Damped and forced harmonic oscillations

This is an article from my home page: www.olewitthansen.dk

Ole Witt-Hansen 2008

5. Damped harmonic oscillations

A harmonic oscillation is a linear movement (along an axis), where the resulting force is always directed against and proportional to the distance to the position of equilibrium.
If the motion is along the x – axis, then the equation of motion is:

(5.1) $$F_{res} = -k \cdot x \quad \Leftrightarrow \quad m\frac{d^2x}{dt^2} = -kx \quad \Leftrightarrow \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

If we put $\omega = \sqrt{\frac{k}{m}}$ (the cyclic frequency), then the equation becomes:

(5.2) $$\frac{d^2x}{dt^2} = -\omega^2 x$$

Which has the familiar solution:

(5.2) $$x = A\cos(\omega \cdot t + \varphi_0)$$

A is the amplitude, ω is the cyclic frequency, and φ_0 is the initial phase.

The period is $$T = \frac{2\pi}{\omega} \quad \Leftrightarrow \quad T = 2\pi\sqrt{\frac{m}{k}} .$$

In the Math classroom, one usually writes the solution in a slightly different way:

(5.3) $$x = c_1 \cos\omega t + c_2 \sin\omega t$$

That this is actually the same solution, one may realize, by applying one of the addition formula, mentioned earlier:

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

to the solution (5.2).

$$x = A\cos(\omega \cdot t + \varphi_0) = A\cos(\varphi_0)\cos(\omega \cdot t) - A\sin(\varphi_0)\sin(\omega \cdot t)$$

If we put $c_1 = A\cos(\varphi_0)$ *and* $c_2 = -A\sin(\varphi_0)$, having the solutions: $\tan\varphi_0 = -\frac{c_2}{c_1}$, $A = \sqrt{c_1^{\ 2} + c_2^{\ 2}}$

We regain the solution (5.2).
If there is friction or viscous forces (drag forces), another term has to be added to (5.1). We shall first assume that the drag force is proportional to the velocity, and directed opposite to the velocity.

The coefficient of proportionality depends on the shape of the body, and the nature of the medium (air, liquid) the body moves in.

(5.4) $$F_{visc} = -\alpha v \quad \Rightarrow \quad F_{visc} = -\alpha\frac{dx}{dt} .$$

Hereafter the differential equation for the damped harmonic oscillator becomes.

(5.5)
$$F_{res} = -k \cdot x + F_{visc} \quad \Leftrightarrow$$
$$m\frac{d^2x}{dt^2} = -\alpha\frac{dx}{dt} - kx \quad \Leftrightarrow$$
$$\frac{d^2x}{dt^2} + \frac{\alpha}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

It is however associated with a bit more ingenuity to solve (5.5), than to solve (5.1).
First we simplify the equation a bit, with the aim of having lesser constants.

(5.6)
$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c \cdot x = 0$$

Where we have put $b = \frac{\alpha}{m}$ *and* $c = \frac{k}{m}$

(5.6) is now a 2. order linear homogenous differential equation with the two constants *b* and *c*.
It is linear, because all terms containing *x* appears in first order, and homogenous, because there are no terms which depend explicit on *t*.

5.1 Solution to the differential equation using complex numbers.

The solution of linearly homogenous differential equations with constant coefficients, can always be reduces to finding the complex roots in the characteristic polynomial of the same order as the differential equation
To solve (5.6), we put $x = e^{z \cdot t}$ where *z* is a complex number to be determined. It then follows:

$$\frac{dx}{dt} = z \cdot e^{z \cdot t} \quad og \quad \frac{d^2x}{dt^2} = z^2 \cdot e^{z \cdot t}$$

Inserting in (5.6) and dividing by $e^{z \cdot t}$, we get a complex quadratic equation:

$$z^2 + b \cdot z + c = 0$$

The discriminant is: $d = b^2 - 4 \cdot c$. The discriminant is real, so if $d > 0$ then quadratic equation has two real solutions.

(5.7)
$$z = -\frac{b}{2} + \frac{\sqrt{b^2 - 4 \cdot c}}{2} \quad \vee \quad z = -\frac{b}{2} - \frac{\sqrt{b^2 - 4 \cdot c}}{2}$$

Returning to the original equation, we notice that $b > 0$ and $c = k/m > 0$, so both solutions (5.7) are negative. The case $d = 0$ reduces to one solution.

