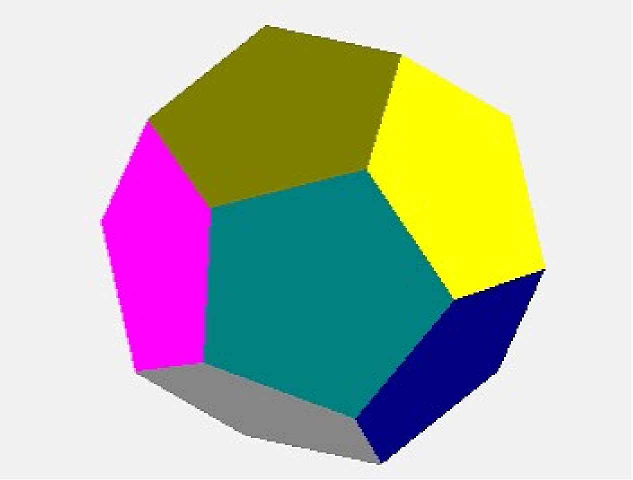
The Platonic solids
The five regular polyhedra



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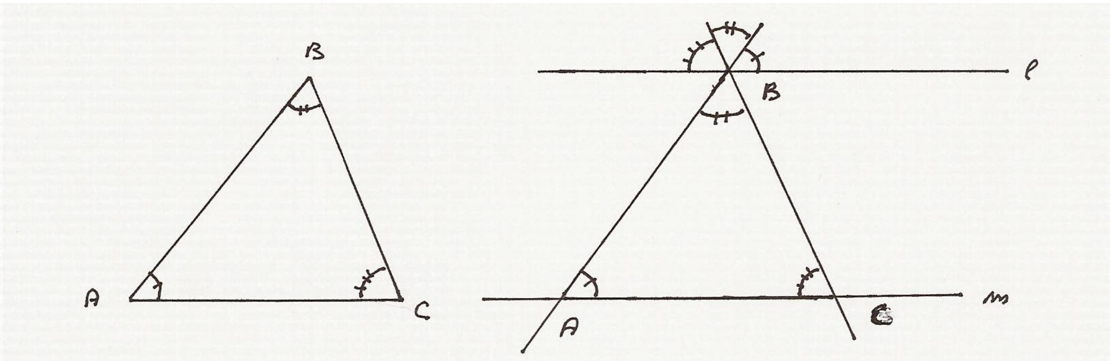
1. Polygons

A polygon is a closed curve, where the border consists of straight line segments. Well known examples are triangles, rectangles and pentagons. The straight line segments are the sides or edges of the polygon, and the intersection of sides are the corners or the angles of the polygon.
The polygon is convex if none of the angles are greater than 180^0.

The sum of the angles in an *n*-sided polygon is (n -2)·180^0 or (n -2)·π. This is straightforward to verify, once you know that the sum of the angles in a triangle is 180^0. That this is actually the case can be verified from the two figures below.
We only have to apply three axioms (from Euclid's elements).

1. When two parallel lines are intersected by a third line, the corresponding angles of intersection are equal.
2. For a given line and a given point outside the line, you may only draw one line through the point parallel to the given line.
3. When two lines intersect, the opposite angles at the intersection point are equal.



The figure shows a triangle *ABC*, which is redrawn, now with a line through *B*, parallel to *AB* (axiom 2). By applying axiom 1 and 3, we can verify that the three angles above *B* are equal to the three angles in the triangle, and their sum is 180^0.

Since an *n*-sided polygon may always be divided into an *n*-1 side polygon and a triangle the formula (n -2)·180^0 for the sum of the angles in an *n*-sided polygon follows.

A polygon is regular if all sides are equal and all angles are equal. The angle in a regular polygon with n corners is therefore.

(1.1) $$v_n = \frac{(n-2)\cdot 180^0}{n} = 180^0 - \frac{360^0}{n} = \pi - \frac{2\pi}{n} = \pi(1-\frac{2}{n})$$

Which shows e.g. That the angle in a regular triangle is $\pi(1-\frac{2}{3}) = \frac{\pi}{3} = 60^0$, and the angle in a regular pentagon is $\pi(1-\frac{2}{5}) = \frac{3\pi}{5} = 108^0$.
A *polyhedron* is a closed solid, where the surface is built by regular polygons. The simplest polyhedron is the tetrahedron, consisting of four regular triangles.

2. Topologically considerations

The famous *polyhedron theorem*, first discovered by Euler is most often proven in topology.

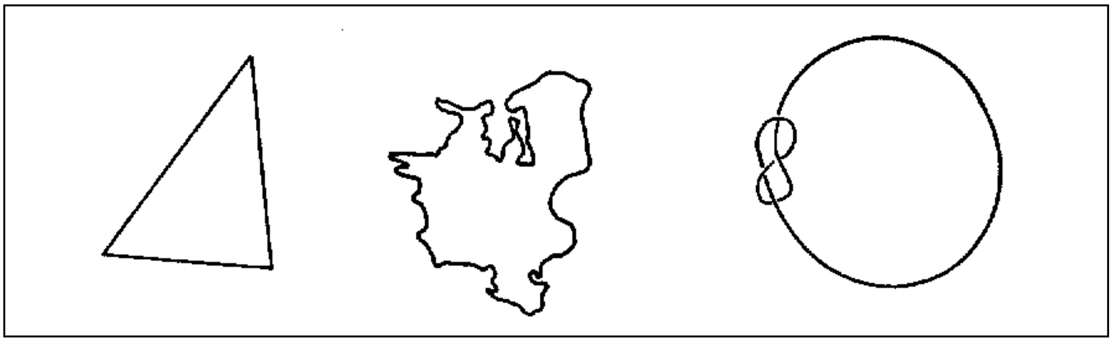
So we wish to map the surface of the polyhedron to a sphere, which circumvents the polyhedron. This will result in a closed net on the sphere, consisting of masks (*m*), corresponding to the polygons in the polyhedron, cords (*s*) corresponding to edges of the polygons and knots (*k*) corresponding to the corners of the polyhedra.

