Damped periodic motion

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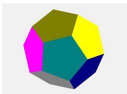

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1. Introduction to the issue

The students in the Danish 9-12 grade high school must hand in a project during the last semester in school. The subject of the project is mainly chosen by the students themselves.
This may result in exotic subjects of which nor the teacher and neither has the student have no knowledge or insight whatsoever. Search in the Internet gives very seldom a tangible theoretical analysis, but since the teacher in the end has the responsibility of formulating the project, this situation often results in that the teacher must himself provide the necessary notes, that the student then can copy more or less.
The articles below are example of the present rather awkward state of the educational system in Denmark.
Some projects are chosen to be concerned with viscous motion in liquids or motion with air drag, but nowadays the theoretical aspect in the Danish high school is almost absent, apart perhaps from establishing the equations of motion (but not solving them by hand).

Instead the projects are focused on making videos with high speed cameras followed by analyzing the videos with the specialized computer programs that have appeared during the last fifteen years.

With the present (very low) theoretical level in mathematics and physics, presenting an analytical solution is not realistic, but nevertheless it may be interesting to make such an analysis, which may be compared to the computer graphs, coming from the high speed cameras.

Searching the Internet, I have not found any theoretical treatment to this kind of problems, and therefore I had (as many times before in similar situations) to make my own analysis, which is, however, not founded on scientific computer programs, but rather on my studies on mathematics and physics at the university of Copenhagen in the sixties.

2. The bouncing ball

When I, for some time ago searched the internet for something different, I encountered accidentally a project made by third year students in the Danish high school.
The headline of the project was “the bouncing ball”
The experiments were naturally recorded motions with a high speed camera, and subsequently analyzed, and graphs produced in advanced computer programs.

It is almost 20 years ago, where simple instructive experiment, (performed to the purpose designed appliences), were replaced with video takes, data sampling and computer programs.

Until about 1990, I made a class experiment to verify Gallilei’s laws of the free fall. Actually it was the first class experiment in the first year of the three year Danish high school.
(After having reviewed the elementary kinematics of course).

In this experiment the students measured the time t for a free fall of a steel ball (with a mechanical Leybold watch) falling different distances s, and only very few students were not able to grasp the purpose of this simple experimental setup.

As a part of the analyzis, the measured points were plotted on *mm*-paper with t^2 on the first axis, and s on the second axis. Visually you could conclude that the points approximately lied on a straight line through (0,0), from which it was concluded that $s = k \cdot t^2$.

The line was drawn, and the slope was calculated According to the laws of Galilei the slope should be ½*g*, which usually was in accordance with experiment, within 5%.

And the purpose of the experiment was to verify the laws of Galilei, not to falsify the laws, that since 2005, has been the pedagogic nonsens guidelines inspired by the German philosopher Karl Popper.

Concerning the fundamental elementary understanding of what physics is about, I find it immensely more instructive to let the students do the measurements directly, and not electronically, and do the analyzing by hand, rather than just feeding the measurements into a computer.

But in this respect I am aware that I belong to the late medieval conception of teaching physics, where the students actually were taught physics and mathematics on their own premises, and not as an accompanying phenomenon in the society ideological upbringing.

Because it was an obligatory part of the curriculum, I actually made a video recording of an oblique throw in 2010, but the project drowned totally in technical problems with the recording, and the subsequent application of the data analyzing computer program, so I estimated the professional benefit of the students to be close to zero.

So it appears that even the most elementary concepts of theoretical physics has been replaced by experiments performed by electronic devices and the results fed into a computer program.

This is at least what I thought, when I read the student generated report on the bouncing ball. There were a lot of computer generated graphs, and computer generated calculation about the loss of energy, but no theoretical argument on how the height reached by the ball depended on the number of impacts with the ground.

In fact it is straightforward to make the assumption that the kinetic energy of the ball: $E_{kin} = \frac{1}{2}mv^2$ (and thus v^2) decreases by a constant factor α after each bounce with the underlay.

If the ball is released from a height h_0, then the velocity v_0, with which it hits the ground given by

(2.1) $$v_0^2 = 2gh_0 \quad \Leftrightarrow \quad h_0 = \frac{v_0^2}{2g}$$

If the energy decreases with a factor α, then the height the ball reaches becomes:

(2.2) $$h_1 = \alpha \frac{v_0^2}{2g} = \alpha h_0$$

And subsequently:

(2.3) $$h_n = \alpha^n \frac{v_0^2}{2g} = \alpha^n h_0$$

Thus the height the ball reaches decreases exponentially with the number of times it hits the ground.