If $d < 0$ then the quadratic equation has no real solutions, but rather two complex solutions.

(5.8) $$z = -\frac{b}{2} + i\frac{\sqrt{4\cdot c - b^2}}{2} \quad \vee \quad z = -\frac{b}{2} - i\frac{\sqrt{4\cdot c - b^2}}{2}$$

Here i is the complex unit. i^2=-1.

In the theory of complex numbers one of the most important formulas is Eulers formula.
Actually one of the most important formulas in the mathematical analysis at all.
If $z = x + i\cdot y$ is a complex number, where x and y are real, Euler's formula reads:

(5.9) $$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

We are only interested in the real part of the solution – naturally!
Furthermore, we notice that when we made the substitution $x = e^{z\cdot t}$, we might as well have written $x = Ae^{z\cdot t + i\varphi_0}$, having two integration constants. necessary for the complete solution of a second order differential equation.

(5.10) $$x(t) = Ae^{-\frac{b}{2}\cdot t}\cos(\omega t + \varphi_0)$$

What we see is that the solution is a harmonic oscillation, but with an amplitude which decreases exponentially with time. This is called damped harmonic oscillations.
If the original values for the constants for b and c: $b = \frac{\alpha}{m}$ *and* $c = \frac{k}{m}$, are substituted back, we get $\omega = \sqrt{\frac{k}{m} - \frac{\alpha^2}{4m^2}}$. Inserted in (5.10) gives:

(5.11) $$x(t) = Ae^{-\frac{\alpha}{2m}\cdot t}\cos(\sqrt{\frac{k}{m} - \frac{\alpha^2}{4m^2}}\cdot t + \varphi_0)$$

α is the coefficient of viscosity, defined by the equation: $F_{visc} = -\alpha\cdot v$, and k is "constant of the spring".
The condition for the validity of the solution (5.11) is that the expression under the square root is positive. Otherwise the oscillating system will never perform one period, and the system will approach the equilibrium exponentially.

5.2 Traditional solution of the same differential equation

We shall again look at the differential equation

(5.12) $$\frac{d^2x}{dt^2} + \frac{\alpha}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

For convenience, as above, we make some abbreviations.

(5.13) $$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c\cdot x = 0 \quad \text{, where} \quad b = \frac{\alpha}{m} \quad og \quad c = \frac{k}{m}$$

We solved earlier the equation by resorting to complex numbers, but here we shall apply a more traditional method, resembling the method used to solve a general linear first order equation.

The method is to introduce an aiding function, equipped to rewrite the differential equation to one, we can solve, that is, the equation for the harmonic oscillator.

(5.14) $$m\frac{d^2y}{dt^2} = -ky \quad \Leftrightarrow \quad \frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \quad \Leftrightarrow \quad \frac{d^2y}{dt^2} + \omega^2 y = 0$$

Having the solution:

(5.15) $$y = A\cos(\omega t + \varphi_0)$$

To obtain this we have put $y = x\,e^{\beta \cdot t}$, where x refers to the solution to the original differential equation (5.13)

(5.16) $$\frac{d^2(xe^{\beta \cdot t})}{dt^2} + \omega^2 xe^{\beta \cdot t} = 0$$

By a suitable choice of β and ω^2, we hope to make (5.16) have the same form as (5.13)

$$\frac{d^2(xe^{\beta \cdot t})}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}e^{\beta \cdot t} + x\beta e^{\beta \cdot t}\right) = \frac{d^2x}{dt^2}e^{\beta \cdot t} + \frac{dx}{dt}\beta\, e^{\beta \cdot t} + \frac{dx}{dt}\beta\, e^{\beta \cdot t} + x\beta^2 e^{\beta \cdot t}$$

$$\frac{d^2(x\,e^{\beta \cdot t})}{dt^2} = \frac{d^2x}{dt^2}e^{\beta \cdot t} + 2\beta\frac{dx}{dt}e^{\beta \cdot t} + x\beta^2 e^{\beta \cdot t}$$

We add the term $\omega^2 xe^{\beta \cdot t}$ to the second derivative of $xe^{\beta \cdot t}$ and put the result to 0.