We are interested in transforming the topological properties of the *net* to the same topological properties of the polyhedron.

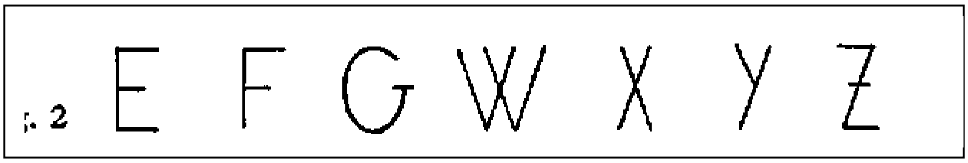
Topological properties are properties, that are invariant upon a continuous one-to-one mapping (function) (injective mapping) . Such a function is called a *homeomorphism.*

Injectivity of a mapping f from a set A into a set B implies that to each element x in A corresponds exactly one element $y = f(x)$ in B (f is a function), and that to each element y in B corresponds exactly one element $x = f^{-1}(x)$ in A.

Two sets which are generated by each other by a *homeomorphism* are said to be topologically equivalent. A simple section of a curve is topologically equivalent to a straight line section. Any closed curve is topologically equivalent to a circle, and any piece of a flat section is topologically equivalent to a circular disc.
Below are shown three figures of which all of them are topologically equivalent to a circle.



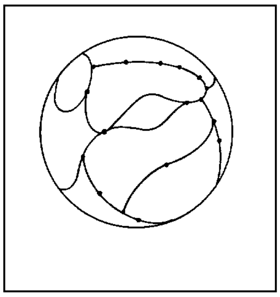
On the figure below E, F and Y are topologically equivalent, while G and Z are topologically equivalent, and W and X are topologically equivalent.

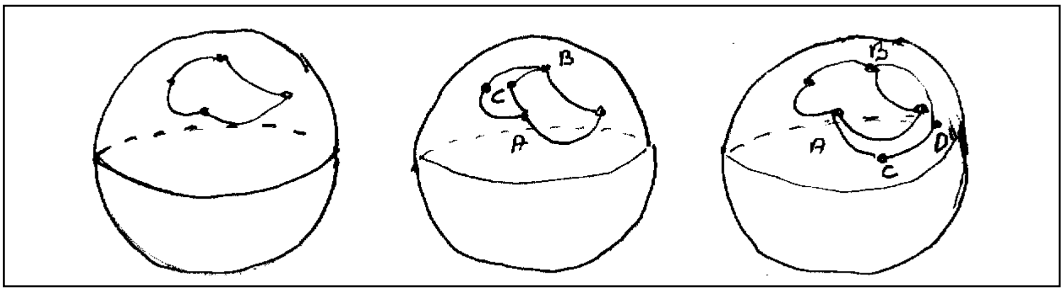


3. Euler's polyhedron theorem

By a *net* on *a sphere* we understand a system consisting of a finite number of flat pieces and a finite number of curve pieces, which are denoted *masks* and cords respectively, which fulfil the following conditions.

1. *Every point on the sphere belongs to at least one mask (inclusive its border), and two different masks do not have any internal points in common.*
2. *The border of each mask is composed of cords*
3. *A point lying on a cord is not an internal point, (that is, not an end point) of any other cord or of any mask*





To the left is shown a net on a sphere, consisting of masks, cords and knots.
A knot is the end point of a cord. The conditions 1 – 3, imply that on the border of each mask in a net there exits at least two knots, and that every *knot* in a *net* lies on the border of exactly two masks.
We shall first consider a simple net consisting of only two masks, as shown on the second figure. The one mask is the one that you see, and the second mask is what is left over on the sphere.
Both masks (*m*) are confined by the same cords (*s*), and the same knots (*k*).
Furthermore $s = k$, since each cord connects two knots, and each knot has two cords attached to it.
Since $s = k$, and there are two masks, the following relation must hold for this most simple net.

(2.1) $$m + k - s = 2$$

In the next figure we have created a new mask by drawing a cord with *p* knots between *A* and *B,* in the inner of the mask. In this manner we have added *p* knots,1 mask and $p+1$ cords, which means that (2.1) is unchanged, as demonstrated below.

$$m + 1 + k + p - (s + p + 1) = m + k - s = 2$$

In the last figure, we have done the same, but now where the cord from *A* to *B* lies outside the mask, but it is easily conceived that the result (2.1) is unchanged.
What we realize is, that whenever we add a mask to a net, subdued to the conditions 1 – 3, the equation (2.1) is unchanged.
We may then create the entire *net* (on the figure to the left) by adding the masks shown, one by one. The relation (2.1) will hold for the complete net, and therefore for any *net* on a sphere that comply to the conditions 1 – 3.

