynamics of deformable bodies

We shall now turn to the concept of *stress*, which is defined as the force per unit area. There must be as many components of stress that there are of strain, since the two physical quantities are related by "Hookes law". For a small cube, the stresses on opposite sides are the same. There are three stresses on each of the three sides.
The stresses acting in the $y$ –$z$ plane are denoted $\sigma_{xx}\sigma_{xy}\sigma_{yx}$ and similar for the two other sides.
Since the stresses are components of a vector, they will form a tensor.

(2.1)
$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

The meaning and directions of the stresses are shown in the figure below.




The small volume has the sides $\Delta x, \Delta y, \Delta z$.
We must verify that the stress tensor is a symmetric tensor, so we setup an expression for the moments around the $z$-axis caused by the stresses $\sigma_{xy}$ , $\sigma_{yx}$

(2.2)
$$M_{xy} = \tfrac{1}{2}\Delta x \cdot F_{xy} = \tfrac{1}{2}\Delta x \cdot \sigma_{xy}\Delta y \Delta z = \tfrac{1}{2}\Delta x \Delta y \Delta z \sigma_{xy} = \tfrac{1}{2}\Delta V \sigma_{xy}$$
$$M_{yx} = \tfrac{1}{2}\Delta y \cdot F_{yx} = \tfrac{1}{2}\Delta y \cdot \sigma_{yx}\Delta x \Delta z = \tfrac{1}{2}\Delta x \Delta y \Delta z \sigma_{xy} = \tfrac{1}{2}\Delta V \sigma_{xy}$$

However in equilibrium we must have $M_{xy} = M_{yx}$ and therefore also $\sigma_{xy} = \sigma_{yx}$

## 2.1 Strain-stress relations. Elastic constants

In the following we shall assume that the forces on the material in question are so small that we can apply a linear relation (Hookes law) between stress and strain. We further assume (trivially) that the elastic body is isotropic, such that its physical properties are the same in all directions.
We shall then analyze the material with the principal axes of the stress ellipsoid.
A volume element cut with sides parallel to the three principal axes of the stress ellipsoid makes a convenient starting point of our analysis.
The three pair of faces are subjected to the three principal stresses $\sigma_1, \sigma_2, \sigma_3$, while in the principal axes system there are no shear stresses, so $\sigma_{ij} = 0 \ \ for\ i \neq j$.
Then there are no angular changes and for that reason the principal axes of the strain tensor must coincide with the principal axes of the stress tensor.
Due to the assumed linearity of strain and stress, we can write

(2.3)
$$\sigma_1 = a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3$$

Since no axis is preferred, then two similar relations can be obtained simply by rotating the indices.

$$\sigma_2 = a\varepsilon_2 + b\varepsilon_3 + c\varepsilon_1 \qquad \sigma_3 = a\varepsilon_3 + b\varepsilon_1 + c\varepsilon_2$$

For symmetry reasons (in an isotropic body): $b$ must be equal to $c$. It is convenient to rewrite the expressions by adding and subtracting $b\varepsilon_1$.

(2.4) $$\sigma_1 = (a-b)\varepsilon_1 + b(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$$

With a change in notation

(2.5) $$\sigma_1 = 2\mu\varepsilon_1 + \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$$

Since there is symmetry in the three principal axes, we have two similar expressions.
And generally:

(2.6) $$\sigma_i = 2\mu\varepsilon_i + \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \ \ ; \ i = 1,2,3$$

To find the strain stress relation in the general case, we must use the transformation $\underline{\underline{T}}$ matrix from the principal axes system to an arbitrary system. Since $\underline{\underline{T}}$ is a transformation from two orthogonal systems, it is a unitary and symmetric matrix, which means:

$$\underline{\underline{T}}\,\underline{\underline{T}}' = \underline{\underline{E}}$$

(2.7) $$\underline{\underline{T}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{where} \quad a_{12} = a_{21}, a_{13} = a_{31}, a_{23} = a_{32}$$

It then follows from the unitary of the transformation

$$\sum_{j=1}^{3} a_{ij} a_{jk} = \delta_{ik}$$

We shall use (*x,y,z*) synonymous with (1,2,3). The rule for transformation of a tensor is. e.g.