I have tried to count the number of times a massive jumping ball hits the ground, when released from a height of 1 meter, and it seems to be 13. Assuming that this corresponds to a height 0.01 h_0, then we have $\alpha^{13} = 0.01$, which gives α =0.7, so that 30% of the energy is lost by each rebound.

Whether the premises are correct, I do not know, but it could have been interesting (having the fine electronic equipment) to investigate the exponential decrease of the height, and determine the factor α.
Under the given assumptions, one may also calculate the time passed before the ball lies still.
If the ball is released from the height h_0, then the time for the fall is given by:

(2.4) $$h_0 = \tfrac{1}{2} g t_0^2 \quad \Rightarrow \quad t_0 = \sqrt{\frac{2h_0}{g}}$$

The next height is $h_1 = \alpha h_0$, and since the ball is going up and down from the same height, the time elapsed is:

(2.5) $$t_1 = 2\sqrt{\frac{2h_1}{g}} = 2\sqrt{\alpha}\, t_0$$

Accordingly, we get: $t_2 = 2(\sqrt{\alpha})^2 t_0$, $t_3 = 2(\sqrt{\alpha})^3 t_0$,.....
The times make a geometric series with the first term $2\sqrt{\alpha}\, t_0$ and the quotient: $\sqrt{\alpha}$.
If we sum up to infinity, and apply the formula for an infinite geometric series:

$$S = a_0 \frac{1}{1-q} ,$$

we have:

(2.6) $$t_\infty = t_0 + 2\sqrt{\alpha}\, t_0 \frac{1}{1-\sqrt{\alpha}}$$

If the ball is released from $h_0 = 1\ m$, then $t_0 = 0.45\ s$, and then we find $t_\infty = 5.1$ s, which is in good accordance with what you can measure with a stopwatch.

3. Damped harmonic oscillations on a flat underlay

If a block is suspended between two springs on a plane underlay, it will perform harmonic oscillations, when it is released from a displacement from equilibrium. In general it will be influenced by a frictional force from the underlay.
If the mass of the block is m, the spring constant is k and the frictional coefficient is μ, then the frictional force is $F_{fric} = \mu mg$, directed against the motion.
We may then establish a differential equation for the motion.

(3.1) $$F_{res} = ma = -kx - signum(v)\mu mg$$

(3.2) $$m\frac{d^2x}{dt^2} = -kx - \frac{v}{|v|}\mu mg$$ where $\frac{v}{|v|}$ is the sign of the velocity.

The first term on the right side indicates the usual harmonic solution, whereas the second term is a bit more unpleasant because of the numeric sign. We must therefore separate the solution in the two cases $v > 0$ and $v < 0$.

(3.3) $$m\frac{d^2x}{dt^2} = -kx - \mu mg \quad \textit{for } v > 0 \qquad m\frac{d^2x}{dt^2} = -kx + \mu mg \quad \textit{for } v < 0$$

We settle by solving the equation for $v > 0$, idet det kun er et fortegn, som skiller de to løsninger. Efter division med m fås:

(3.4) $$\frac{d^2x}{dt^2} = -\frac{k}{m}x - \mu g \quad \Leftrightarrow$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}\left(x + \frac{\mu mg}{k}\right)$$

If we put: $$x_1 = x + \frac{\mu mg}{k} \quad \Rightarrow \quad \frac{d^2x_1}{dt^2} = \frac{d^2x}{dt^2}$$

then

$$\frac{d^2x_1}{dt^2} = -\frac{k}{m}x_1$$

Which has the solution:

(3.5) $$x_1 = A\cos\left(\sqrt{\frac{k}{m}}\,t + \varphi_0\right)$$

If we insert $x = x_1 - \frac{\mu mg}{k}$, we get:

(3.6) $$x = A\cos\left(\sqrt{\frac{k}{m}}\,t + \varphi_0\right) - \frac{\mu mg}{k}$$

For $v < 0$ the solution becomes:

(3.7) $$x = A\cos\left(\sqrt{\frac{k}{m}}\,t + \varphi_0\right) + \frac{\mu mg}{k}$$

At a glance it may look strange with a constant term in a harmonic motion, but we should recall that the initial conditions (the amplitude) changes for each half period.

If we put $\varphi_0 = 0$, and initiate the motion at $x = A$, where $t = 0$, then we denote the amplitude in the first half period with A_0.