If a sphere encloses a polyhedron, then a central projection of the polyhedron from the centre of the sphere to the surface of the sphere will be a homeomorphism, which conserves the topology. The *sides* in the polyhedron, the *corners* of the polyhedron and the *flat pieces* in the polyhedron will correspond to the *cords*, *knots* and the *masks* of the net with respect to their numbers and topological relations.
Consequently the same relations between *sides*, *corners* and *flat pieces* (polygons) of the polyhedron will hold true as for the *cords*, *knot* and *masks* of the net.
This leads to Euler's Polyhedron theorem.

For any polyhedron, that is homeomorph with a net on a sphere, holds true that the number of flat pieces (polygons) (m), plus the number of corners (k), minus the number of cords(s) are equal to 2.

(2.2) $$m + k - s = 2$$

3. Regular nets on a sphere

A *net* on a sphere is called regular if there exits two numbers λ and μ, both being at least 3, and such that, from each knot leaves λ cords, and the border of each mask consists of μ cords.
We consider a regular *net* on a sphere, where the masks (*m*), knots (*k*) and cords (*s*) bear the same meaning as before. Since every mask is limited by μ cords , and each cord belong to two masks, the following must hold.

$$s = \frac{\mu m}{2} \qquad \text{and} \qquad s = \frac{\lambda k}{2}$$

Which we rewrite as.

$$m = \frac{2s}{\mu} \qquad \text{and} \qquad k = \frac{2s}{\lambda}$$

When inserting these expressions into Euler's polyhedron theorem $m + k - s = 2$, we get:

$$\frac{2s}{\mu} + \frac{2s}{\lambda} - s = 2$$

And by division by $2s$:

(3.1)
$$\frac{1}{\mu} + \frac{1}{\lambda} = \frac{1}{2} + \frac{1}{s}$$

Leading to the inequality:

$$\frac{1}{\mu} + \frac{1}{\lambda} \geq \frac{1}{2}$$

From which follows that both numbers λ and μ cannot be greater than 3, as $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
Furthermore it follows, that none of the numbers λ and μ can be greater than 5, since they are at least 3 and:

$$\frac{1}{\mu} + \frac{1}{\lambda} \leq \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

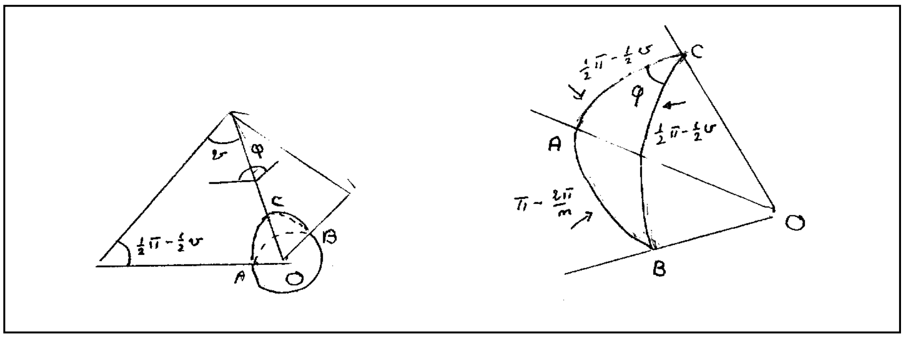
A polyhedron is *regular*, if it consists of congruent regular polygons, so that all angles and all sides in the polygon are equal. For a regular polyhedron, one can easily convince oneself that there are only 5 possibilities for a regular polyhedron, shown below in the table (3.2)

λ	**μ**	***s***	***m***	***k***	***polyhedron***
3	3	6	4	4	Tetrahedron
3	4	12	6	8	Cube
3	5	30	12	20	Dodecahedron
4	3	12	8	6	Octahedron
5	3	30	20	12	Icosahedron

4. The dihedral angle in a regular pyramid

Aiming at calculating the dihedral angle, that is, the angle between two adjacent polygons which form the regular polyhedron, we shall first find the dihedral angle in a regular pyramid. Concerning the tetrahedron, cube and octahedron this can be done using plane geometry, but for the dodecahedron and the icosahedron, it is not possible, since I have found no reference to the issue in the mathematical literature.

I have however found a reference to the subject in a mathematical textbook used for the 3. year of the Danish 9 – 12 grade high school (gymnasium), but it involves spherical geometry in a fairly advanced level [1] .
The trick is to intersect a corner at the base of a pyramid by a sphere, having its centre at a corner of the base of the pyramid. The point of doing this is of course that any corner of a polyhedron can be cut of to form a regular pyramid.
In the two drawings below is shown a corner of a regular pyramid, having *n*-corner surfaces and *n*-edges. The top angle in the pyramid is denoted *v* and the dihedral angle is φ.
As the corner surfaces are isosceles, the angle between a side in the corner surface and a side in the base is $\frac{1}{2}(\pi - v) = \frac{1}{2}\pi - \frac{1}{2}v$.
From the corner *O* is drawn a sphere that intersects the pyramid in the two corner surfaces as well as in the base. So the sphere intersects the pyramid in the two great circle arcs *AC* and *BC,* and it intersects the base in the great circle arc *AB*, which according to (1.1) is equal to $AB = \pi - \frac{2\pi}{n}$.



