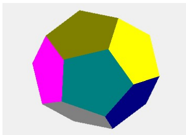
About balancing or tipping over

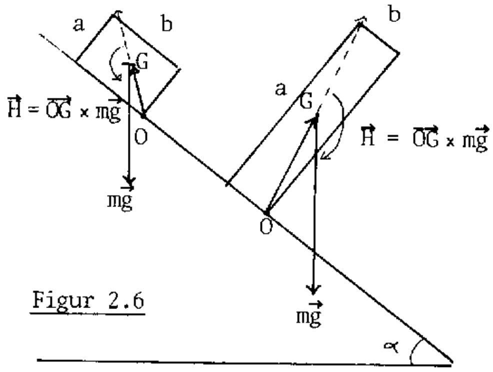
This is an article from my home page: www.olewitthansen.dk



Ole Witt-Hansen 2019

Contents

1. Equilibrium condition for a box on a slope



Figur 2.6

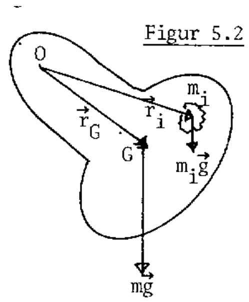
The figure shows two boxes placed on a slope, where the friction from the underlay prevents them in sliding down the slope. An observer will (by experience) probably claim that the one box will remain where it is, but that the second box will overturn.
We shall now apply the definition of the torques acting on the boxes to determine the condition that the box tilts or not.
The gravity is acting at *G,* the centre of mass (geometric centre) of the box.

With regard to the first box, it is seen from the figure that the torque from gravity $\vec{H} = \overrightarrow{OG} \times m\vec{g}$ will turn the box counter clockwise, towards the underlay, while the same torque on the second box, will turn the box clockwise and turn it over.

The limit between the two cases is of course when $\vec{H} = 0 \quad \Leftrightarrow \quad \overrightarrow{OG} \times m\vec{g} = 0 \quad \Leftrightarrow \quad \overrightarrow{OG} \,||\, m\vec{g}$.
In this case it is seen, that the inclination of the slope is equal to the angle between *OG* and the front side of the box. This angle may, however, be calculated from: $\tan\alpha = b/a$.
We thus find that the condition that the box is overturned:

$$\tan\alpha > b/a$$

2. Centre of gravity equals centre of mass



Figur 5.2

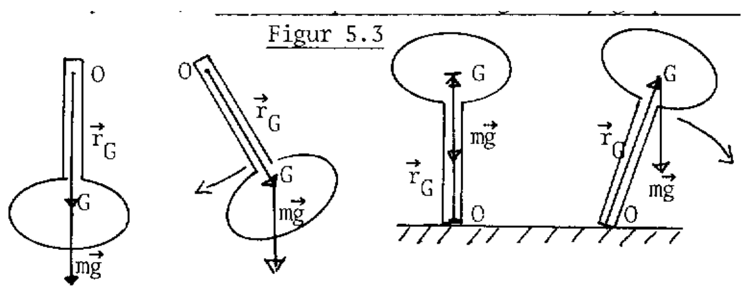
The centre of gravity of a rigid body is usually described as a point of suspension, where the body is balanced (equilibrium), with respect to every displacement from this position.
The condition of equilibrium is obviously that the sum of torques acting on the body is zero.
The figure shows a body suspended in a point *O* (or on a rod through *O*) in the gravity field.
The sum of torques from gravity acting on the mass elements m_i situated at $\vec{r}_i$ with respect to *O*, is calculated from:

$$(2.1) \qquad \vec{H} = \sum \vec{r}_i \times m_i\vec{g} = \left(\sum m_i\vec{r}_i\right) \times \vec{g} \quad \Leftrightarrow \quad \vec{H} = m\vec{r}_G \times \vec{g} = \vec{r}_G \times m\vec{g}$$

From (10.1) it appears that the entire torque (moment of force) can be found, as if the whole mass of the body is situated in the *CM* of the body.

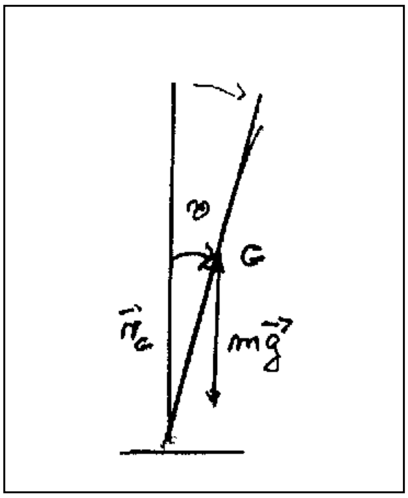
Especially if $O = G$, that is, the body is suspended in its centre of mass, $\vec{r}_G = \vec{0}$ and therefore $\vec{H} = 0$, the body will be in equilibrium turned in any position, and this will also hold if the body is suspended on an axis through the CM.