Since the motion for is directed against the equilibrium position, we apply the solution where $v < 0$. Then we have:

(3.8) $$A_0\cos(0) + \frac{\mu mg}{k} = A \quad \Rightarrow \quad A_0 = A - \frac{\mu mg}{k}$$

In the first half period, we then have:

$$(3.9) \qquad x = A_0 \cos\sqrt{\frac{k}{m}}\,t + \frac{\mu mg}{k} \qquad \Rightarrow \qquad x = (A - \frac{\mu mg}{k})\cos\sqrt{\frac{k}{m}}\,t + \frac{\mu mg}{k}$$

After half a period: $x = -A_1$. The Amplitude A_1 is given by *cos* π, and A_1 is then inserted in the solution for $v > 0$.

$$(3.10) \qquad x = -A_1 = \cos\pi \cdot (A - \frac{\mu mg}{k}) + \frac{\mu mg}{k} \qquad \Rightarrow A_1 = A - \frac{2\mu mg}{k}$$

And the solution for the next half period is:

$$(3.11) \qquad x = (A - \frac{2\mu mg}{k})\cos\sqrt{\frac{k}{m}}\,t - \frac{\mu mg}{k}$$

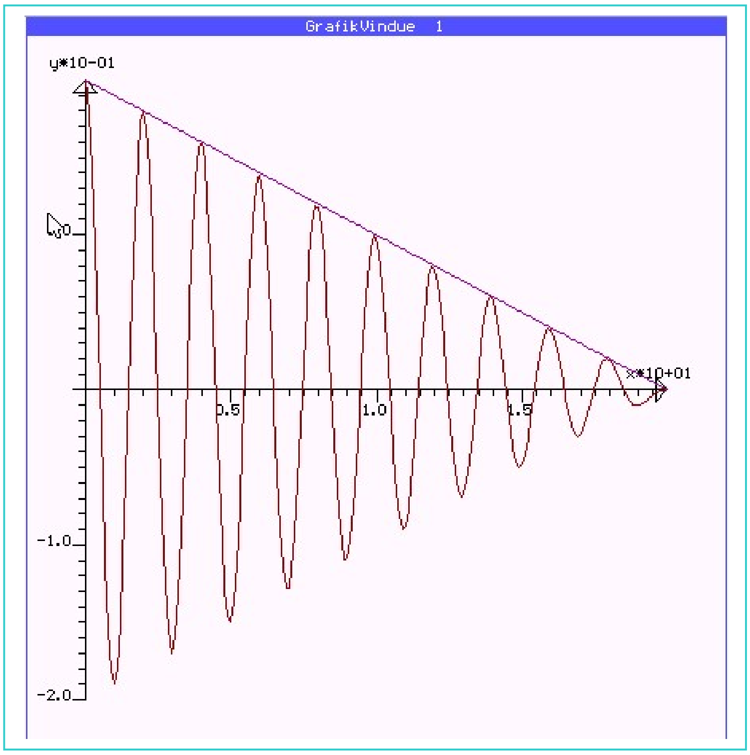
The expression for the n'th half period becomes:

$$(3.12) \qquad x = (A - n\frac{\mu mg}{k})\cos\sqrt{\frac{k}{m}}\,t \pm \frac{\mu mg}{k}$$

The oscillations will fade out when the amplitude approaches 0. According to the equations above it will happen, when:

$$(3.13) \qquad A - n\frac{\mu mg}{k} = 0 \qquad \Leftrightarrow \qquad n = \frac{Ak}{\mu mg}$$

It is no serious problem to draw the graphs for $n = 1, 2, 3\ldots$, but it is easier to solve the basic differential equation numerically, and let the mathematical program draw the solution.
Not surprisingly the amplitude decreases linearly according to the function: $y = A - \frac{2\mu mg}{k}x$, which also applies for the analytic solution. The computer generated graph is shown below.



4. Damped harmonic oscillations with damping proportional to *v*

The differential equation for the damped harmonic oscillations is probably one of the most well known equation of motion in physics. It appears in numerous connections e.g. in the electric oscillator.
Linear differential equations of degree *n* with constants coefficients can, as you know, be reduced to finding the complex roots in an *n*-degree polynomial equation by using the complex exponential function, whereas the traditional solution is somewhat more complex.

For the harmonic oscillator, we shall first assume, that the air drag is proportional to and opposite directed the velocity *v*, such that $F_{air} = -\alpha \cdot v$.