The arcs $AC = BC = \frac{1}{2}\pi - \frac{1}{2}v$ are the angles in the isosceles triangle, which has the top angle *v*.
The arcs *AC, BC*, *AB* form a spherical triangle with $AC = \frac{1}{2}\pi - \frac{1}{2}v$, $BC = \frac{1}{2}\pi - \frac{1}{2}v$, $AB = \pi - \frac{2\pi}{n}$, and where $C = \varphi$ is the dihedral angle of the pyramid.
The dihedral angle can then be found from the spherical triangle *ABC*, where we first write the cosine-relation in the familiar form, using the sides *a, b, c*, corresponding to the angles *A, B, C.*

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \quad \Leftrightarrow$$

$$\cos AB = \cos BC \cos AC + \sin BC \sin AC \cos \varphi \quad \Leftrightarrow$$

$$\cos(\pi - \tfrac{2\pi}{n}) = \cos(\tfrac{\pi}{2} - v)\cos(\tfrac{\pi}{2} - v) + \sin(\tfrac{\pi}{2} - v)\sin(\tfrac{\pi}{2} - v)\cos\varphi \quad \Leftrightarrow$$

$$-\cos(\tfrac{2\pi}{n}) = \sin^2(\tfrac{v}{2}) + \cos^2(\tfrac{v}{2})\cos\varphi \quad \Leftrightarrow$$

$$\cos\varphi = -\frac{\cos(\frac{2\pi}{n}) + \sin^2(\frac{v}{2})}{\cos^2(\frac{v}{2})}$$

[1] www.olewitthansen.dk Spherical Geometry

From the formula: $\sin\frac{\varphi}{2}=\sqrt{\frac{1-\cos\varphi}{2}}$, derived from $\cos 2x = 1-2\sin^2 x$, we get when inserting the expression for cos φ.

$$\sin\frac{\varphi}{2}=\sqrt{\frac{1}{2}\left(1+\frac{\cos(\frac{2\pi}{n})+\sin^2(\frac{v}{2})}{\cos^2(\frac{v}{2})}\right)}=\sqrt{\frac{\cos^2(\frac{v}{2})+\cos(\frac{2\pi}{n})+\sin^2(\frac{v}{2})}{2\cos^2(\frac{v}{2})}}=\sqrt{\frac{1+\cos(2\frac{\pi}{n})}{2\cos^2(\frac{v}{2})}}$$

Where we have applied the relation: $\cos^2 x+\sin^2 x=1$, and the formula:

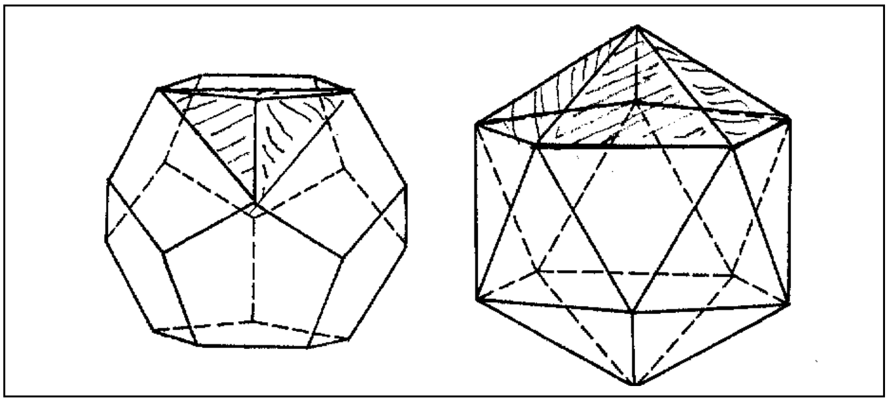
$$\cos 2x = 2\cos^2 x - 1 \quad\Leftrightarrow\quad 1+\cos 2x = 2\cos^2 x$$

In the numerator of the square root. Thus we find:

(4.1) $$\sin\frac{\varphi}{2}=\frac{\cos(\frac{\pi}{n})}{\cos(\frac{v}{2})}$$

5. Evaluation of the dihedral angles and the radii in the inscribed and circumscribed spheres of the five regular polyhedra

The dihedral angle is the angle between two adjacent polygons.
If one cuts a corner of a polyhedron along the edges of the corner one has a regular pyramid, where the top angle is the angle in the regular polygons, that constitute the polyhedron, and where the sides in the base in the pyramid can be found from the regular polygon, and where the pyramid has the same dihedral angle as the regular polyhedron!
This is illustrated below for a dodecahedron and a icosahedron.



We start out from (4.1)

$$\sin\frac{\varphi}{2}=\frac{\cos(\frac{\pi}{n})}{\cos(\frac{v}{2})}$$

n the number of sides in the pyramid, being same as the number of edges from a corner of a regular polyhedron, and as we in section 3 denoted by λ. The number of corners in the polygons that constitutes the polyhedron we have previously denoted μ.

In the formula (4.1) ½v is half the top angle in the pyramid, being the same as the angle in the regular polygon that constitutes the polyhedron According to (1.1) $v = \pi - \frac{2\pi}{n}$.

With these notations, we find for the half of the dihedral angle:

(5.1) $$\sin\frac{\varphi}{2} = \frac{\cos(\frac{\pi}{\lambda})}{\cos(\frac{\pi}{2} - \frac{\pi}{\mu})} \quad \Rightarrow \quad \sin\frac{\varphi}{2} = \frac{\cos(\frac{\pi}{\lambda})}{\sin(\frac{\pi}{\mu})}$$

The values for λ and μ are given in table (3.2), which we for convenience repeat below.

