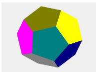
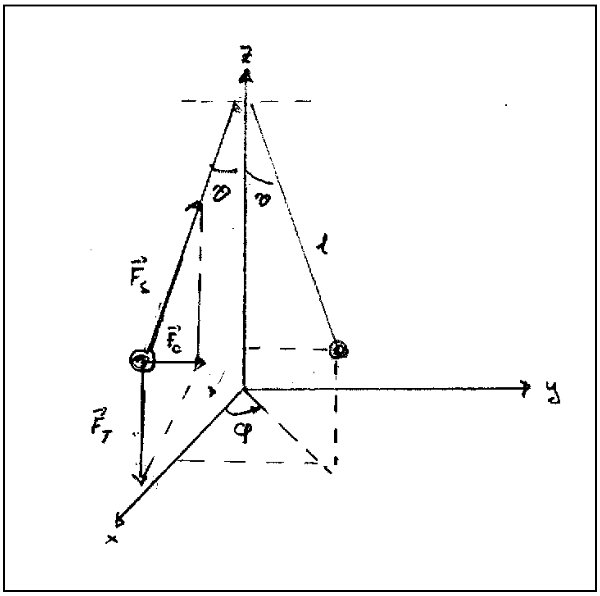
The conical and the chaotic pendulum



Ole Witt-Hansen 2019 (1997)

1. Dynamic analysis of the conic pendulum



The figure to the left shows a conic pendulum, that is, a mass, which is suspended in a cord, fixed on the *z* axis, and where the length of the cord is *l*. Using polar coordinates, we have for the coordinates of the mass (*x, y, z)*.

$$x = l\sin\theta\cos\varphi, \quad y = l\sin\theta\sin\varphi, \quad z = l(1-\cos\theta)$$

The kinetic energy is:

$$E_{kin} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

And the potential energy:

$$E_{pot} = mgz$$

It is well known, (but a bit circumstantially), to show, that the velocities along the two polar angles are: $v_\theta = r\dot{\theta}$ and $v_\varphi = r\sin\theta\,\dot{\varphi}$, and that we furthermore have: $v^2 = {v_\theta}^2 + {v_\varphi}^2$ (since the two directions are orthogonal).
To determine the equations of motion, we shall apply the Lagrange formalism. Here the kinetic energy is traditionally labelled *T*, and the potential energy *U*. The Lagrange function is:

$$L = T - U$$

Using generalized coordinates: $q_i, \dot{q}_i$, and where a bullet over a variable as usual denotes differentiation with respect to time. The equations of motion are then derived from the Lagrange equations.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

$$T = \tfrac{1}{2}m(l^2\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\varphi}^2) = \tfrac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) \quad and \quad U = mgl(1-\cos\theta)$$

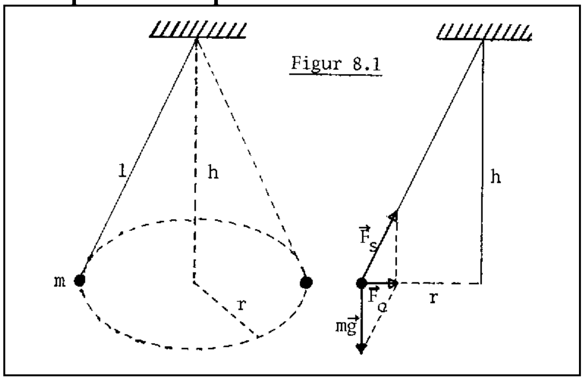
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \quad \Leftrightarrow \quad ml^2(\ddot{\theta} - \cos\theta\sin\theta\dot{\varphi}^2) + mgl\sin\theta = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0 \quad \Leftrightarrow \quad ml^2(2\cos\theta\sin\theta\dot{\varphi}\dot{\theta}) + \sin^2\theta\ddot{\varphi}) = 0$$

The equations of motion then become:

$$\ddot{\theta} = \cos\theta\sin\theta\dot{\varphi}^2 - \frac{g\sin\theta}{l} \qquad \text{and} \qquad \ddot{\varphi} = -\frac{2\cos\theta\dot{\varphi}\dot{\theta}}{\sin\theta}$$

Example. Conical pendulum



A weight that is suspended in the field of gravity can be brought to perform a uniform circular motion. It is then called a conical pendulum. The weight is affected by gravity $\vec{F} = m\vec{g}$ and the force from the wire: $\vec{F}_s$. The sum of these two forces deliver the necessary force to uphold the uniform circular motion, with the centripetal force: $F_c = m\omega^2 r$.
The length of the wire is *l*. The height from the point of suspension to the plane of the circular motion is *h*, and *r* is the radius in the circular motion.
From the two even angled triangles in the figure we have.

$$\frac{F_c}{mg} = \frac{r}{h} \quad \Leftrightarrow \quad \frac{m\omega^2 r}{mg} = \frac{r}{h} \quad \Rightarrow \quad \omega^2 = \frac{g}{h} \quad \Rightarrow \quad \omega = \sqrt{\frac{g}{h}}$$

(8.1.2) $$\omega = \sqrt{\frac{g}{h}} \quad \wedge \quad T = \frac{2\pi}{\omega} \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{h}{g}}$$

This result may be used to determine the initial angular velocities, so that the equations of motion results in a uniform circular motion.
Below are shown two graphical solutions to the equations of motion above. They are fabricated from a (home) 1996 Turbo 7.0 program.

The figure to the left corresponds to a uniform circular motion. The data used are:

$$l = 2.0\ m,\ h = l\cos\theta = ½l,\quad \theta = 60^0.\ \omega = \sqrt{\frac{g}{h}} = 3.367\ s^{-1}$$

The figure to the right, corresponds to initial angular velocities $\dot{\theta}$ *and* $\dot{\varphi}$, which deviates (much) for that of the uniform circular motion. The motion is neither periodic nor completely chaotic.