On queue accumulation on highways and before traffic lights

A theoretical analysis

This is an article from my home page: www.olewitthansen.dk

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1. Introduction

Driving in highways in Europe – especially in Germany – one sometime encounters, what I would call paradoxical queues (rather a cork), where the cars stand still or move with walking speed.

Usually the arise of traffic queues are caused by roadwork, blocking a lane, or traffic accidents, which deliver a natural explanation of the queues, but sometimes it occurs, that a queue is formed, several kilometres long, were the causes are obscure.
On the other hand, when the queue is formed, it is perfectly understandable that it will persist or even grow.
In the more mysterious highway queues, cars are leaving the front of the queue and quickly accelerate to 130 -160 *km/h*, and cars are entering the rear of the queue with the same speed.
But in the queue the cars stand still, irrespectively of the absence of blockings of lanes along the queue.
The question is therefore how a traffic cork can be formed, when there are no external circumstances, that is, road work or accidents that have caused them.

However, the model described below shows that a traffic cork can actually be formed, if the speed of the cars in the highway for some reason or another is slowed down, and thereby the density of cars is increased or if the density of cars are increased, because of an approach lane.

It can appear paradoxical that a queue can stop completely, and at the same time cars in front of the queue and behind the queue drive normally on the highway.

Another, but more understandably phenomenon is the growing queues in front of traffic lights.
And we shall resolve that problem, dependent of how long the green light must be on to avoid growing queues before traffic lights.

2. Definitions and considerations over car traffic

We shall now be occupied with some mathematical models, which concerns arise of traffic queues on highways and before traffic lights. First we introduce some concepts and definitions.

(2.1)

$v = v(x,t)$: The speed of a car at the position x at time t.

$\rho = \rho(x,t)$: The density of cars at the position x at time t.

ρ can e.g. be measured as the number of cars in one lane per 100 *m*.

The number of cars in one lane passing a given position is called the *frequency*, and is denoted f.
If there is more than one lane, the frequency must be multiplied by the number of lanes.

The frequency is equal to the density times the speed:

(2.2) $$f = f(x,t) = \rho\cdot v$$

This is easy to realize, since the number of cars that will pass the position x in the time interval Δt, are the same number of cars being at the length $\Delta s = v\Delta t$ before the position x.

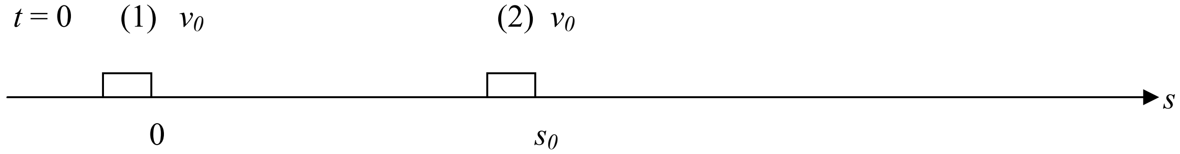
But this number is $\Delta n = \rho\, v\Delta t,$ from which it follows that:

(2.2) $$f = \frac{\Delta n}{\Delta t} = \rho v$$

We shall assume that when you are driving in a queue you are keeping a constant (safety) distance (dependent of your speed) to the car driving in front of you.

A fair assessment of the safety distance is that you should at least drive this distance with speed v in the reaction time t_r, which passes from you register that the car in front of you begins to brake until you hit the brake yourself. The safety distance is thus: $l_s = v\cdot t_r$.

Although this choice of safety distance appears obvious, we shall give a kinematics founded reasoning, that this is the only sensible choice.

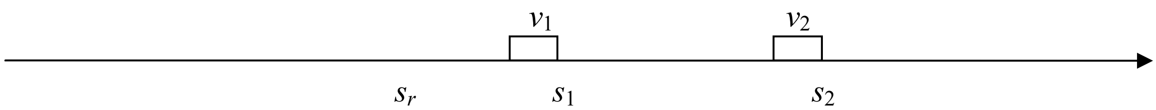


In the figure two cars (1) and (2) are driving with the same speed v_0. (1) has the initial position $s = 0$, while (2) has the position s_0.
At the time $t = 0$, (2) begins to brake, while (1) only starts to brake after the reaction time t_r, and at that time (1) will have the position $s_r = v_0 t_r$.

Now clearly if $s_r = s_0$, and if the two cars have the same braking distance, then (1) will smash into the rear of (2), when (2) has come to rest. Should this be avoided, then the distance between the two cars must at least be $s_r + w = v_0 t_r + w$, where w is the average length of a car.

We shall now using kinematics fix the positions and velocities of the two cars in various critical positions.

