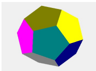
Solution of the third degree equation



Ole Witt-Hansen 2018 (1966)

1. Polynomials of the third degree

Our purpose is to derive a solution formula for the roots of a third degree polynomial.

(1.1) $$P(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

Where the coefficients are real numbers and a_3 is non-zero, while z may be a complex number.

We know that a third degree polynomial always has one real root, due to the fact that a third degree polynomial is a continuous function, which has both positive and negative values, and therefore also must have the value 0.
A third degree polynomial can have 1, 2 or 3 *real* roots, but according to the *fundamental theorem of algebra*, it has always 3 (complex) roots.

Our aim is to derive a formula for finding the 3 roots.

To begin with, we do a simple rewriting, as we remove the second order term, by setting $z = w + a$.

$$P(w) = a_3(w+a)^3 + a_2(w+a)^2 + a_1(w+a) + a_0$$

(1.2) $$\begin{aligned} P(w) &= a_3(w^3 + 3w^2 a + 3wa^2 + a^3) + a_2(w^2 + 2aw + a^2) + a_1(w+a) + a_0 = \\ &= a_3 w^3 + (3aa_3 + a_2)w^2 + (...)w + (...) \end{aligned}$$

If we choose $a = -\dfrac{a_2}{3a_3}$, the second order term disappears, resulting in a third degree polynomial having the form:

(1.3) $$P(w) = a_3(w^3 + pw + q) = a_3 Q(w)$$

Where p and q may be calculated from (1.2). So if $Q(w)$ has the roots w_1, w_2, w_3, then $Q(w)$ may be written as:

$$Q(w) = (w - w_1)(w - w_2)(w - w_3)$$

And thereby

$$P(z) = a_3(z - w_1 - a)(z - w_2 - a)(z - w_3 - a)$$

To determine the roots in an arbitrary third degree polynomial, we may settle for determining the roots in a polynomial of the form:

(1.4) $$P(z) = z^3 + pz + q$$

We can assume that both p and q are non-zero, since otherwise the solution becomes trivial.
Making the substitution: $z = u + v$, we get:

(1.5)
$$P(z) = (u+v)^3 + p(u+v) + q = u^3 + v^3 + 3u^2v + 3uv^2 + p(u+v) + q$$
$$P(z) = u^3 + v^3 + (3uv + p)(u+v) + q$$

Now for each z we may choose u and v, such that: $u + v = z \quad \wedge \quad uv = -\frac{p}{3}$, since these equations just imply that u and v are roots in the quadratic equation: $x^2 - zx - \frac{p}{3} = 0$.

So as you probably remember from high school:

In the ordered and reduces quadratic equation, $x^2 + bx + c = 0$ *the sum of the roots is equal to minus the coefficient to x, and their product is equal to the last term in the equation.*

For this specific choice of u and v, we find:

(1.6)
$$P(z) = u^3 + v^3 + q$$

And the condition that $z = u + v$ is a root is therefore $u^3 + v^3 = -q$.

When two numbers which fulfil the conditions: $u^3 + v^3 = -q$ and $uv = -\frac{p}{3}$ it is equivalent to fulfilling the conditions:

(1.7)
$$u^3 + v^3 = -q \quad \text{and} \quad u^3v^3 = -\frac{p^3}{27}$$

According to the theorem about the sum and the product of the roots in a quadratic equation, this means that u^3 *and* v^3 are roots in the quadratic equation:

(1.8)
$$x^2 + qx - \frac{p^3}{27} = 0 \quad \Leftrightarrow \quad x = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

If $\frac{q^2}{4} + \frac{p^3}{27} < 0$, however, the two roots should be replaced by:

(1.9)
$$x = -\frac{q}{2} \pm i\sqrt{-\frac{q^2}{4} - \frac{p^3}{27}}$$

However we shall continue our writing as if:
$\frac{q^2}{4} + \frac{p^3}{27} \geq 0$, but with the significance above, if $\frac{q^2}{4} + \frac{p^3}{27} < 0$.

We therefore proceed taking the third root of the solution (1.8), as if we were dealing with real numbers.