λ	μ	*s*	*m*	*k*	*polyhedron*
3	3	6	4	4	Tetrahedron
3	4	12	6	8	Cube
3	5	30	12	20	Dodecahedron
4	3	12	8	6	Octahedron
5	3	30	20	12	Icosahedron

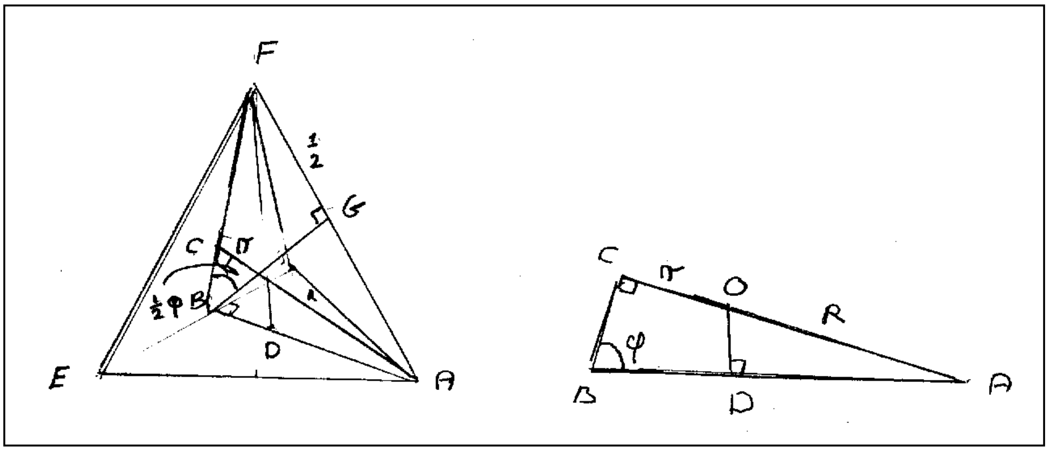
5.1 The Tetrahedron

According to the table above $(\lambda, \mu) = (3,3)$, and we find for the dihedral angle.

(5.2) $$\sin\tfrac{\varphi}{2} = \frac{\cos(\frac{\pi}{3})}{\sin(\frac{\pi}{3})} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3} \Rightarrow \frac{\varphi}{2} = 35{,}27^0$$

$$\varphi_{tetrahedron} = 70{,}54^0$$

The dihedron and the radii for the inscribed and circumscribed spheres may relatively easy be calculated using plane trigonometry. Below is shown a tetrahedron, where some lines have been drawn to that purpose. To the right, the triangle *ABC* has been extracted, which is used to calculate the radii of the inscribed and circumscribed spheres.



For symmetry reasons the heights in a regular triangle are also the medians of the triangle.
If you draw the height from a corner, then the bottom point will lie in the intersection point of the heights in the base triangle.

The centres for the inscribed and circumscribed spheres, will lie in the common intersection of the three heights from the corners. For simplicity we shall put the length of the side in the polyhedra equal to 1.

The height in the regular triangles can be found from Pythagoras: $h = FB = \sqrt{1^2 - (\frac{1}{2})^2} = \dfrac{\sqrt{3}}{2}$

From ΔABF we find the dihedral angle, by drawing the height from B to the opposite edge at G.

(5.3) $$\sin\frac{\varphi}{2} = \frac{AG}{AB} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3} \quad \Rightarrow \quad \frac{\varphi}{2} = 35^0.27 \qquad \varphi_{tetrahedron} = 70.54^0$$

To determine the radii r and R of the inscribed and circumscribed spheres, we apply the two one angled triangles ABC and AOD.

Here $AB = h = \dfrac{\sqrt{3}}{2}$ and since the intersection point of the medians divides the median in the fraction 1:2 then: $AD = \frac{2}{3}AB = \frac{2}{3}\dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{3}}{3}$.

Furthermore: $AC = \sqrt{AF^2 - FC^2} = \sqrt{1^2 - (\frac{2}{3}\frac{\sqrt{3}}{2})^2} = \sqrt{1^2 - \frac{3}{9}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{2}{3}}$

From the two one angled triangles ABC and AOD, we thus find, since $R : r = 2 : 1$.

(5.4) $$\frac{AO}{AB} = \frac{AD}{AC} \quad \Leftrightarrow \quad \frac{R}{\frac{\sqrt{3}}{2}} = \frac{\frac{\sqrt{3}}{3}}{\sqrt{\frac{2}{3}}} \quad \Rightarrow \quad R = \frac{\sqrt{6}}{4}$$
$$r = \tfrac{1}{2}R = \frac{\sqrt{6}}{8}$$

The last equation because O divides AC in the same proportion as D divides AB.

5.1.1 *Surface* and the *volume* of the tetrahedron

The plane *surface* and the *volume* of the tetrahedron, can rather easily be determined, but this time we put the length of the edge to a.

