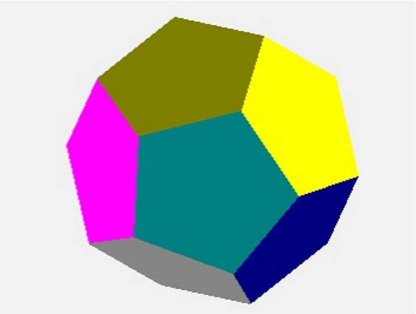
Submarine hunting and the Logarithmic spiral



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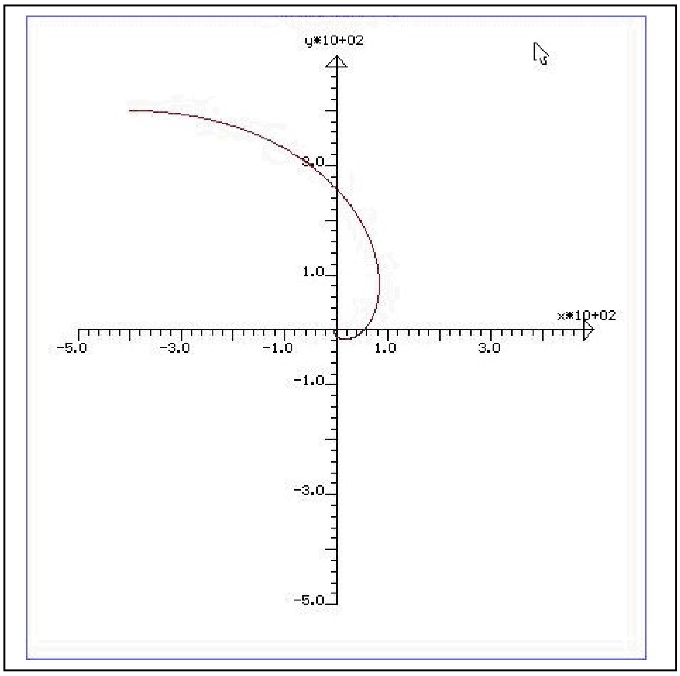
1. The logarithmic spiral.

The logarithmic spiral is a curve, where the radius grows proportional to a parameter *t*, while the rotation angle grows proportional to the logarithm of the parameter *t*.
Because of the logarithmic dependence of the angle, the logarithmic spiral takes almost "infinitely long" time making a round.
Below is shown a computer generated study of the logarithmic spiral, having the parametric equation.

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0.1t\cos\ln t \\ 0.1t\sin\ln t \end{pmatrix}$$



2. Submarine hunting.

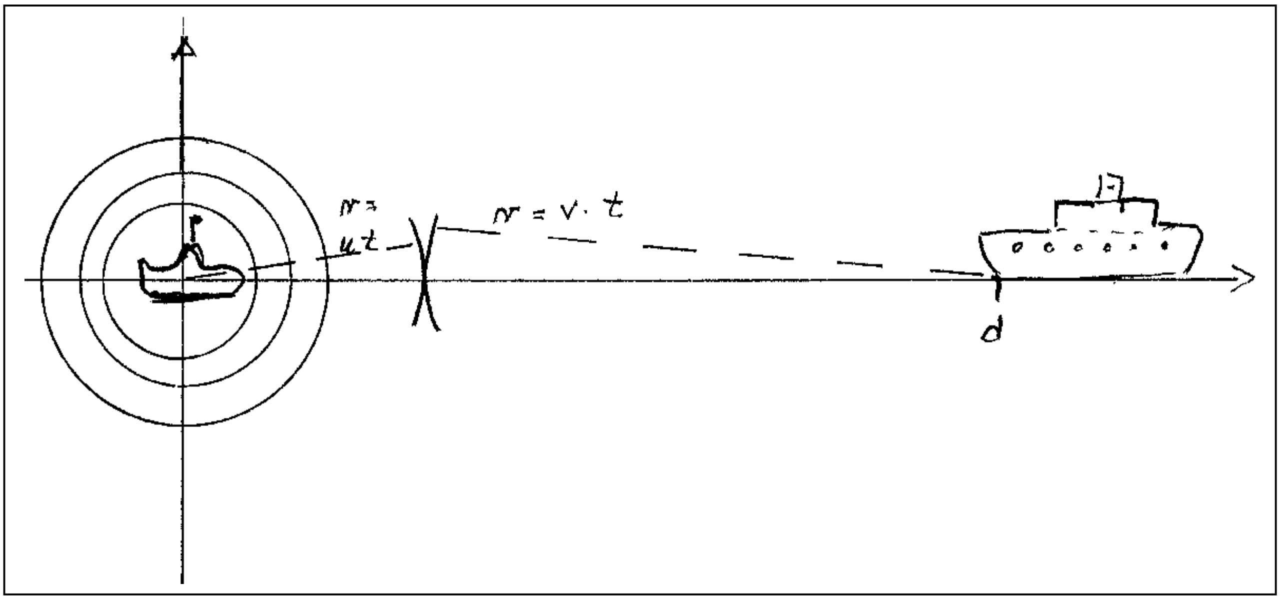
One of the reasons why we are interested in the logarithmic spiral is that this curve appears as a solution to some problems.
This specific problem may be formulated in various ways, but it is less abstract, if we consider a destroyer hunting a submarine.