(2.8) $$\varepsilon'_{ij} = \sum_k a_{ik} a_{lj} \varepsilon_{kl}$$

The transformation matrix is the same for the strain and the stresses, since the principal axes are the same for the strain and the stresses.
Using the principal axes: $\varepsilon_{ij} = \varepsilon_i \delta_{ij}$ and $\sigma_{ij} = \sigma_i \delta_{ij}$
For the transformed strain and stresses from the principal system to an arbitrary system, we can put $k = l$ since $\varepsilon_{ij} = \varepsilon_i \delta_{ij}$, using the symmetry of the two tensors, such that.

(2.9) $$\varepsilon'_{ij} = \sum_k a_{ik} a_{kj} \varepsilon_k$$

(2.10)
$$\varepsilon_{xx} = \varepsilon'_{11} == \sum_k a_{1k} a_{1k} \varepsilon_k = a^2_{11}\varepsilon_1 + a^2_{12}\varepsilon_2 + a^2_{13}\varepsilon_3$$
$$\varepsilon_{yy} = \varepsilon'_{22} == \sum_k a_{2k} a_{2k} \varepsilon_k = a^2_{21}\varepsilon_1 + a^2_{22}\varepsilon_2 + a^2_{23}\varepsilon_3$$
$$\varepsilon_{zz} = \varepsilon'_{33} == \sum_k a_{3k} a_{3k} \varepsilon_k = a^2_{31}\varepsilon_1 + a^2_{32}\varepsilon_2 + a^2_{33}\varepsilon_3$$

And correspondingly:

(2.11) $$\sigma_{xx} = \sigma'_{11} = \sum_k a_{1k} a_{1k} \sigma_k = a^2_{11}\sigma_1 + a^2_{12}\sigma_2 + a^2_{13}\sigma_3$$

(2.12) $$\sigma'_{ij} = \sum_k a_{ik} a_{kj} \sigma_k$$

Using: $\sigma_i = 2\mu\varepsilon_i + \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ ; $i = 1,2,3$ , we find for $i = x = 1$,when inserting in (2.12).

(2.11) $$\sigma_{xx} = \sigma'_{11} = \sum_k a_{1k} a_{1k} (2\mu\varepsilon_k + \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3))$$

Inserting the expression $\varepsilon_{xx} = \sum_k a_{1k} a_{1k} \varepsilon_k = a^2_{11}\varepsilon_1 + a^2_{12}\varepsilon_2 + a^2_{13}\varepsilon_3$ , and $\Theta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$

and noticing that $\sum_{j=1}^{3} a_{ij} a_{jk} = \delta_{ik}$ we find:

(2.12)
$$\sigma_{xx} = 2\mu\varepsilon_{xx} + \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})$$
$$\sigma_{yy} = 2\mu\varepsilon_{yy} + \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})$$
$$\sigma_{zz} = 2\mu\varepsilon_{zz} + \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})$$

For $\sigma_{ij}$ where *i* and *j* are different, we take the same starting point as before. Inserting the expression: $\sigma_k = 2\mu\varepsilon_k + \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ in $\sigma'_{ij} = \sum_k a_{ik} a_{kj} \sigma_k$ gives:

(2.12)
$$\sigma'_{ij} = \sum_k a_{ik} a_{kj} (2\mu\varepsilon_k + \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3))$$
$$\sigma'_{ij} = 2\mu \sum_k a_{ik} a_{kj} \varepsilon_k + \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \sum_k a_{ik} a_{kj}$$

And replacing $\sum_k a_{ik} a_{kj} \varepsilon_k = \varepsilon'_{ij}$ we get:

(2.13) $$\sigma'_{ij} = 2\mu\varepsilon'_{ij} \quad or \quad \sigma_{xy} = 2\mu\varepsilon_{xy}$$

In condensed writing:

(2.14) $$\sigma'_{ik} = 2\mu\varepsilon'_{ik} + \lambda\delta_{ik}\Theta$$

The inverse relations are found by summing the first three relations:

$$\Sigma = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = (2\mu + 3\lambda)(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) = (2\mu + 3\lambda)\Theta$$

$$\Theta = \frac{\Sigma}{(2\mu + 3\lambda)}$$

This permits one to express the first scalar $\Theta$ by the first scalar $\Sigma$. On introducing $\Sigma$ in (2.14) $\sigma'_{ik} = 2\mu\varepsilon'_{ik} + \lambda\delta_{ik}\Theta$ we have:

$$\sigma'_{ik} = 2\mu\varepsilon'_{ik} + \lambda\delta_{ik}\frac{\Sigma}{(2\mu + 3\lambda)}$$

and solving for $\varepsilon'_{ik}$

(2.15) $$\varepsilon'_{ik} = \frac{1}{2\mu}(\sigma'_{ik} - \delta_{ik}\frac{\lambda}{(2\mu + 3\lambda)}\Sigma)$$ and since $i, j, k = x, y, z$

(2.16) $$\varepsilon_{xx} = \frac{1}{2\mu}(\sigma_{xx} - \frac{\lambda}{(2\mu + 3\lambda)}\Sigma) = 2\mu'\sigma_{xx} + \lambda'\Sigma$$

$$\varepsilon_{yy} = \frac{1}{2\mu}(\sigma_{yy} - \frac{\lambda}{(2\mu + 3\lambda)}\Sigma) = 2\mu'\sigma_{yy} + \lambda'\Sigma$$

$$\varepsilon_{zz} = \frac{1}{2\mu}(\sigma_{zz} - \frac{\lambda}{(2\mu + 3\lambda)}\Sigma) = 2\mu'\sigma_{zz} + \lambda'\Sigma$$

or in condensed writing:

(2.17) $$\varepsilon_{ij} = 2\mu'\sigma_{ij} + \lambda'\delta_{ij}\Sigma$$

These are the general stress-strain relations expressed by the constants $\mu$ *and* $\lambda$, which is known as Lamé's moduli.

(2.18) $$2\mu' = \frac{1}{2\mu}, \quad \lambda' = -\frac{1}{2\mu}\frac{\lambda}{2\mu + 3\lambda}$$

The quantities $\mu$ *and* $\lambda$ shall now be expressed by two other constants that refer to the simplest type of compression or tension experiment.
A vertical prismatic bar with cross section *A*, the upper end which is rigidly fixed, is subjected to a load *F*, which acts at the lower end. Let the load be uniformly distributed over the cross section *A*. The tension across any section is

(2.19) $$\sigma = \frac{F}{A}$$

The stress $\sigma$ is the principal stress, since there are no side forces, and therefore no shear stresses. The associated principal extensions (strains) $\varepsilon$ equals $\Delta l / l$ where *l* is the length of the bar, and $\Delta l$ Is the displacement of the end cross section. It turns out that the ratio between the stress and the extension is a material constant, that is, independent of *F, A* and *l* as long as the loading is not excessive. The ratio is denoted by *E* and is known as *Young's module.*

(2.20) $$E = \frac{\sigma}{\varepsilon}$$

$E$ has the same dimension [force/area] as the stress $\sigma$, since $\varepsilon = \frac{\Delta l}{l}$ is dimensionless.

Tension, that is, elongation or contracting is, however, not the only response to a force acting on a bar. Actually a tension or contraction is always accompanied by a contraction/extension of the cross section of the bar. It is the same contraction, which on a much larger scale when a rubber band is stretched, and we take it as a natural model of a material as a respond to the loading.
If we denote the contraction, which is the same for all fibre of a homogenous material by $-\delta$, in this case a positive number, we define generally.

$$-\frac{\delta}{\varepsilon} = \nu \qquad \text{or using} \qquad E = \frac{\sigma}{\varepsilon} \qquad \text{to give}$$

