The Brachistochrone And The Tautochrone

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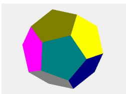

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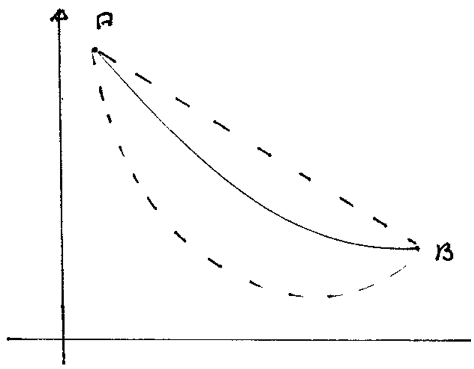
This is probably the most notorious problem first proposed by Johan Bernoulli, and solved by Euler in his invention of the calculus of variations.



The problem is to determine the trajectory a particle will chose, if it without friction should move from a higher position *A* to a lower position *B* in the shortest possible time.
Instantaneously, one might think that the shortest path (being a straight line) also would be the fastest path, but this is not necessarily the case since, if the trajectory is steeper in the beginning, the particle will gain more speed to traverse the rest of the path.
The solution to the problem is in fact rather surprising.

From kinematics, we know, that $ds = vdt$ (distance = velocity x time)

At the same time $ds = \sqrt{1+y'^2}\,dx \quad \Rightarrow \quad dt = \frac{\sqrt{1+y'^2}}{v}dx$

Conservation of energy in a free fall in the gravitational field gives: $\frac{1}{2}mv^2 = mgy \Leftrightarrow v = \sqrt{2gy}$
When inserted in the expression for *dt*, it gives:

(1.1)
$$dt = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}dx$$

Thus the exercise is then to determine the minimum value of the functional:

(1.2)
$$t_{AB} = \int_a^b \sqrt{\frac{1+y'^2}{2gy}}dx$$

Which is a *variation problem* with the function:

(1.3)
$$F(y', y'x) = \sqrt{\frac{1+y'^2}{2gy}}$$

Since *F* does not depend explicitly on *x*, we shall apply the simplified version of the Euler-Lagrange equation: $y'\frac{\partial F}{\partial y'} - F = C$, Where we insert the expression for *F*, omitting the factor $\sqrt{2g}$.

(1.4)
$$y'\frac{2y'}{2\sqrt{y(1+y'^2)}} - \sqrt{\frac{1+y'^2}{y}} = C$$

Multiplying (3.19) by $\sqrt{y(1+y'^2)}$ gives:

$$y'^2-(1+y'^2)=C\sqrt{y(1+y'^2)} \quad \Leftrightarrow \quad -1=C\sqrt{y(1+y'^2)}\,,$$

Taking the square of the equation, it leaves us with the non linear first order differential equation:

(1.5) $$y(1+y'^2)=c$$ (Where c is a new constant)

Since (*tan x*)' = 1 + *tan*^{2}*x* , it is tempting to try with the substitution. *y'= tan θ,*

Then $$1+y'^2 = 1+tan^2\theta = \frac{1}{\cos^2\theta}.$$

When inserting this, the equation becomes: (We have used $2\cos^2\theta = 1+\cos 2\theta$)

(1.6) $$y=\frac{c}{1+y'^2}=c\cdot\cos^2\theta=\frac{c}{2}(1+\cos 2\theta)$$

The substitution *y'= tan θ,* however, only gives one half the parametric of the trajectory.
To obtain $x = x(\theta)$, we shall make the following rewriting:

(1.7) $$\frac{dx}{d\theta}=\frac{dx}{dy}\frac{dy}{d\theta}=-\frac{c}{y'}\sin 2\theta=-c\frac{2\sin\theta\cos\theta}{\tan\theta}=-c2\cos^2\theta=-c(1+\cos 2\theta)$$

To arrive at the sought solution.