The area of the regular triangle is $T = \frac{1}{2}ah = \dfrac{\sqrt{3}}{4}a^2$, and the surface of the 4 triangles is therefore:

$$O_{tetrahedron} = \sqrt{3}a^2$$

The volume of a pyramid is as it is well known 1/3 height of the pyramid times the area of the base. We have the height $AC = \sqrt{\frac{2}{3}}a$, and the volume becomes:

$$V_{tetrahedron} = \tfrac{1}{3}\sqrt{\tfrac{2}{3}}a\frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{2}}{12}a^3$$

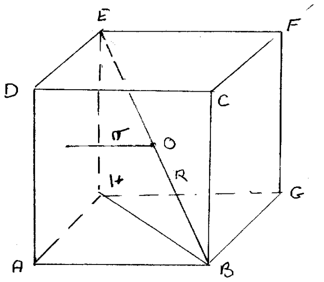
5.2 The Cube

The dihedral angle of the cube is 90^0, which also can be calculated from the formula (5.1)

$$\sin\frac{\varphi}{2} = \frac{\cos(\frac{\pi}{\lambda})}{\cos(\frac{\pi}{\mu})} \quad \text{with} \quad (\lambda, \mu) = (3,4).$$

(5.5) $$\sin\tfrac{\varphi}{2} = \frac{\cos(\frac{\pi}{3})}{\sin(\frac{\pi}{4})} = \frac{\frac{1}{2}}{\frac{\sqrt{2}}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad \frac{\varphi}{2} = 45^0 \qquad \varphi_{cube} = 90^0$$

We put the length of the side in the polygons to 1.
In this case it is very easy to determine the radii *r* and *R* of the inscribed and circumscribed spheres, since their centre is the centre of symmetry, so the radius of the inscribed sphere is
.



(5.6) $$r = \tfrac{1}{2}$$

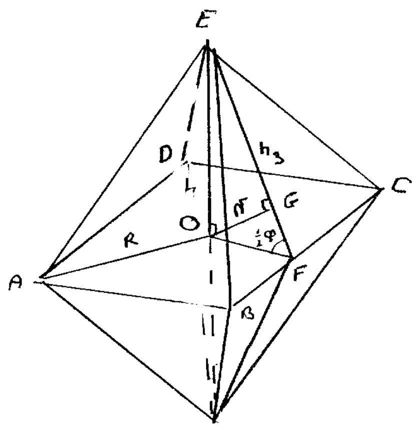
The radius of the circumscribed sphere is simply half the diagonal *BE*. The diagonal of the square is: $BH = \sqrt{1^2 + 1^2} = \sqrt{2}$, and therefore: $BE = \sqrt{BH^2 + EH^2} = \sqrt{\sqrt{2}^2 + 1^2} = \sqrt{3}$, so

(5.7) $$R = \tfrac{1}{2} BE = \frac{\sqrt{3}}{2}$$

The surface and volume of the cube are of course:

$$O_{Cube} = 6a^2 \quad \text{and} \quad V_{Cube} = a^3$$

5.3 The Octahedron



On the figure to the left is shown an octahedron.
We wish to determine the dihedral angle φ and the radii *r* and *R* of the inscribed and circumscribed spheres.
In the figure to the left, we have drawn the height h_3 from *E* on *BC* in the regular triangle *BCE,* and we have drawn the height *h* from *E* on *ABCD.*
The centre *O* the inscribed and circumscribed spheres, lies for symmetry reasons in the centre of the square *ABCD.*
We find the half dihedral angle as the angle *OFE*.

$$EF = \sqrt{1^2 - (\tfrac{1}{2})^2} = \frac{\sqrt{3}}{2} \quad \text{and} \quad OF = \frac{1}{2}$$

Thus we find:

(5.8) $$\cos\tfrac{1}{2}\varphi = \frac{OF}{EF} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3} \Rightarrow \sin\tfrac{1}{2}\varphi = \sqrt{1 - \tfrac{3}{9}} = \sqrt{\tfrac{2}{3}} \quad \Rightarrow \quad \varphi = 109^0{,}48$$

This may be compared with the formula (5.1) $\sin\frac{\varphi}{2} = \dfrac{\cos(\frac{\pi}{\lambda})}{\sin(\frac{\pi}{\mu})}$ med $(\lambda, \mu) = (4,3)$.

$$\sin\frac{\varphi}{2} = \frac{\cos(\frac{\pi}{4})}{\sin(\frac{\pi}{3})} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{3}}{2}} \quad \Rightarrow \quad \varphi = 109^0{,}48$$

Radius in circumscribed sphere is simply half the diagonal of the square *ABCD*.

(5.9) $$R = \tfrac{1}{2}\sqrt{1^2 + 1^2} = \frac{\sqrt{2}}{2}$$

Radius r in the inscribed sphere can be determined from the triangle *OFG*, where $r = OG$, since the point of contact with the sides must be placed in the intersection point of the medians (and the heights) of the regular triangle. We therefore have: $FG = \frac{1}{3}EF = \frac{1}{3}\frac{\sqrt{3}}{2}$.