(1.10)
$$\left.\begin{matrix} u \\ v \end{matrix}\right\} = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

And thus finding the (possible complex) solutions: $z = u + v$

(1.11) $$z = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

To determine the other (possible complex) solutions, we shall recall the binome equation:

(1.12) $$z^n = a$$

Where $z = |z|(\cos x + i \sin x)$ and $a = |a|(\cos v + i \sin v)$

(1.13) $$z^n = a \Leftrightarrow |z|^n (\cos nx + i \sin nx) = |a|(\cos v + i \sin v)$$

If the direction angles for a are: $v + p2\pi, \quad p = 0, 1, 2...$ we get immediately:

$$|z| = \sqrt[n]{|a|} \quad \text{and} \quad nx = v, v + 2\pi, v + 4\pi, v + (n-1)2\pi \quad \Leftrightarrow$$

$$|z| = \sqrt[n]{|a|} \quad \text{and} \quad x = \frac{v}{n} + p\frac{2\pi}{n}, \quad p = 0,1,2...,(n-1)$$

So the complete solution becomes:

(1.14) $$|z| = \sqrt[n]{|a|}(\cos(\frac{v}{n} + p\frac{2\pi}{n}) + i \sin(\frac{v}{n} + p\frac{2\pi}{n})), \quad p = 0,1,2...,(n-1)$$

For the equation $z^3 = a$, where a is a real number, we therefore find:

(1.15) $$z = \sqrt[3]{a}, \quad z = \sqrt[3]{a}(\cos(\frac{2\pi}{3}) + i \sin(\frac{2\pi}{3}), \quad z = (\cos(\frac{4\pi}{3}) + i \sin(\frac{4\pi}{3})$$

Or

$$z = \sqrt[3]{a}, \quad z = \sqrt[3]{a}(-\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}), \quad z = \sqrt[3]{a}(-\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2})$$

It is seen that: $(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) = (-\frac{1}{2} + i\frac{\sqrt{3}}{2})^2$, so we put $\alpha = (-\frac{1}{2} + i\frac{\sqrt{3}}{2})$ and $\alpha^2 = (-\frac{1}{2} - i\frac{\sqrt{3}}{2})$
The three roots in the equation then becomes, (since $\alpha^3 = 1$):

(1.16) $$z = \sqrt[3]{a}, \quad z = \alpha \cdot \sqrt[3]{a}, \quad z = \alpha^2 \cdot \sqrt[3]{a}$$

If we return to the equation: $x = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$, which is equivalent with the two binome equations:

(1.17) $$u^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \quad \text{and} \quad v^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

Then if both right hand sides are real, we find according to (1.16).

(1.18)

$$u_0 = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \vee \quad u_1 = \alpha \cdot \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \vee \quad u_2 = \alpha^2 \cdot \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

$$v_0 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \vee \quad v_1 = \alpha \cdot \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \vee \quad v_2 = \alpha^2 \cdot \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

One might think that it should result in six solutions to the equation $z = u + v$, but we recall that the solutions u and v must obey the equation: $uv = -\frac{p}{3}$. That this applies to $u_0 v_0$ is seen from:

(1.19)

$$u_0 v_0 = \left(\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}\right)\left(\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}\right) =$$

$$\sqrt[3]{\left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)\left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)} = \sqrt[3]{-\frac{p^3}{27}} = -\frac{p}{3}$$

A simple calculation shows that the three solutions, which comply with the claim: $uv = -\frac{p}{3}$ are:

(1.20) $\quad u_0 + v_0 \;, \alpha u_0 + \alpha^2 v_0 \;\; and \;\; \alpha^2 u_0 + \alpha v_0$, where $\alpha = (-\frac{1}{2} + i\frac{\sqrt{3}}{2})$

This follows from $\alpha^2 = (-\frac{1}{2} + i\frac{\sqrt{3}}{2})^2 = (-\frac{1}{2} - i\frac{\sqrt{3}}{2}) \;\; and \;\; \alpha^3 = (-\frac{1}{2} - i\frac{\sqrt{3}}{2})(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = \frac{1}{4} + \frac{3}{4} = 1$.

Finally the solutions to the equation:

$$z^3 + pz + q = 0$$

Are then given by:

(1.21)

$$z = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \vee$$

$$z = \alpha \cdot \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \alpha^2 \cdot \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \vee$$

$$z = \alpha^2 \cdot \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \alpha \cdot \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Since every third degree equation: $a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0$ can be transformed into the equation $z^3 + pz + q = 0$ by the substitution $z \to z + a$ with $a = -\frac{a_2}{3a_3}$, we have proven that every third degree equation has the solutions shown above.

Notice, however, we have used the square root, as if the argument was a positive number, but that is not necessarily the case, which we shall look into now.

We shall then consider the 3 cases:

1) $\frac{q^2}{4}+\frac{p^3}{27}=0$. Here we must have $p < 0$: As u_0 and v_0 we may apply the real value of $\sqrt[3]{-\frac{q}{2}}$, and we therefore find, according to the solution formula above.

(1.22) $z_1 = u_0 + v_0 = 2\sqrt[3]{-\frac{q}{2}}$, together wit the double root: $z_2 = z_3 = -\sqrt[3]{-\frac{q}{2}}$

2) $\frac{q^2}{4}+\frac{p^3}{27}>0$. The numbers $-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}$ are then real, and we may apply the real cubic roots for u_0 and v_0. We then have the one real root $z_1 = u_0 + v_0$ and the two complex conjugate roots.