Let us assume that a destroyer and a submarine have visual contact when they are at a distance *d* from each other.
The submarine dives immediately and takes the flight in a certain but unknown direction and it continues with the speed *u* in that direction without changing course.

The destroyer, however, can sail at speed *v*, and we shall assume that $v > u$.

The problem is this: Is it possible for the destroyer to navigate such that it will meet the submarine, no matter what course the submarine decides for?

The situation is illustrated below where we have also placed a suitable coordinate system.



No matter what, then (according to the assumptions) the submarine will be located on the periphery of a circle with radius $r = ut$.
And the solution for the destroyer is then that is must navigate so that it will always be located at the same periphery as the submarine until it has done one round.

First the destroyer must sail to the point of the circle, where the submarine will be at time t.
But that is easy, since we must have: $ut_0 + vt_0 = d$, so that $t_0 = d/(u+v)$, and the destroyer must sail the distance vt_0 in the direction, where the submarine was observed.
We start our analysis, at that point, where we put $t = 0$.

The task of determining the curve for the destroyer is simplified, if we make our calculations in polar coordinates.
For the position of the destroyer, we then have the coordinates: $(x, y) = (r\cos\varphi, r\sin\varphi)$.
The parametric position of the destroyer is then:

$$f(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} r(t)\cos\varphi(t) \\ r(t)\sin\varphi(t) \end{pmatrix}$$

The radial velocity is $r'(t)$. The arc ds traversed, corresponding to the increment in angle $d\varphi$ is then $ds = r(t)\,d\varphi$.
From this follows that the tangential velocity is: $\frac{ds}{dt} = r(t)\frac{d\varphi}{dt} = r(t)\varphi'(t)$

Since the destroyer at all times must be at the same periphery of a circle as the submarine, its radial velocity must be the same as that for the submarine. Consequently: $r'(t) = u$ or $r(t) = ut$.

The speed of the destroyer is the square root of the squares of the tangential and the radial velocities.

$$v = \sqrt{u^2 + (ut\varphi'(t))^2}$$

If this equation is solved with respect to $\varphi'(t)$, we find:

$$\varphi'(t) = \sqrt{\frac{v^2 - u^2}{u^2}}\frac{1}{t} = \alpha\frac{1}{t} \quad \Rightarrow \quad \varphi(t) = \alpha \ln\frac{t}{t_0} \qquad \text{Where we have put:} \quad \sqrt{\frac{v^2 - u^2}{u^2}} = \alpha$$

At time t_0 the hunt along the circle periphery begins. For simplicity we put $t_0 = 1$, and we find the simple equation:

$$\varphi(t) = \alpha \ln t$$

Inserting this into the parametric for the position of the destroyer, we find:

$$f(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} ut\cos(\alpha \ln t) \\ ut\sin(\alpha \ln t) \end{pmatrix}$$

And we realize that the curve that the destroyer must follow is a *logarithmic spiral*.

We may (with caution) try to estimate how long it will take for the destroyer to make a whole round, and how far away the submarine then has escaped.

We shall therefore assume that $u = 12$ knot and $v = 25$ knot, and we get $\alpha = 1.83$.

We must the solve the equation:

$$\alpha\ln(t) = 2\pi \quad \Leftrightarrow \quad 1.83\ln(t) = 2\pi \quad \Rightarrow \quad t = 30.98$$

and at that time the submarine has sailed 12·30.98 *sm* = 371,8 *sm* = 689 *km*, so this hunting method, although theoretically impeccable, is probably not applicable for the marine.