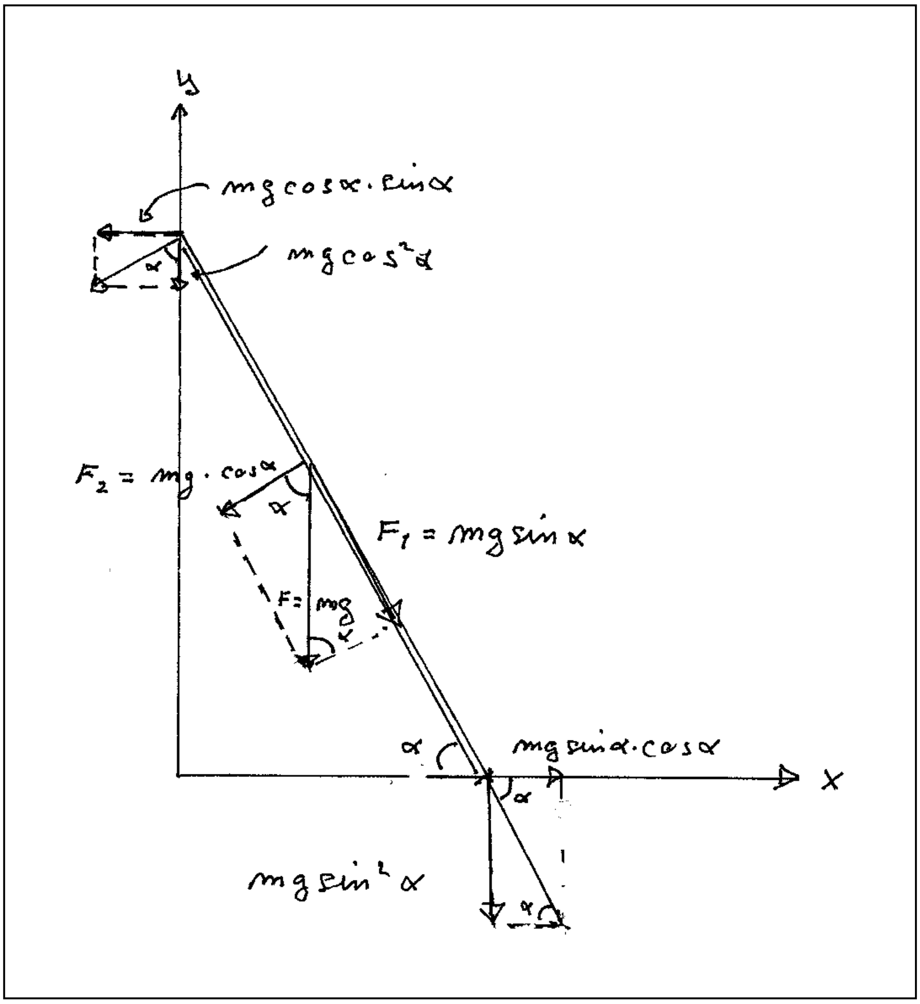
The practical application of a theoretical explanation on which angle a ladder placed up at wall will skid can easily be overlooked, but the example is rather chosen to demonstrate some analytic techniques in classical mechanics, that might be forgotten and replaced by computer programs. This also includes the "principle of virtual forces", which was earlier used in mechanical problems related to statics.

Firstly we shall conduct a classical analysis to the problem, by resolving the components of the acting forces. Subsequent we shall demonstrate that we get the same result by applying the principle of virtual forces, which belong to the analytical mechanics, formulated by the Lagrange formalism.



1. Classical mechanical analysis

The figure shows a ladder leaning to a wall. Under some general assumptions, we shall try to find the angle α with the *x*-direction, where the ladder starts to skid. This is of course dependent on the friction coefficient μ between the top and the foot of the ladder and the underlay.

Static dynamics is sometimes more challenging than dynamic ones, because the resulting force is zero in every point.
We have only drawn the ladder, but we shall later show that it makes no difference, whether there is a person climbing the ladder or not.

We resolve gravity from the centre of mass of the ladder in one component F_1 along the ladder and one component F_2 perpendicular to the ladder. From the figure we can see:

$$F_1 = mg\sin\alpha \qquad og \quad F_2 = mg\cos\alpha$$

We then place F_1 at the foot of the ladder and F_2 at the top of the ladder.

Both F_1 and F_2 are then resolved after the directions of x and y.
This gives, as it appear from the figure:

$$F_{1x} = mg\sin\alpha\cos\alpha \quad F_{1y} = mg\sin^2\alpha \quad and \quad F_{2x} = mg\sin\alpha\cos\alpha \qquad F_{2y} = mg\cos^2\alpha$$

The ladder is affected by the two forces F_{1x} and F_{2y}, which make the ladder skid, and the two opposite directed friction forces, which come from the two normal forces F_{1y} and F_{2x}.

$$F_x = mg\cos\alpha\sin\alpha + mg\cos^2\alpha \quad\Leftrightarrow\quad F_x = mg\cos\alpha(\sin\alpha+\cos\alpha)$$

$$F_{fric} = \mu mg\sin^2\alpha + \mu mg\cos\alpha\sin\alpha \quad\Leftrightarrow\quad F_{fric} = \mu mg\sin\alpha(\sin\alpha+\cos\alpha)$$

The ladder will not skid as long as,

$$F_x < F_{fric}. \quad\Leftrightarrow\quad mg\cos\alpha(\sin\alpha+\cos\alpha) < \mu mg\sin\alpha(\sin\alpha+\cos\alpha)$$

Which can be reduced to:

$$\tan\alpha > \frac{1}{\mu}$$

A surprising simple result that confirms our expectations, at least for a very big and a very small friction coefficient. If the friction coefficient for example is 0.75, then the critical angle is 53.1^0.