(4.1) $$F_{res} = -k \cdot x + F_{air} \quad \Leftrightarrow \quad m\frac{d^2x}{dt^2} = -\alpha\frac{dx}{dt} - kx \quad \Leftrightarrow$$

(4.2) $$\frac{d^2x}{dt^2} + \frac{\alpha}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

We shall rewrite it in a more general form.

(4.4) $$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c \cdot x = 0 \quad \text{, where } b = \frac{\alpha}{m} \quad and \quad c = \frac{k}{m}$$

When the equation (4.4) is solved in a traditional manner, the methods differ somewhat, but I shall apply a method, which is similar to the solution of a general 1. order linear differential equation.

We make our starting point in a 2. order differential equation, where we know the solution, namely the differential equation for the harmonic oscillator.

(4.5) $$m\frac{d^2y}{dt^2} = -ky \quad \Leftrightarrow \quad \frac{d^2y}{dt^2} + \frac{k}{m}y = 0\,,$$

We then introduce an auxiliary function $y = x\,e^{\beta \cdot t}$, where x is a solution the equation (4.4), and where y is a solution to (4.5). We put $\gamma = \frac{k}{m}$

(4.6) $$\frac{d^2y}{dt^2} + \gamma \cdot y = 0 \qquad \Leftrightarrow \qquad \frac{d^2(x \cdot e^{\beta \cdot t})}{dt^2} + \gamma \cdot x \cdot e^{\beta \cdot t} = 0$$

The purpose of this substitution is to transform (4.6) equation into the original equation (4.4) by a suitable choice of the constants β and γ . We evaluate therefore:

(4.7) $$\frac{d^2(x \cdot e^{\beta \cdot t})}{dt^2} + \gamma \cdot x \cdot e^{\beta \cdot t} = 0 \quad \Leftrightarrow$$

$$\frac{d}{dt}\left(\frac{dx}{dt} \cdot e^{\beta \cdot t} + x \cdot \beta \cdot e^{\beta \cdot t}\right) + \gamma \cdot x \cdot e^{\beta \cdot t} = \frac{d^2x}{dt^2} \cdot e^{\beta \cdot t} + \frac{dx}{dt} \cdot \beta \cdot e^{\beta \cdot t} + \frac{dx}{dt} \cdot \beta \cdot e^{\beta \cdot t} + x \cdot \beta^2 \cdot e^{\beta \cdot t} + \gamma \cdot x \cdot e^{\beta \cdot t}$$

$$\frac{d^2(x \cdot e^{\beta \cdot t})}{dt^2} + \gamma \cdot x \cdot e^{\beta \cdot t} = 0 \quad \Leftrightarrow \quad \frac{d^2x}{dt^2} \cdot e^{\beta \cdot t} + 2 \cdot \beta \cdot \frac{dx}{dt} \cdot e^{\beta \cdot t} + x \cdot \beta^2 \cdot e^{\beta \cdot t} + \gamma \cdot x \cdot e^{\beta \cdot t} = 0$$

(4.8) $$\frac{d^2(x \cdot e^{\beta \cdot t})}{dt^2} + \gamma \cdot x \cdot e^{\beta \cdot t} = 0 \qquad \Leftrightarrow$$

$$\frac{d^2x}{dt^2} \cdot e^{\beta \cdot t} + 2 \cdot \beta \cdot \frac{dx}{dt} \cdot e^{\beta \cdot t} + x \cdot \beta^2 \cdot e^{\beta \cdot t} + \gamma \cdot x \cdot e^{\beta \cdot t} = 0$$

The equation is reduced by division by $e^{\beta \cdot t}$.

(4.9) $$\frac{d^2x}{dt^2} + 2 \cdot \beta \cdot \frac{dx}{dt} + (\beta^2 + \gamma) \cdot x = 0$$

And this equation is then compared with the original equation (4.4).