The centre of gravity is the same as the centre of mass.



From (10.1) it is seen that a suspended body is in balance, (that is, when $\vec{H} = 0$), when $\vec{r}_G$ is parallel with $\vec{g}$.
It therefore follows that the body is balanced, if the point *O* of suspension or of support is right below or above the centre of mass G. If *G* is below *O*, the equilibrium is *stable*, since the body will move towards the equilibrium position from any minor displacement. However, if *G* is vertically above *O*, the equilibrium is *unstable*, since even a minor displacement from the equilibrium position, will generate a torque, turning (with increasing strength) the body away from the equilibrium, which is a well known experience from everyone, including balancing artists. This is illustrated in figure (5.3).

3. Balancing a vertical rod



The figure shows schematically a vertical rod, supported in the bottom end.
It is a general experience that it is easier to balance with a long rod, than on a short one, especially if most of the mass is placed on the upper end.
No one can balance a pencil on a finger, but someone may balance an upside down broom, and as we have experienced in a circus, the artists can balance a heavy object placed on a rod of several meters.
We shall here deliver a simple explanation of this phenomenon.
In the figure the rod is tilted a (small) angle from equilibrium.
As explained above the gravitational force can be calculated as if the whole mass is placed in the centre of mass.

The torque on the rod is $\vec{H} = \vec{r}_G \times m\vec{g}$ directed into the paper, and having the size: $H = mgr_G \sin\theta$.

Newton's law for rotation is: $H = I\dfrac{d\omega}{dt} = I\dfrac{d^2\theta}{dt^2}$, where ω is the angular velocity, and *I* is the moment of inertia with respect to the axis of rotation. We find thus:

(3.1) $$I\frac{d\omega}{dt} = mgr_G \sin\theta$$

If the rod is homogenous, and having the length *l* the moment of inertia with respect to one of the end points is: $I = \frac{1}{3}ml^2$. With $r_G = \frac{1}{2}l$, we therefore find:

(3.2) $$\tfrac{1}{3}ml^2 \frac{d\omega}{dt} = \tfrac{1}{2}mgl\sin\theta \quad \Leftrightarrow \quad \frac{d\omega}{dt} = \tfrac{3}{2}\frac{g\sin\theta}{l}$$

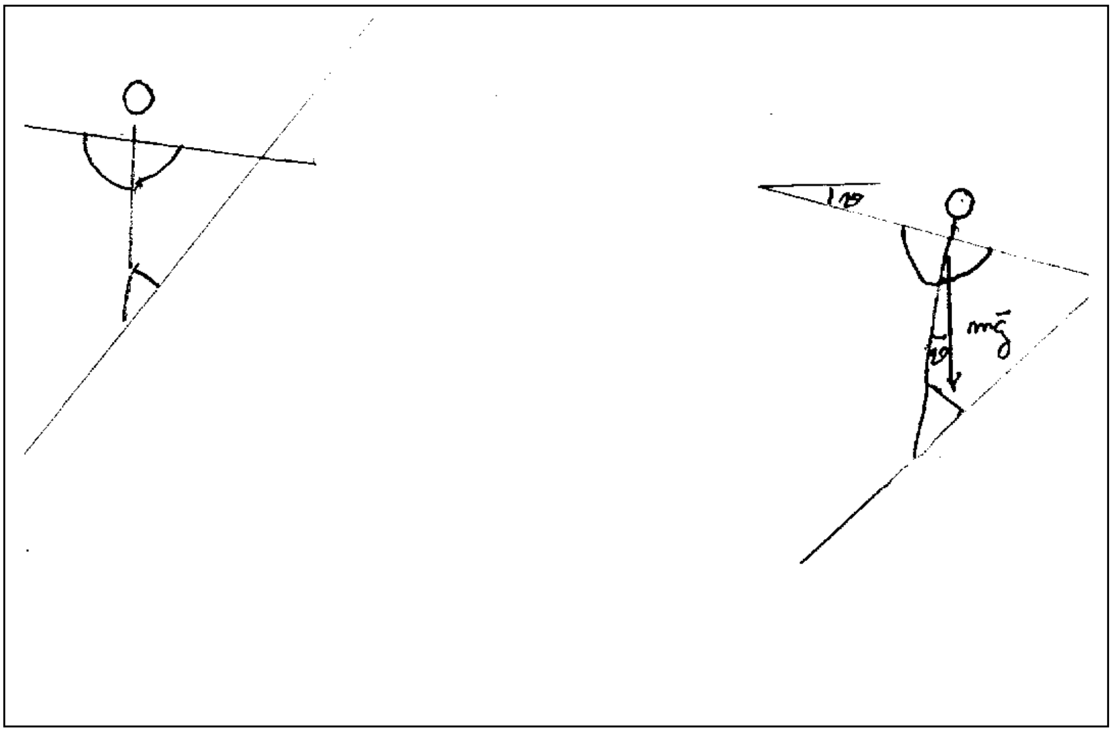
What we see is that the angular acceleration is inversely proportional to the length of the rod.