The coefficients at the top of the ladder, and at the bottom may be different, but that changes only the expression for the frictional force, but it does not alter the result fundamentally.

$$F_{fric} = \mu_1 mg\sin^2\alpha + \mu_2 mg\cos\alpha\sin\alpha \quad\Leftrightarrow\quad F_{fric} = mg\sin\alpha(\mu_1\sin\alpha+\mu_2\cos\alpha)$$

F_x is the same as before, so we have the same condition that the ladder does not skid if:

$$F_x < F_{gn} \quad\Leftrightarrow\quad mg\cos\alpha(\sin\alpha+\cos\alpha) < mg\sin\alpha(\mu_1\sin\alpha+\mu_2\cos\alpha)$$

$$\cos\alpha(\sin\alpha+\cos\alpha) < \sin\alpha(\mu_1\sin\alpha+\mu_2\cos\alpha)$$

By division with $\cos^2\alpha$ we find:

$$\tan\alpha + 1 < \tan\alpha(\mu_1\tan\alpha+\mu_2)$$

And for the corresponding quadratic equation in tan α.

$$\tan\alpha + 1 = \tan\alpha(\mu_1 \tan\alpha + \mu_2) \quad \Leftrightarrow \quad \mu_1 \tan^2\alpha + (\mu_2 - 1)\tan\alpha - 1 = 0$$

The discriminator for the equation is: $d = (\mu_2 - 1)^2 + 4\mu_1$. The solution is

$$\tan\alpha = \frac{-(\mu_2 - 1) \pm \sqrt{(\mu_2 - 1)^2 + 4\mu_1}}{2\mu_1}$$

If we put $\mu_2 = \mu_1 =$ then $d = (\mu + 1)^2$ and we find

$$\tan\alpha = \frac{1 - \mu + 1 + \mu}{2\mu} = \frac{1}{\mu}$$, as we should have.

2. The pinciple of virtual forces

If U denotes the potential energy of a body, then we have, as it is well known that the force on the body is equal to minus the gradient of U.

$$\vec{F} = -\nabla U \qquad \vec{F} = -(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z})$$

In the analytical mechanics, where the poential energy is expressed in generalized coordinates, One may determine the force in the direction along r as:

$$F_r = -\frac{\partial U}{\partial r}$$

If we have an expression for the potential energy, we may take advantage of this equation as an alternative to resolve the forces in components.

We then make an addition to the problem by placing a person the distance d up the ladder, which itself has the length $2L$. We shall use the same drawing as shown above. The ladder has its foot at $(x,0)$ and the top of the ladder is at $(0,y)$.
We then have: $x = 2L\cos\alpha \;\; and \;\; y = 2L\sin\alpha$. The height h, where the person is lifted above the ground is $h = d\sin\alpha$. The person has the mass M, and the ladder has the mass m.

We then set the expression for the potential energy from the formula mgh.

$$U = mg\,L\sin\alpha + Mgd\sin\alpha$$

From this expression we are able to determine the force in the x-direction, (which makes the ladder skid), and the force in the y-direction, which caused by the frictional force tend to impede the skidding of the ladder. We shall apply the following rewriting:

$$\frac{\partial U}{\partial x}=\frac{\partial U}{\partial \alpha}\frac{\partial \alpha}{\partial x} \quad \Rightarrow \frac{\partial U}{\partial x}=\frac{\partial U}{\partial \alpha}(\frac{\partial x}{\partial \alpha})^{-1}$$

And likewise for y.

$$\frac{\partial U}{\partial y}=\frac{\partial U}{\partial \alpha}\frac{\partial \alpha}{\partial y} \quad \Rightarrow \quad \frac{\partial U}{\partial y}=\frac{\partial U}{\partial \alpha}(\frac{\partial y}{\partial \alpha})^{-1}$$

Since: $\frac{\partial x}{\partial \alpha}=-2L\sin\alpha \quad and \quad \frac{\partial y}{\partial \alpha}=2L\cos\alpha$, we then find directly:

$$F_x=-\frac{\partial U}{\partial x}=-\frac{\partial U}{\partial \alpha}(\frac{\partial x}{\partial \alpha})^{-1}=\frac{mgL\cos\alpha+Mgd\cos\alpha}{2L\sin\alpha}=\frac{mgL+Mgd}{2\tan\alpha}$$

$$F_y=-\frac{\partial U}{\partial y}=\frac{\partial U}{\partial \alpha}(\frac{\partial y}{\partial \alpha})^{-1}=-\frac{mgL\cos\alpha+Mgd\cos\alpha}{2L\cos\alpha}=-\frac{mgL+Mgd}{2L}$$

The condition that the ladder does not overturn is: $F_x < \mu F_y$, which results in the inequality:

$$\frac{mgL+Mgd}{2\tan\alpha}<\mu\frac{mgL+Mgd}{2L} \quad \Leftrightarrow \quad \tan\alpha>\frac{1}{\mu}$$

We can see that we get exactly the same condition, as we did doing a classical analysis.
But the result is achieved, perhaps in a more elegant, but also less transparent manner.
We can also conclude that it makes no difference (for the physics) whether there is person on the ladder or not, when it skids.