(1.8) $$x=-\frac{c}{2}(2\theta+\sin 2\theta)+x_0 \quad and \quad y=\frac{c}{2}(1+\cos 2\theta)$$

Introducing *A=½c* and *t = -2θ* , we can write it in a more common form.

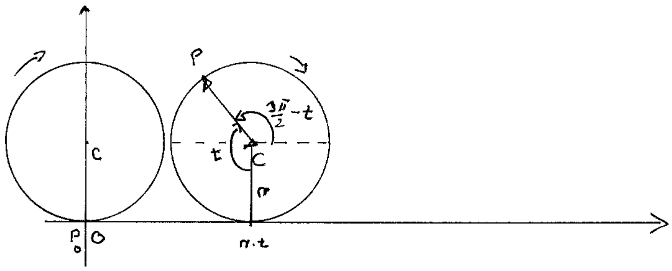
(1.9) $$x = A(t+\sin t)+x_0 \qquad y = A(1+\cos t)$$

This we recognize as the parametric of a cycloid, which is called the Brachistochrone.

The curve is sketched below.
It is the trajectory that a fixed point on a circle rolling on the *x*-axis follows.

If we choose $x_0 = A(½\pi +1)$, then a point , which at $t = ½\pi$, move until $t = \pi$, will move from the position $(x,y) = (0, A)$ to $(x,y) = (A(\pi -1), 0)$.

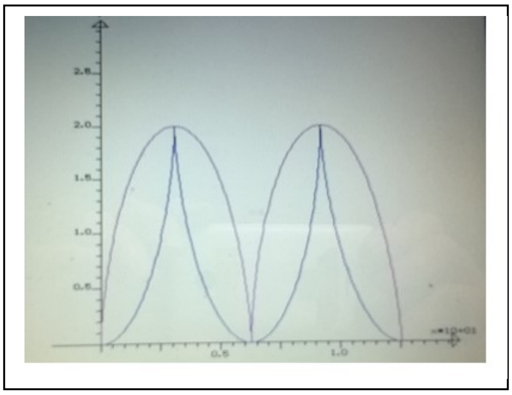
(1.10) **The parametric for the cycloid**



From the figure above, we see: $\vec{OP} = \vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \vec{OC} + \vec{CP} = \begin{pmatrix} rt \\ r \end{pmatrix} + \begin{pmatrix} r\cos(\frac{3}{2}\pi - t) \\ r\sin(\frac{3}{2}\pi - t) \end{pmatrix} = \begin{pmatrix} rt - r\sin t \\ r - r\cos t \end{pmatrix}$

And the parametric becomes: $\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} rt - r\sin t \\ r - r\cos t \end{pmatrix}$

Below is shown the graph of the mathematical cycloid, and the graph for the solution to the differential equation



2. The Tautochrone

The tautochrone is a curve on which a body can roll back and fourth in a free fall of gravity, but in such a way that it takes the same time to reach the bottom, no matter from how high it is started.

This kind of behaviour is well known from the harmonic oscillator, where the period $T = \sqrt{\frac{k}{m}}$ is independent of the amplitude. k is the spring constant.
The equation of motion for the harmonic oscillator is: $m\ddot{x} = -kx$. The force is proportional to the distance from the equilibrium position x, and the potential energy of the spring is $E_{pot} = \frac{1}{2}kx^2$.
When the force that drives the body back and forth is not a spring but gravity, we have a mathematical pendulum.
The lot performs a periodic motion along ac arc of a circle for small fluctuations $\sin\theta \approx \theta$, and therefore the circular arc is for small oscillations a tautochrone.

If we generalize this to an arbitrary curve $\vec{r} = (x, y)$, where $(x, y) = (x(t), y(t))$ and $ds^2 = dx^2 + dy^2$ we again claim that $m\ddot{s} = -ks$, the result will certainly be a tautochrone, but is not palatable to solve the equation for the curve *s*.
Instead one could use the equivalent expression for the potential energy $E_{pot} = \frac{1}{2}ks^2$.
And put it equal to the gravitational potential energy: *mgy*(*s*).