If the whole mass is placed on the top of the rod, then $I = ml^2$, and $r_G = l$ we find a angular acceleration 1.5 times less.

$$ml^2 \frac{d\omega}{dt} = mgl \sin\theta \quad \Leftrightarrow \quad \frac{d\omega}{dt} = \frac{g \sin\theta}{l}$$

If we exemplify this with a pencil of 16 *cm*, and broom of 1.80 m (for simplicity we assume that the whole mass is placed at the end), we get the following angular accelerations, at an angle of two degrees. For the pencil: 3.21 rad/s^2 , and for the broom 0.216 rad/s^2.
This explains why no one can balance with a pencil, but possibly with a broom.

4. Balancing on a rope



Tightrope on a rope is not for everyone, but requires long practicing. For amateurs one usually use out stretched arms to compensate for the inevitable tilts away from stability.
For more professional purposes, one uses a long rod, which may be rotated to compensate for an angular acceleration away from equilibrium.

If the person is tilted an angle θ away from vertical there will be an angular acceleration, which is given by:

(4.1) $$I_{person} \frac{d\omega_p}{dt} = m_{person} g r_G \sin\theta$$

If the person must avoid being overthrown, he must generate an opposite torque to compensate.
If he does so by turning the rod, we have:

(4.2) $$H_{rod} = I_{rod} \frac{d\omega_{rod}}{dt}$$

So to keep the balance, we must have:

(4.3) $$I_{rod} \frac{d\omega_{rod}}{dt} = m_{person} g r_G \sin\theta$$

If we approximate the person with a box 1.8 *m* times 0.5 *m*, with the centre of mass 1 *m* from the ground, and a weight of 70 *kg*, the moment of inertia, can be calculated from: $I_{box} = \frac{1}{12} m(a^2 + b^2)$. This gives a moment of inertia 20.4 *kg* m^2, and thus:

$$\frac{d\omega_{person}}{dt} = \frac{m_{person} g r_G \sin\theta}{I_{person}} = 34 \cdot \sin\theta \;\; rad/s^2$$

If the homogenous rod has the mass m_{rod} = 4 *kg*, and the length l = 4 *m*, it has a moment of inertia $I_{rod} = \frac{1}{12} m_{rod} l^2$, which evaluates to 5.3 *kg* m^2. If we insert this in the equation above, we get:

$$\frac{d\omega_{rod}}{dt} = \frac{m_{person} g r_G \sin\theta}{I_{rod}} = 130 \cdot \sin\theta \;\; rad/s^2$$

This result could also have been obtained simply by stating the conservation of angular momentum.

$$I_{person}\omega_{person} = I_{rod}\omega_{rod} \quad \Rightarrow \quad \omega_{rod} = \frac{I_{person}}{I_{rod}} \omega_{person}$$

$$\omega_{rod} = 3.8\omega_{person}$$

If we have a tilt of 1^0 it gives $\frac{d\omega_{person}}{dt} = \frac{m_{person} g r_G \sin\theta}{I_{person}} = 34 \cdot \sin\theta \;\; rad/s^2 = 0.59 \;\; rad/s^2$

And therefore $\omega_{rod} = 3.8\omega_{person} = 2.24 \; rad/s$

As it is clear, the correction to regain stability must be very quick, and that a deviation from stability on more than one degree is probably fatal.

Tightrope on a rope is not for everyone, but requires long practicing. If you look into the physics, it is amazing that it is possible at all.