(5.17) $$\frac{d^2(xe^{\beta \cdot t})}{dt^2} + \omega^2 xe^{\beta \cdot t} = 0 \quad \Leftrightarrow$$

$$\frac{d^2x}{dt^2}e^{\beta \cdot t} + 2\beta\frac{dx}{dt}e^{\beta \cdot t} + x\beta^2 e^{\beta \cdot t} + \omega^2 xe^{\beta \cdot t} = 0$$

The equation is simplifying by dividing by $e^{\beta \cdot t}$.

(5.18) $$\frac{d^2x}{dt^2} + 2\beta \cdot \frac{dx}{dt} + (\beta^2 + \omega^2) = 0$$

We then compare (5.18) with the original differential equation:

(5.19) $$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c \cdot x \cdot = 0$$

And we see that the two differential equations are identical if and only if:

$$\beta = \frac{b}{2} = \frac{\alpha}{2m} \quad \text{and} \quad \beta^2 + \omega^2 = c \quad \Leftrightarrow \quad \omega^2 = c - \frac{b^2}{4} \quad \Rightarrow \quad \omega^2 = \frac{k}{m} - \frac{\alpha^2}{4m}.$$

However, we can solve (5.17) directly. If we put $y = x \cdot e^{\beta \cdot t}$, the equation takes the form:

(5.20) $$\frac{d^2 y}{dt^2} = -\omega^2 y \quad \text{having the solution} \quad y = A\cos(\omega t + \varphi_0)$$

So we find:

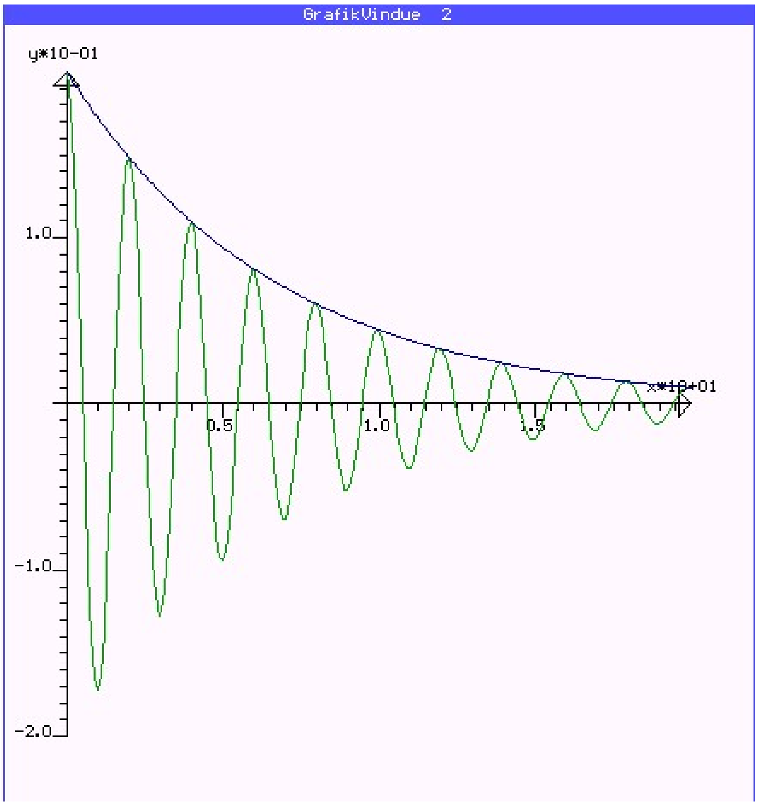
(5.21) $$y = x \cdot e^{\beta \cdot t} = A\cos(\omega t + \varphi_0) \quad \Leftrightarrow \quad x = A \cdot e^{-\beta \cdot t}\cos(\omega t + \varphi_0)$$

Inserting the constants: $\beta = \dfrac{\alpha}{2m}$ and $\omega^2 = \dfrac{k}{m} - \dfrac{\alpha^2}{4m}$ we arrive at the solution to (5.12)

(5.22) $$x(t) = Ae^{-\frac{\alpha}{2m} \cdot t}\cos\left(\sqrt{\frac{k}{m} - \frac{\alpha^2}{4m^2}} \cdot t + \varphi_0\right).$$

The solution is a harmonic oscillation with an exponential decreasing amplitude.

Below is shown an example of a solution, where the exponential envelope curve is also shown.



Damped harmonic oscillations turn up in many fields of physics, and therefore it is not without interest to be able to solve the associated differential equations.