We then get from the right angle triangle *OFG:*

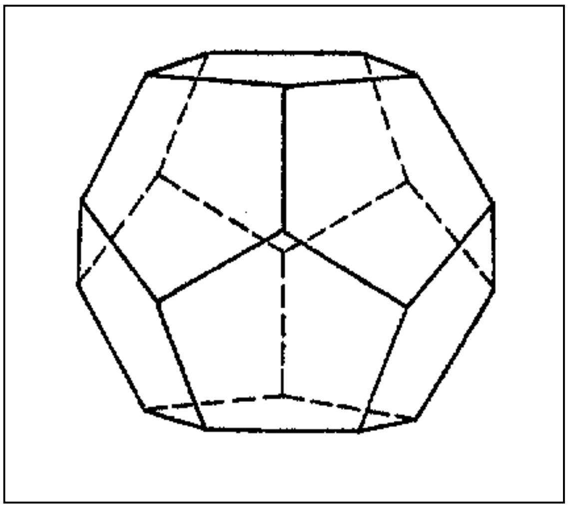
(5.10) $$r = \sqrt{(\tfrac{1}{2})^2 + (\tfrac{\sqrt{3}}{6})^2} = \sqrt{\tfrac{1}{6}}$$

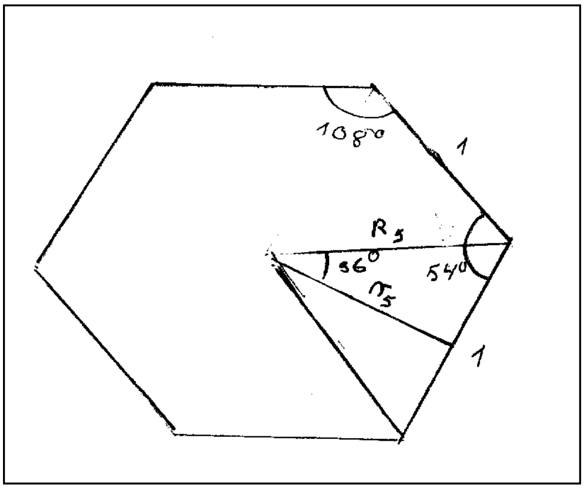
5.3.1 Surface and the volume of the octahedron

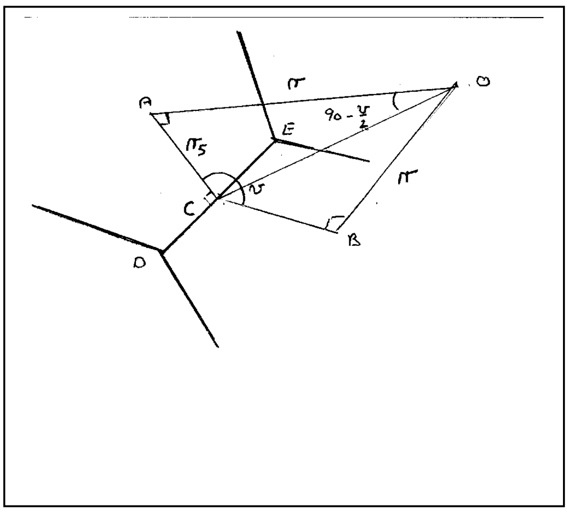
The surface of the octahedron is simply 8 times the area of a regular triangle and the volume of the octahedron is 2 times the volume of a regular 4-sided pyramid with base area a^2. The height of the pyramid equals the radius in the circumscribed sphere.

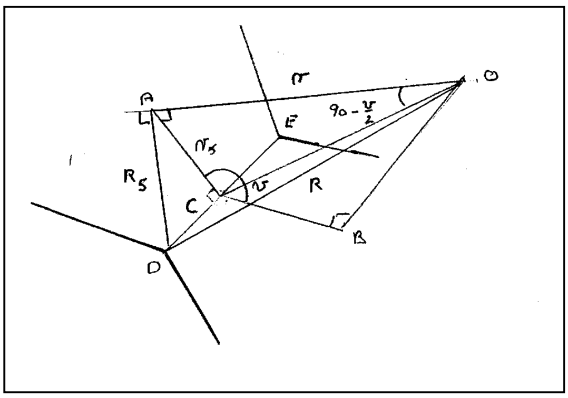
$$O_{octahedron} = 8\frac{\sqrt{3}}{4}a^2 = 2\sqrt{3}a^2 \quad \text{and} \quad V_{octahedron} = \frac{2}{3}\frac{\sqrt{2}}{2}a^3 = \frac{\sqrt{2}}{3}a^3$$

5.4 The Dodecahedron









As remarked earlier, there is probablyno way to find the dihedral angle, and consequently the radii in the inscribed and circumscribed spheres from elementary trigonometry. We therefore resort to formula (5.1).

$$\sin\frac{\varphi}{2}=\frac{\cos(\frac{\pi}{\lambda})}{\sin(\frac{\pi}{\mu})}\quad\text{with } (\lambda,\mu)=(3,5)$$

(5.11) $$\sin\frac{v}{2}=\frac{\cos(\frac{\pi}{3})}{\sin(\frac{\pi}{5})}\quad\Rightarrow\quad v=116^0,60\,.$$

To determine radii in the inscribed and circumscribed spheres, we need to know the distance r_5 from the centre of the inscribed and circumscribed circle of the regular pentagon to the side, and the distance R_5 from this centre to a corner of the pentagon.
As previously, we put the side of the pentagon to 1.
The angle of the pentagon is $\frac{(5-2)180}{5}=108^0$, and half the angle is therefore 54^0.
The centre angle to a side is therefore 72^0, and its half is 36^0. From the drawing of the pentagon at the top to the right is seen:

$$R_5=\frac{\frac{1}{2}}{\sin 36}=\frac{1}{2\sin 36}(=0.85)\qquad\text{og}\quad r_5=\tfrac{1}{2}\tan 54(=0.69)$$

Radius in the inscribed sphere is determined by the 3. figure above to the left, which shows a section of the dodecahedron, with two pentagons intersecting with the dihedral angle. We have drawn the two perpendiculars from the centre *O* for the sphere on the two pentagons having the common edge *DE*. The angle *ACB* is equal to the dihedral angle *v* of the dodecahedron and the angle *O* is *O* =180 – *v*.