(2.1) $$\tfrac{1}{2}ks^2 = mgy(s) \quad \Leftrightarrow \quad s^2 = \alpha y \quad \text{where } \alpha = \frac{2mg}{k}$$

By taking the differential of both sides:

$$2sds = \alpha dy \quad \Rightarrow \quad 4s^2 ds^2 = \alpha^2 dy^2$$

We then eliminate *s*, since $ds^2 = dx^2 + dy^2$ and $s^2 = \alpha y$

$$4\alpha y(dx^2 + dy^2) = \alpha^2 dy^2$$

Dividing by dy^2:

(2.2) $$4y(\frac{dx^2}{dy^2} + 1) = \alpha \quad \Leftrightarrow \quad (\frac{dx}{dy})^2 = \frac{\alpha - 4y}{4y} \quad \Leftrightarrow \quad \frac{dx}{dy} = \sqrt{\frac{\alpha - 4y}{4y}}$$

Making the substitution: $t^2 = 4y \quad \Rightarrow \quad 2tdt = 4dy \quad \Rightarrow \quad 2dy = tdt$ gives

(2.3) $$dx = \sqrt{\frac{\alpha - 4y}{4y}}dy = \tfrac{1}{2}\sqrt{\frac{\alpha - t^2}{t^2}}tdt = \tfrac{1}{2}\sqrt{\alpha - t^2}\,dt = \tfrac{1}{2}\sqrt{\alpha}\sqrt{1 - (\frac{t}{\sqrt{\alpha}})^2}\,dt$$

Replacing $(\frac{t}{\sqrt{\alpha}})$ by *t'* gives $dt = \sqrt{\alpha}dt'$

$$dx = \tfrac{1}{2}\alpha\sqrt{1 - t'^2}\,dt'$$

We do the integral: $\int_{t_0}^{t} \sqrt{1 - t^2}\,dt$, which is the area under a unit circle, by the substitution $t = \sin u$

$$\int \sqrt{1 - t^2}\,dt = \int \sqrt{1 - \sin^2 u}\,\cos u du = \int \cos^2 u du$$

Since $\cos 2x = 2\cos^2 x - 1 \quad \Rightarrow \quad \cos^2 x = \dfrac{\cos 2x + 1}{2}$, we find:

$$\int \cos^2 u du = \tfrac{1}{2}\int (\cos 2u + 1) = \tfrac{1}{4}\sin 2u + \tfrac{1}{2}u + c$$

$$t' = \sin u \quad \Leftrightarrow \quad u = \sin^{-1} t' = \sin^{-1}(2\sqrt{\frac{y}{\alpha}})$$

$$x = \tfrac{1}{2}\alpha(\tfrac{1}{4}\sin 2(\sin^{-1}(2\sqrt{\frac{y}{\alpha}})) + \tfrac{1}{2}\sin^{-1}(2\sqrt{\frac{y}{\alpha}})) + c$$

If we put $\sin t = 2\sqrt{\frac{y}{\alpha}}$ then:

$$x = \tfrac{1}{2}\alpha(\tfrac{1}{4}\sin 2(\sin^{-1}\sin t) + \tfrac{1}{2}\sin^{-1}(\sin t) + c \qquad \text{and} \qquad y = \tfrac{1}{4}\alpha\sin^2 t$$

$$\cos 2x = 1 - 2\sin^2 x \quad \Rightarrow \quad \sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

(2.4) $$x = \tfrac{1}{2}\alpha(\tfrac{1}{4}\sin 2t + \tfrac{1}{2}t + c) \qquad y = \tfrac{1}{8}\alpha(1 - \cos 2t)$$

Finally putting $2t = \theta$ and $c = 0$ gives:

(2.5) $$x = \tfrac{1}{8}\alpha(\theta + \sin\theta) \qquad \text{and} \qquad y = \tfrac{1}{8}\alpha(1 - \cos\theta)$$

Which is one parametric for the cycloid.

Below is shown the graph of the mathematical cycloid, and the graph for the solution to the differential equation