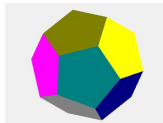


The infinite sum of the reciprocals of n -square

The values of Zeta(2), Zeta(4) and Zeta(6)

This is an article from my home page: www.olewitthansen.dk



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1. The infinite sum of the reciprocals of n-square. Zeta(2).

The *Zeta* function is defined by the series:

$$(1.1) \quad \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \quad x > 1$$

For $x = 1$ it is the harmonic series, which easily can be shown to diverge, but for $x > 1$ it is convergent. The aim of this note is to find the exact value of $\zeta(2)$, $\zeta(4)$ and $\zeta(6)$

$$(1.2) \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

There are in fact many exact values of the *Zeta* that have been found, but we shall be concerned with the series above for $x = 2, 4, 6$, where the sum may be determined by applying Fourier series.

The coefficients of the Fourier series of a function $f(x)$ are in the interval $[-\pi, \pi]$ defined by the integral:

$$(1.3) \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \quad \text{where} \quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{is the Fourier series for } f(x)$$

The norm of $f(x)$, which is written as $|f(x)|$, is defined by the expression:

$$(1.4) \quad |f|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) f(x)^* dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

We shall also use Parseval's theorem.

$$(1.5) \quad |f|^2 = \sum_{n=-\infty}^{\infty} c_n c_n^* = \sum_{n=-\infty}^{\infty} |c_n|^2$$

(For details see www.olewithhansen.dk: Fourier series and Fourier integrals)

We shall then apply the formulas (1.3) and (1.4) for the function $f(x) = x$. This gives:

$$(1.6) \quad \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx \quad \Rightarrow \quad \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx$$

$$(1.6) \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \quad \Rightarrow \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx$$

We shall first do the coefficients c_n .

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x (\cos nx - i \sin nx) dx$$

And we must therefore do the two integrals:

$$\int_{-\pi}^{\pi} x \cos nx dx \quad \text{and} \quad \int_{-\pi}^{\pi} x \sin nx dx$$

Since $x \cos nx$ is an odd function, that is, $f(-x) = -f(x)$ the first integral is zero. However, the integrand in the second integral is even, since $(-x)\sin(-nx) = x\sin nx$.

That the integral of an odd function over a symmetric interval vanishes is obvious from a geometrical point of view, but it can also be proved directly: Let f be an odd function.

$$\begin{aligned} \int_{-a}^a f(x) dx &= \int_{-a}^0 f(x) dx + \int_0^a f(x) dx = \\ \int_a^0 f(-x) d(-x) + \int_0^a f(x) dx &= \int_a^0 f(x) dx + \int_0^a f(x) dx = -\int_0^a f(x) dx + \int_0^a f(x) dx = 0 \end{aligned}$$

In the first integral we have made the substitution: $x \rightarrow -x$, and changed sign for the limits: Finally we have switched the upper and lower limits, resulting in a minus sign.

The integration is done by parts.

$$\begin{aligned} (1.7) \quad \int_{-\pi}^{\pi} x \sin nx dx &= -\frac{1}{n} \int_{-\pi}^{\pi} x d \cos nx = -\frac{1}{n} [x \cos nx]_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} \cos nx dx \\ &= -\frac{1}{n} (\pi \cos n\pi + \pi \cos n\pi) = \frac{2\pi}{n} (-1)^{n+1} \end{aligned}$$

The last integral: $\int_{-\pi}^{\pi} \cos nx dx = 0$, because the integral of $\cos nx$ over a period of 2π is always zero for $n \neq 0$. So we have:

$$(1.8) \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x (\cos nx - i \sin nx) dx = \frac{(-1)^n}{n} i \quad \text{for } n \neq 0 \quad \text{and } c_0 = 0$$

$$\text{since } c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x dx = 0$$

Therefore: $|c_n|^2 = \frac{1}{n^2}$. According to Parseval's theorem:

$$\begin{aligned} (1.9) \quad \sum_{n=-\infty}^{\infty} |c_n|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx, \quad \text{So we find:} \\ \sum_{n=-\infty}^{\infty} |c_n|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi} \left[\frac{1}{3} x^3 \right]_{-\pi}^{\pi} = \frac{\pi^2}{3}, \quad \text{but also: } \sum_{n=-\infty}^{\infty} |c_n|^2 = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

From (1.9) we therefore have: $2 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3}$, which gives us the value of $\zeta(2)$:

$$(1.10) \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

2. The value of Zeta(4)

We shall now do the same line of calculations to determine

$$(2.1) \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

Applying the Fourier transformation formulas:

$$(2.2) \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \quad \text{where} \quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

And using Parseval's theorem: $\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$ with the function $f(x) = x^2$.

We have:

$$(2.3) \quad \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 dx \quad \text{and} \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx} dx$$

We begin by evaluating c_n .

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 (\cos nx - i \sin nx) dx$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 (\cos nx - i \sin nx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \cos nxdx - \frac{i}{2\pi} \int_{-\pi}^{\pi} x^2 \sin nxdx$$

The last integral is zero, because $x^2 \sin nx$ is an odd function, so we shall only evaluate the first integral by parts.