(2.21) $$\delta = -\frac{\nu}{E}\sigma$$

The quantity $\nu$ is also a constant of the material. It is called Poisson's ratio of transverse contraction or longitudinal extension.
Poisson did not only introduce this number, but actually experimentally determined its ratio to 0.25. Attempts to improve this value, has revealed other values, but up till now there exits no universal theoretical value.
For engineering applications the general accepted value for iron is $\nu = 1/m = 0.3$.
When the bar is under compression instead of tension, the longitudinal contraction is accompanied by transverse dilatation, But $E$ and $\nu$ remains the same in spite of the change of sign that have occurred, provided of course that the load is in the region where Hooke's law apply.
We now apply, what we have learned in the simple tension example to analyze the general state of stress, which we consider as the superposition of three unidirectional stresses in the principal directions, where the effect of each stress is the same as in the simple bar experiment.
Because of the linearity of the elastic stress relations we may superimpose the stresses as well as the associated strains. However the extension along the principal axis (1) is not entirely determined by the stress $\sigma_1$, but in addition by the stresses in the two other principal directions. (2) and (3).

Applying this, we obtain from (2.20) $E = \frac{\sigma}{\varepsilon}$ and (2.21) $\delta = -\frac{\nu}{E}\sigma$

(2.22) $$\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\nu}{E}(\sigma_2 + \sigma_3) = \frac{1+\nu}{E}\sigma_1 - \frac{\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)$$ or in general

(2.23) $$\varepsilon_i = \frac{1+\nu}{E}\sigma_i - \frac{\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)$$

Summing the relation for $i = 1,2,3$ we find:

(2.24) $$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3) \qquad \text{or} \qquad \Theta = \frac{1-2\nu}{E}\Sigma$$

And by substituting $\Sigma$ in (2.23), we find the result:

(2.25) $$\sigma_i = \frac{E}{1+\nu}\left(\varepsilon_i + \frac{\nu}{1-2\nu}\Theta\right)$$

The fundamental equation, which we wish to find, is found from a comparison between (2.25) and (2.6) $\sigma_i = 2\mu\varepsilon_i + \lambda(\varepsilon_1+\varepsilon_2+\varepsilon_3)$ ; $i=1,2,3$. This leads to:

(2.26) $$2\mu = \frac{E}{1+\nu} \text{ and } \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)},$$

Which expresses Lamé's parameters by $E$ and $\nu$, whose physical significance is, however, not immediate.
Let us now take up another particular simple case of a stress field, namely equal stresses in all directions, as it is realized inside a cylinder with a hydraulic press.
The strain an stress ellipsoids are then both spheres, so

$$\sigma_1 = \sigma_2 = \sigma_3 = -p \qquad \sigma_1+\sigma_2+\sigma_3 = \Sigma = -3p \qquad \varepsilon_1+\varepsilon_2 = -\varepsilon_3 - \tfrac{1}{3}|\Theta|$$

Equation (2.25) $\sigma_i = \frac{E}{1+\nu}\left(\varepsilon_i + \frac{\nu}{1-2\nu}\Theta\right)$ gives in this case:

(2.27) $$p = \frac{E\Theta}{3(1-2\nu)}$$

We define now the modulus of compression $K$ by the equation.

(2.28) $$K = \frac{p}{|\Theta|},$$

And from (2.27) and (2,28), we infer:

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(2.29) $$K = \frac{1}{3}\frac{E}{1-2\nu}$$

The incompressible case $\Theta = 0$ corresponds to $K=\infty$ or $\nu = \frac{1}{2}$. This value represents the upper limit for the possible values of Poisson ratio.
For $\nu > \frac{1}{2}$ the behaviour of the body becomes instable. The response to an external pressure will be an increase of the volume instead of a decrease.
We shall then look at another particular stress phenomenon, namely a pure shear stress.

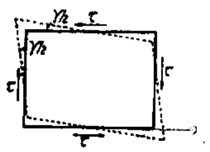


FIG. 12a. Change of shape in pure shear loading.

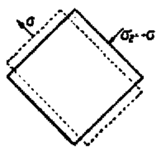


FIG. 12b. The principal stresses in pure shear loading are equal and opposite.