The angle *AOC* is therefore 90 - ½*v*. The radius *r = OA=OB,* and from the triangle *AOC* is seen

(5.12) $$r_5=r\tan(90-\tfrac{1}{2}v)\quad\Rightarrow\quad r=r_5\tan\tfrac{1}{2}v\ \ (=0.69\cdot\tan 58{,}30=1.12)$$

From the triangle *ADO ,* shown at the figure to the right, we find:

(5.13) $$R^2=\sqrt{r_5^{\,2}+r^2}\,(=1.32)$$

5.4.1 Surface and the volume of the dodecahedron

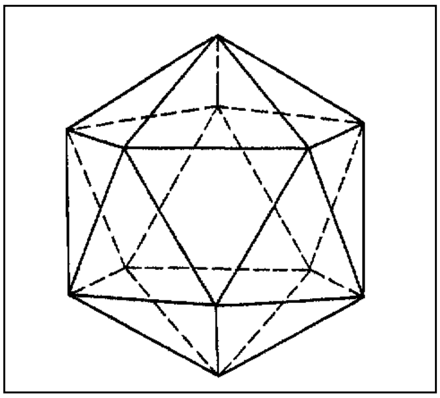
The plane surface of the dodecahedron can be determined as 12 times the area of an regular pentagon which is 5 times a triangular section with the side a: $T_{pentagon}=5\tfrac{1}{2}r_5a\tfrac{1}{2}a=\tfrac{5}{4}a^2\tan 54$

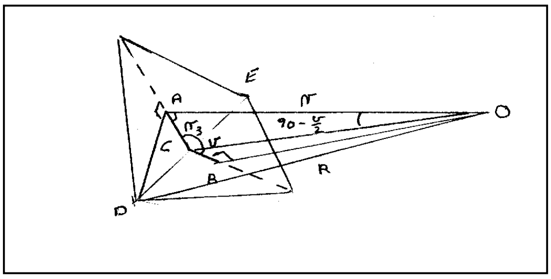
$$O_{dodecahedron}=15\tan 54a^2$$

In the same manner, the volume determined as 12 times the volume of a regular pentagon pyramid with a height equal to the radius of the inscribed sphere.

$$V_{dodecahedron} = 12\tfrac{1}{3}ar\tfrac{5}{4}a^2\tan 54 = 5r\tan 54a^3$$

5.5 The Icosahedron





As it is the case with the dodecahedron, it is not possible to determine the dihedral angle of the icosahedron from pure geometry. We therefore use the formula (5.1).

$$\sin\tfrac{\varphi}{2} = \frac{\cos(\frac{\pi}{\lambda})}{\sin(\frac{\pi}{\mu})} \quad \text{med } (\lambda,\mu) = (5,3).$$

(5.14) $$\sin\tfrac{v}{2} = \frac{\cos(\frac{\pi}{5})}{\sin(\frac{\pi}{3})} = \frac{\cos 36}{\sin 60} \quad \Rightarrow \quad v = 138^0.20.$$

Concerning the radii of the inscribed and circumscribed spheres, we shall apply the same method as we did for the dodecahedron .

$r_3 = AC$ is the radius of the inscribed circle in the regular triangle. $r_3 = \tfrac{1}{3}h_3 = \tfrac{1}{3}\tfrac{\sqrt{3}}{2} = \tfrac{\sqrt{3}}{6}$.

$R_3 = AD$ is the radius in the circumscribed circle in the regular triangle. $R_3 = \tfrac{2}{3}h_3 = \tfrac{2}{3}\tfrac{\sqrt{3}}{2} = \tfrac{\sqrt{3}}{3}$

From the triangle OAC is seen:

(5.15) $$r_3 = r\sin(90 - \tfrac{v}{2}) \quad \Rightarrow \quad r = \frac{r_3}{\cos\frac{v}{2}} (= 0.81)$$

And from the triangle OAD is seen:

(5.16) $$R = OA = \sqrt{(\tfrac{2}{3}h_3)^2 + r^2} = 0.88$$

5.5.1 Surface and the volume of the icosahedron

The plane surface and the volume of the icosahedron are determined in the same manner as for the dodecahedron, as 20 times the area of the regular triangle, and 20 times the volume of a 3 sided pyramid with the height equal to the radius of the inscribed sphere.

$$O_{i\cos ahedron} = 20\frac{\sqrt{3}}{4}a^2 = 5\sqrt{3}a^2$$

$$V_{i\cos ahedron} = 20\tfrac{1}{3}\frac{r_3}{\cos\frac{v}{2}}a\frac{\sqrt{3}}{4}a^2 = 20\cdot\tfrac{1}{3}\frac{\frac{\sqrt{2}}{3}}{\cos\frac{v}{2}}a\frac{\sqrt{3}}{4}a^2$$

$$V_{i\cos ahedron} = \frac{5}{9}\frac{\sqrt{6}}{\cos\frac{v}{2}}a^3$$

This ends our treatment of the five regular polyhedra.