6. Forced harmonic oscillations without damping

We shall consider a forced oscillation without damping, where the mass m, besides the "spring force" i.e. obeys Hookes law $-kx$, is driven by an external time dependent force $f(t)$.
The results are directly applicable to an electrical circuit consisting of a capacitor and a coil, and driven by an alternating current.

$$F_{res} = -k \cdot x + F_{ext} \quad \Leftrightarrow$$

(6.1) $$m\frac{d^2x}{dt^2} = -kx + F_{ext}(t) \quad \Leftrightarrow$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{F_{ext}(t)}{m}$$

We shall assume that the external force varies harmonically: $\frac{F_{ext}(t)}{m} = \frac{f_0}{m}e^{i\omega t}$.
The solution to (6.1) is (as well known from the theory of differential equations) a particular solution to the non homogeneous equation plus the complete solution to the corresponding homogeneous equation:

(6.2) $$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Which has the solution:

$$x = A\cos(\omega_0 t + \varphi) \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

Since the differential equation

(6.3) $$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{f_0}{m}e^{i\omega t} \quad \Leftrightarrow \quad \frac{d^2x}{dt^2} + \omega_0{}^2 x = \frac{f_0}{m}e^{i\omega t}$$

is of second order with constant coefficients, we may write a particular solution as: $x = Ae^{i\omega t}$ (where ω is the enforced frequency), when inserted in (6.3) gives:

(6.4) $$-\omega^2 Ae^{i\omega t} + \omega_0{}^2 Ae^{i\omega t} = \frac{f_0}{m}e^{i\omega t},$$

When solved with respect to A gives: $A = \frac{\frac{f_0}{m}}{\omega_0{}^2 - \omega^2}$

The complete solution to (6.3) can thereafter be written as the particular solution plus the solution to the homogeneous equation:

(6.5) $$x = A_0\cos(\omega_0 t + \varphi) + \frac{\frac{f_0}{m}}{\omega_0{}^2 - \omega^2}\cos(\omega t)$$

Writing this as: $x = A\cdot\cos(\omega_0 t+\varphi)+ B\cdot\cos(\omega t)$, we can in the case where $A = B$ apply the first of the logarithmic formulas for addition of two cosine functions:

$$\cos u + \cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2} \quad \text{og} \quad \cos u - \cos v = -2\sin\frac{u+v}{2}\sin\frac{u-v}{2}$$

(6.6) $$x = 2A\cos(\frac{\omega_0+\omega}{2}t + ½\varphi)\cos(\frac{\omega_0-\omega}{2}t + ½\varphi)$$

The system will perform oscillations with a frequency $\frac{\omega_0+\omega}{2}$ with an amplitude

$2A\cos(\frac{\omega_0-\omega}{2}t + ½\varphi)$, which is time dependent and varies between the values -2A and 2A.
A phenomenon, which is familiar for sound waves, and goes under the name modulation and beats.

The beat frequency is $\frac{\omega_0-\omega}{2}$, and when $\omega_0 - \omega << \omega_0$ the signal will sound like a sirene.
In general the two amplitudes A and B are not equal, but it changes only the situation in the sense that the signal will have two beat frequencies instead of one.
We may namely always determine two numbers C and D, so that $A = C+D$ and $B = C - D$, and solve for C and D:

$$C = \frac{A+B}{2} \quad og \quad D = \frac{A-B}{2}$$

$$A\cdot\cos(\omega_0 t+\varphi)+ B\cdot\cos(\omega t)=(C+D)\cos(\omega_0 t+\varphi)+ (C-D)\cos(\omega t) =$$

$$C\cdot\cos(\omega_0 t+\varphi)+ C\cdot\cos(\omega t)+ D\cdot\cos(\omega_0 t+\varphi)- D\cdot\cos(\omega t)$$

Rewriting the solution using the logarithmic formulas then gives:

(6.6) $$x = 2C\cos(\frac{\omega_0+\omega}{2}t + ½\varphi)\cos(\frac{\omega_0-\omega}{2}t + ½\varphi) - 2D\sin(\frac{\omega_0+\omega}{2}t + ½\varphi)\sin(\frac{\omega_0-\omega}{2}t + ½\varphi)$$

The result is two modulations with the same frequency, but where the amplitudes (the beats) are $\frac{\pi}{2}$ out of phase.