A theorem on the sum and product of roots in second and third degree polynomials

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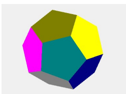

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Contents

1. Factorizing a quadratic polynomial.

A number r is said to be a *root* in a polynomial $p(x)$ if $p(r) = 0$.
Determining the roots of a polynomial is the same as finding the intersection of $p(x)$ with the x-axis.
We shall then show some theorems about the roots in a quadratic polynomial, which as we know, may have two roots if $d > 0$, only one root if $d = 0$ or no roots if $d < 0$.

First we look at the case where $d > 0$, where the polynomial has two roots r_1 and r_2.

r_1 and r_2 are roots in: $p(x) = ax^2 + bx + c \quad \Leftrightarrow \quad ax^2 + bx + c = 0 \quad \Leftrightarrow$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \Leftrightarrow \quad x = r_1 \ \vee \ x = r_2 \quad \Leftrightarrow$$

$$x - r_1 = 0 \ \vee \ x - r_2 = 0 \quad \Leftrightarrow \quad (x - r_1)(x - r_2) = 0$$

By multiplying the parentheses and collecting the terms, we have:

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0$$

If we compare it to: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, we see that we must have:

$$r_1 + r_2 = -\frac{b}{a} \quad og \quad r_1 \cdot r_2 = \frac{c}{a}$$

This can be formulated as:

In the ordered (*by decreasing powers of x*) *and reduced* (*the coefficient to* x^2 *is* 1) *quadratic equation, the sum of the roots is equal to the coefficient to x with opposite sign, and the product of the roots is equal to the last term in the equation.*

This theorem is often used to guess the roots in a quadratic equation

Example

1) Guess the roots in the quadratic equation: $x^2 + 2x - 15$. We should think of two numbers having the sum -2 and the product -15. The only possibility is 3 and -5 (Since the equation can have at most 2 roots)

2) If the roots are not integral numbers it is only a little more difficult.
   $x^2 + \frac{3}{2}x - 1 = 0$. We can see that: $-2 + \frac{1}{2} = -\frac{3}{2}$ *and* $-2 \cdot \frac{1}{2} = -1$, so the roots are -2 and $\frac{1}{2}$.

We have established above that:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = (x - r_1)(x - r_2)$$

Multiplying the equation by a, we find:

$$ax^2 + bx + c = a(x - r_1)(x - r_2)$$

This is called the *factorization* of the quadratic polynomial, and it is a special case of a more comprehensive theorem about factorization of higher degree polynomials .

2. A theorem about the sum and product of the roots in a third degree polynomial.

A third degree polynomial can be written as: $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

The polynomial is said to be normalized if $a_3 = 1$ in which case we have:

$p(x) = x^3 + a_2x^2 + a_1x + a_0$

Clearly, if α is a root in the normalized polynomial, so it is in the polynomial itself.

Let us assume that the polynomial has three roots: $\alpha_1, \alpha_2, \alpha_3$ then:

$$x^3 + a_2x^2 + a_1x + a_0 = 0 \quad \Leftrightarrow \quad x = \alpha_1 \quad \vee \quad x = \alpha_2 \quad \vee \quad x = \alpha_3 \quad \Leftrightarrow$$

$$x - \alpha_1 = 0 \quad \vee \quad x - \alpha_2 = 0 \quad \vee \quad x - \alpha_3 = 0 \quad \Leftrightarrow$$

$$(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) = 0 \quad \Leftrightarrow$$

$$(x^2 - (\alpha_1 + \alpha_2)x + \alpha_1\alpha_2)(x - \alpha_3) = x^3 - (\alpha_1 + \alpha_2 + \alpha_3)x^2 + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)x - \alpha_1\alpha_2\alpha_3$$

Which leads to the theorem:

In a normalized and ordered 3. degree equation, having 3 roots, the sum of the roots is equal to the coefficient to x^2 with opposite sign. And the product of the roots is the constant term with opposite sign.

2.1 Example

A normalized polynomial having the roots $\alpha_1 = 1,\ \alpha_2 = -2,\ \alpha_3 = 3$ *can be written as:*

$p(x) = x^3 - 2x^2 - 5x + 6$

It can be verified directly that $\alpha_1 = 1,\ \alpha_2 = -2,\ \alpha_3 = 3$ *are roots, and that*

$\alpha_1 + \alpha_2 + \alpha_3 = 2$ *and* $\alpha_1\alpha_2\alpha_3 = -6$.

A similar theorem can be obtained from polynomial of higher degree, but they are not interesting, since it does not help much to guessing the roots.

However as it is shown in the section of polynomials that integral roots are divisors in the constant term of the polynomial.