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \cos nxdx = \frac{1}{2\pi n} \int_{-\pi}^{\pi} x^2 d \sin nx = \frac{1}{2\pi n} [x^2 \sin nx]_{-\pi}^{\pi} - \frac{1}{2\pi n} \int_{-\pi}^{\pi} 2x \sin nxdx$$

The first term vanishes because $\sin n\pi = 0$. So we evaluate the second integral by parts.

$$-\frac{1}{\pi n} \int_{-\pi}^{\pi} x \sin nxdx = \frac{1}{\pi n^2} \int_{-\pi}^{\pi} x d \cos nx = \frac{1}{\pi n^2} [x \cos nx]_{-\pi}^{\pi} - \frac{1}{\pi n^2} \int_{-\pi}^{\pi} \cos nxdx$$

The second integral vanishes because the integral of $\cos nx$ over a period of 2π is always zero.

$$(2.5) \quad \frac{1}{\pi n^2} [x \cos nx]_{-\pi}^{\pi} = \frac{1}{\pi n^2} (\pi \cos n\pi - (-\pi \cos n\pi)) = \frac{2}{n^2} \cos n\pi = \frac{2}{n^2} (-1)^n$$

So we find: $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx} dx = \frac{2}{n^2} (-1)^n$ for $n \neq 0$ and $c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}$

Thus

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{\pi^4}{9} + 4 \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{9} + 4 \sum_{n=-\infty}^{-1} \frac{1}{n^4} + 4 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{9} + 8 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

On the other hand:

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 dx = \frac{1}{2\pi} \left[\frac{1}{5} x^5 \right]_{-\pi}^{\pi} = \frac{\pi^4}{5}$$

So we find: $\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 dx \Leftrightarrow \frac{\pi^4}{9} + 8 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{5} \Leftrightarrow 8 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{5} - \frac{\pi^4}{9} = \frac{4\pi^4}{45}$

(2.6) $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

3. The value of Zeta(6)

As we did in the previous cases, we use the Fourier transformation formulas:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \quad \text{where} \quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

And we apply also Parseval's theorem:

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

This time with the function $f(x) = x^3$. We then have:

(3.1) $\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^6 dx$ and $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^3 e^{-inx} dx$

We shall first evaluate c_n

(3.2) $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^3 e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^3 (\cos nx - i \sin nx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^3 \cos nxdx - \frac{i}{2\pi} \int_{-\pi}^{\pi} x^3 \sin nxdx$

The first term is zero, because $x^3 \cos nx$ is an odd function, so we shall only evaluate the second integral by parts.

$$c_n = -\frac{i}{2\pi} \int_{-\pi}^{\pi} x^3 \sin nxdx = \frac{i}{2\pi n} \int_{-\pi}^{\pi} x^3 d \cos nx = \frac{i}{2\pi n} [x^3 \cos nx]_{-\pi}^{\pi} - \frac{3i}{2\pi n} \int_{-\pi}^{\pi} x^2 \cos nxdx$$

$$c_n = \frac{i}{2\pi n} (\pi^3 \cos n\pi - (-\pi)^3 \cos n\pi) - \frac{3i}{2\pi n} \int_{-\pi}^{\pi} x^2 \cos nxdx$$

$$(3.3) \quad c_n = \frac{i}{\pi n} \pi^3 (-1)^n - \frac{3i}{2\pi n} \int_{-\pi}^{\pi} x^2 \cos nxdx$$

However, we have already evaluated the integral above, and the result was:

$$(3.4) \quad \frac{3i}{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \cos nxdx = \frac{3i}{n} \left(\frac{2}{n^2} (-1)^n \right)$$

So we get for c_n .

$$c_n = \frac{i}{\pi n} \pi^3 (-1)^n - \frac{3i}{n} \left(\frac{2}{n^2} (-1)^n \right) = i \left(\frac{\pi^2}{n} (-1)^n - \frac{6}{n^3} (-1)^n \right) \quad \text{and} \quad c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^3 dx = 0$$

$$|c_n|^2 = \frac{\pi^4}{n^2} + \frac{36}{n^6} - \frac{12\pi^2}{n^4}$$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \sum_{n=-\infty}^{\infty} \frac{\pi^4}{n^2} + \sum_{n=-\infty}^{\infty} \frac{36}{n^6} - \sum_{n=-\infty}^{\infty} \frac{12\pi^2}{n^4} = \pi^4 \sum_{n=-\infty}^{\infty} \frac{1}{n^2} + 36 \sum_{n=-\infty}^{\infty} \frac{1}{n^6} - 12\pi^2 \sum_{n=-\infty}^{\infty} \frac{1}{n^4} =$$

Changing the lower limit from minus infinity to 1 and multiplying by 2

$$(3.5) \quad \sum_{n=-\infty}^{\infty} |c_n|^2 = 2\pi^4 \sum_{n=1}^{\infty} \frac{1}{n^2} + 72 \sum_{n=1}^{\infty} \frac{1}{n^6} - 24\pi^2 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

However, we have already evaluated the two sums: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$, so

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{\pi^6}{3} + 72 \sum_{n=1}^{\infty} \frac{1}{n^6} - \frac{4\pi^6}{15}$$

On the other hand: $\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^6 dx = \frac{\pi^6}{7}$ so we have:

$$\frac{\pi^6}{3} + 72 \sum_{n=1}^{\infty} \frac{1}{n^6} - \frac{4\pi^6}{15} = \frac{\pi^6}{7} \quad \Leftrightarrow \quad 72 \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{4\pi^6}{15} + \frac{\pi^6}{7} - \frac{\pi^6}{3} = \frac{15\pi^6}{105} + \frac{28\pi^6}{105} - \frac{35\pi^6}{105} = \frac{8\pi^6}{105}$$

And we finally get:

$$(3.6) \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$