(1.23)
$$\left.\begin{matrix} z_2 \\ z_3 \end{matrix}\right\} = \left.\begin{matrix} \alpha u_0 + \alpha^2 v_0 \\ \alpha^2 u_0 + \alpha v_0 \end{matrix}\right\} = -\frac{u_0+v_0}{2} \pm i\sqrt{3}\,\frac{u_0-v_0}{2}$$

3) $\frac{q^2}{4}+\frac{p^3}{27}<0$. Here we must have $p < 0$. For u and v, we apply:

$$\left.\begin{matrix} u^3 \\ v^3 \end{matrix}\right\} = -\frac{q}{2} \pm i\sqrt{-\frac{q^2}{4}-\frac{p^3}{27}}\,.$$

It appears that the two roots are complex conjugates, and therefore have the same modulus.

(1.24)
$$\sqrt{\left(-\frac{q}{2}\right)^2+\left(\sqrt{-\frac{q^2}{4}-\frac{p^3}{27}}\right)^2} = \sqrt{-\frac{p^3}{27}}\,.$$

We then get the solutions to u and v, as we notice that:

$$\sqrt[3]{\sqrt{-\frac{p^3}{27}}} = \sqrt{-\frac{p}{3}} = |u| = |v|\,.$$

If we put:

$$-\frac{q}{2} \pm i\sqrt{-\frac{q^2}{4}-\frac{p^3}{27}} = \sqrt{-\frac{p^3}{27}}(\cos\varphi \pm i\sin\varphi), \qquad \textit{where} \qquad \cos\varphi = \frac{-\frac{q}{2}}{\sqrt{-\frac{p^3}{27}}}$$

Then the applicable solutions to the equations becomes.

(1.25)
$$\left.\begin{matrix} u^3 \\ v^3 \end{matrix}\right\} = -\frac{q}{2} \pm i\sqrt{-\frac{q^2}{4}-\frac{p^3}{27}}$$

Equal to:

$$\left.\begin{matrix} u_0 \\ v_0 \end{matrix}\right\} = \sqrt{-\frac{p}{3}}(\cos\tfrac{\varphi}{3} \pm i\sin\tfrac{\varphi}{3})$$

(1.26) $$\left.\begin{matrix} \alpha u_0 \\ \alpha^2 v_0 \end{matrix}\right\} = \sqrt{-\frac{p}{3}}(\cos\tfrac{\varphi+2\pi}{3} \pm i\sin\tfrac{\varphi+2\pi}{3})$$

$$\left.\begin{matrix} \alpha^2 u_0 \\ \alpha v_0 \end{matrix}\right\} = \sqrt{-\frac{p}{3}}(\cos\tfrac{\varphi+4\pi}{3} \pm i\sin\tfrac{\varphi+4\pi}{3})$$

So in this case we get three *real* roots!

(1.27) $$z_1 = 2\sqrt{-\frac{p}{3}}\cos\frac{\varphi}{3}\ ,\quad z_2 = 2\sqrt{-\frac{p}{3}}\cos\frac{\varphi+2\pi}{3}\ ,\quad z_3 = 2\sqrt{-\frac{p}{3}}\cos\frac{\varphi+4\pi}{3}$$

Determining the roots in a concrete example

To determine the numeric solutions to a specific cubic equation e.g. $z^3 - 2z^2 - 5z + 6 = 0$, (which is seen to have the roots {-2,1, 3}), is rarely possible analytically, mostly because of the factor $\cos\frac{\varphi}{3}$.

To express $\cos\frac{\varphi}{3}$ by $\cos\varphi$ one has namely to solve a 3. degree equation in $\cos\frac{\varphi}{3}$.

For the equation: $z^3 - 2z^2 - 5z + 6 = 0$ we find:

$$p = 3(\tfrac{2}{3})^2 - 4(\tfrac{2}{3}) - 5 = \tfrac{4}{3} - \tfrac{8}{3} - 5 = -\tfrac{19}{3}$$

$$q = (\tfrac{2}{3})^3 - 2(\tfrac{2}{3})^2 - 5(\tfrac{2}{3}) + 6 = \tfrac{8}{27} - \tfrac{8}{9} - \tfrac{10}{3} + 6 = \tfrac{8-24-90}{27} + 6 = \tfrac{-106}{27} + 6 = \tfrac{56}{27}$$

$$\cos\varphi = \frac{-\dfrac{q}{2}}{\sqrt{-\dfrac{p^3}{27}}} = \frac{-\frac{56}{54}}{\sqrt{\dfrac{(\frac{19}{3})^3}{27}}}\ .$$

Therefore there seem to be only one way to get the solutions, namely by numerical calculations.

Reference: Børge Jessen. Lectures on complex numbers. Mat 2. 1965 -1966. (Handwritten notes).