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c \cdot x \cdot = 0$$

We can then see that the two equations are identical if and only if:

$$\beta = \frac{b}{2} = \frac{\alpha}{2m} \quad \text{and} \quad \beta^2 + \gamma = c \quad \Leftrightarrow \quad \gamma = c - \frac{b^2}{4} \quad \Rightarrow \quad \gamma = \frac{k}{m} - \frac{\alpha^2}{4m} .$$

The equation (4.6) can, however, be solved directly.
If we put $y = x \cdot e^{\beta \cdot t}$, then the equation takes the form:

(4.10) $$\frac{d^2 y}{dt^2} + \gamma \cdot y = 0 \quad \Leftrightarrow \quad \frac{d^2 y}{dt^2} = -\gamma \cdot y$$

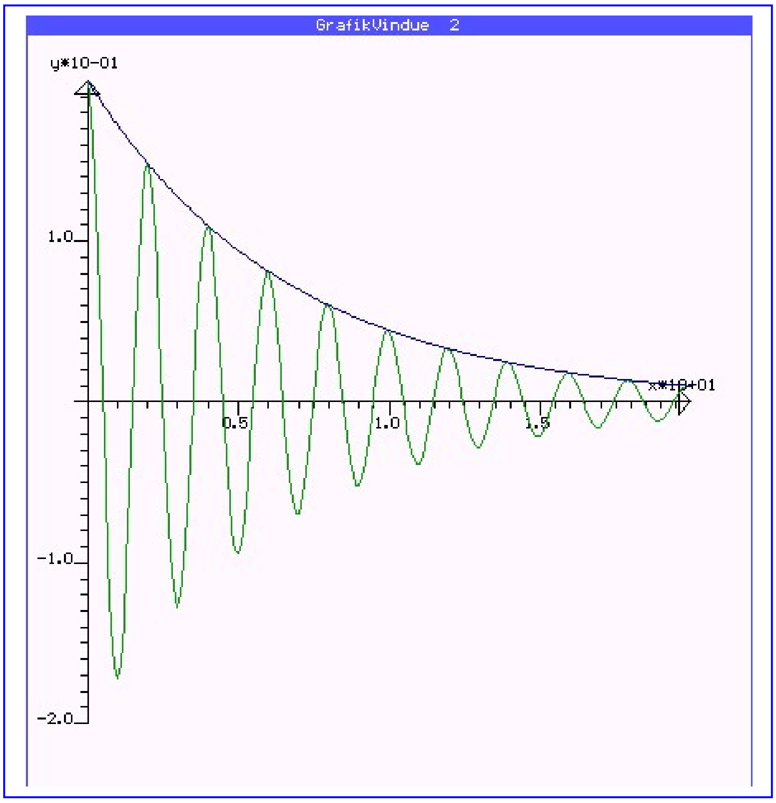
If $\gamma > 0$, then the differential equation (4.6) has the solution: $y = A\cos(\sqrt{\gamma} \cdot t + \varphi_0)$, so we find:

(4.11) $$y = x \cdot e^{\beta \cdot t} = A\cos(\sqrt{\gamma} \cdot t + \varphi_0) \quad \Leftrightarrow \quad x = A \cdot e^{-\beta \cdot t} \cos(\sqrt{\gamma} \cdot t + \varphi_0)$$

If we then substitute the constants from the original equation, where $\beta = \dfrac{\alpha}{2m}$ and $\gamma = \dfrac{k}{m} - \dfrac{\alpha^2}{4m}$,
we find:

(4.12) $$x(t) = A \cdot e^{-\frac{\alpha}{2m}t} \cos\left(\sqrt{\frac{k}{m} - \frac{\alpha^2}{4m^2}} \cdot t + \varphi_0 \right)$$

What we find is a harmonic oscillation, having an exponentially decreasing amplitude.
Below is shown the graph for a solution made by a computer program. The exponential envelope for the solution is also shown.
(In the program the first axis is always named *x*, and the second axis is always named *y*. This should of course be understood as *t* and *x*.)



5. Harmonic motion with damping proportional to v^2

If the drag force is proportional to v^2, then $F_{drag} = \eta v^2$, but there is, (as far as I know), no analytic solution to that equation of motion. The differential equation becomes in this case.

$$\frac{d^2x}{dt^2} + \frac{\eta}{m}\frac{dx}{dt}\left|\frac{dx}{dt}\right| + \frac{k}{m}x = 0 \quad \Leftrightarrow \quad \ddot{x} + \frac{\eta}{m}\dot{x}|\dot{x}| + \frac{k}{m}x = 0 \;\; \textit{or alternatively} \quad y'' + \frac{\eta}{m}y'|y'| + \frac{k}{m}y = 0$$

As in the first case, the equation must be solved separately for $v > 0$ or $v < 0$.
A numerical solution is depicted below.
It appears that the damping is most significant for high velocities, but that it almost disappears for small velocities, as it should be expected.