If we take a rectangular parallelepiped, where the $z$ – surface(in the plane of the paper) is free from stress, while the shearing stress $\sigma_{xy} = \sigma_{yx} = \tau$ acts on the $x$ and $y$-surface.
We assume that the stresses are independent of $z$, and they can simply be represented by drawing a cross perpendicular to the $z$-axis as in fig 12$a$. The deformation due to the stresses changes the rectangular into a rhomboid, one pair of right angles being diminished and, the other pair being increased by angle $\gamma$. The stress quadric, is like the strain quadric in pure shear, a cylinder parallel to the $z$ –axis, the basis of which is an equilateral hyperbola. The principal axes form an angle of $45^{0}$ with the $x$ and $y$-axis, and the two principal stresses are numerically equal with opposite sign. $\sigma_2 = -\sigma_1$, whereas $\sigma_3$ is zero. This is a consequence of the invariance of the first scalar of the stress tensor $\Sigma$ for the element considered originally (Fig. 12$a$) $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$ by hypothesis, hence an element out parallel to the principal axes must take tension and compression in equal amounts to make $\Sigma = \sigma_1 + \sigma_2 = 0$.
The state of pure stress serves to define the *shear modulus*, which we provisionally denote by $G$, a notation which is generally accepted in engineering practice. So we set

(2.30) $$G = \frac{\tau}{\gamma}$$

Where $\tau$ is the shear stress and $\gamma$ is the angular change.
To determine $G$, that is, to express it by the elastic constants, already introduced, we return to the general relation $\sigma_{xy} = 2\mu\varepsilon_{xy}$, which reads in our present notation $\tau = \gamma\mu$. Thus we have

(2.31) $$G = \mu = \frac{E}{2(1+\nu)}$$

The shear modulus $G$ is thus identical with Lamé's modulus $\mu$, so the notation $G$ is no longer needed.
We can then determine the stress from the equilibrium condition $Div\sigma + \vec{F} = 0$.
However the three equilibrium conditions are insufficient for a complete determination of the six stress components $\sigma_{xy}$. If we express the stresses $\sigma_{xy}$ by the tensions $\varepsilon_{xy}$ and write for the $\varepsilon_{xy}$, the original definitions in terms of the displacements, we obtain three equations, which is sufficient to determine the displacement vector. Cartesian coordinates are used. The $x$-component of the vector divergence transforms, according to the stress-strain relation into.

$$Div_x\sigma = 2\mu\left(\frac{\partial\varepsilon_{xx}}{\partial x} + \frac{\partial\varepsilon_{yx}}{\partial y} + \frac{\partial\varepsilon_{zx}}{\partial z}\right) + \lambda\frac{\partial\Theta}{\partial x}$$

Which becomes when we reinstall the expressions for the displacements (1.15).

$$Div_x\sigma = \mu\left(2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x\partial z}\right) + \lambda\frac{\partial\Theta}{\partial x}$$

This can further be rearranged into

$$Div_x\sigma = \mu\nabla^2 u + (\lambda + \mu)\frac{\partial\Theta}{\partial x}$$

By rotating the letters using the equilibrium condition, $\left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z}\right) + F_x = 0$, we finally have.

(2.32)
$$\mu\nabla^2 u + (\lambda + \mu)\frac{\partial\Theta}{\partial x} + F_x = 0$$
$$\mu\nabla^2 v + (\lambda + \mu)\frac{\partial\Theta}{\partial y} + F_y = 0$$
$$\mu\nabla^2 w + (\lambda + \mu)\frac{\partial\Theta}{\partial z} + F_z = 0$$

These three fundamental equations determine the displacement vector (*u, v, w*) everywhere in the interior of a body, if the behaviour along the body surface is determined by appropriate boundary conditions.
However, there is no general solution to these equation, and generally they cannot be solved, without resorting to symmetry of the problem and rewriting the equations in curve-linear coordinates. See e.g. http://olewitthansen.dk/Mathematics/Partial_differential_equations.pdf

Reference: Arnold Sommerfeld: Mechanics of deformable bodies