$t_r < t < t_2(stop)$



By applying the usual kinematics formulas for the constant accelerated motion with acceleration $-a$, we may write the velocities and positions during the braking of both cars for $t > t_r$.

(2.3)
$$v_1 = v_0 - a(t - t_r) \qquad v_2 = v_0 - a\cdot t$$
$$s_1 = s_r + v_0(t - t_r) - \tfrac{1}{2}a(t - t_r)^2 \qquad s_2 = s_r + v_0\cdot t - \tfrac{1}{2}a\cdot t^2$$

From these equations, we may e.g. read their relative velocity, and the distance between them.

$$v_2 - v_1 = -a\cdot t_r \qquad s_2 - s_1 = s_0 - s_r + v_0\cdot t_r + \tfrac{1}{2}a\cdot t_r^2 - a\cdot t_r\cdot t$$

But since $s_r = v_0\cdot t_r$ we have: $$s_2 - s_1 = s_0 + \tfrac{1}{2}a\cdot t_r^2 - a\cdot t_r\cdot t$$

We notice that their relative velocity is negative, so the two cars approach each other. Also the distance between them $s_2 - s_1$ decreases linearly with time:

(2) has stopped, when: $v_2 = 0 \Leftrightarrow v_0 - at_2 = 0 \Leftrightarrow t_2 = \frac{v_0}{a}$

(1) has stopped, when: $v_1 = 0 \Leftrightarrow v_0 - a(t_1 - t_r) = 0 \Leftrightarrow t_1 = t_r + \frac{v_0}{a}$

When (2) has stopped, the distance between the two cars is:

$$s_2 - s_1 = s_0 + \tfrac{1}{2}at_r^2 - at_r\frac{v_0}{a} = s_0 + \tfrac{1}{2}at_r^2 - v_0 t_r$$

Using $s_r = v_0 t_r$ we have:

$$s_2 - s_1 = s_0 - s_r + \tfrac{1}{2}a \cdot t_r^2.$$

We notice that this distance is positive as long as $s_0 - s_r > 0$.
We can find the positions of the two cars, when both have stopped, but it is easier to use the formulas.

$$2a(s - s_0) = v^2 - v_0^2 \quad with \;\; v = 0$$

(2.4)
$$(1)\;\; s_{stop}(1) - s_r = \frac{v_0^2}{2a} \Rightarrow s_{stop}(1) = s_r + \frac{v_0^2}{2a}$$
$$(2)\;\; s_{stop}(2) - s_0 = \frac{v_0^2}{2a} \Rightarrow s_{stop}(2) = s_0 + \frac{v_0^2}{2a}$$

Thus we end up with the result that we already found in the beginning that when the cars have stopped the distance between the two fronts of the cars is the same as when they began to brake, (but at different times).

$$s_{stop}(2) - s_{stop}(1) = s_0 - s_r \quad \text{Where we have} \;\; s_r = v_0 t_r$$

To avoid harmonica collisions this distance must be bigger than the length of a car.
If w as before denotes the length of a car, then driving in a queue, where everybody holds the same safety distance, there will be exactly one car at the length $vt_r + w$ of the lane.
The density of cars is consequently:

(2.5)
$$\rho(v) = \frac{1}{vt_r + w}$$

We define a (*mathematical*) *queue as a movement of cars, where the relation between speed and density is given by the relation* (2.5) *above.*

For example we can see that $\rho(0) = 1/w$ and $\rho(\infty) = 0$. If the speed is zero then the distance between the cars is also zero, and if the length of a car $w = 5$, then there are 1/5 car per meter.

When dealing with mathematical queues, we shall always assume that the density is given by the expression (2.5). Generally we shall assume that the length of a car is $w = 5$ *m*, while we may adjust the reaction time t_r from 1 *s* to 2 *s*. If we differentiate $\rho(v)$ we have:

$$\rho'(v) = -\frac{t_r}{(vt_r + w)^2}$$

As one could expect the density decreases almost quadratic with the speed.

Hereafter the frequency is given by the formula:

(2.5) $$f(v) = \rho \cdot v = \frac{v}{vt_r + w}$$

2.6 Example
As an example we shall calculate the frequency of cars moving in a queue with the speed 50 *km/h* = 13.9 *m/s*.
We put the reaction time to 1 *s*, and we find: $f(13.9) = 0.73$ cars per second. If the reaction time is 2 *s*, then we have: $f(13.9) = 0.42$ cars per second. The safety distance in the two cases are 13.9 *m* and 27.8 *m*.

The frequency is a function of the form: $f(x) = \frac{x}{ax+b}$ *where* $x \geq 0.$ and a and b are positive numbers. We see immediately that $f(0) = 0$. No cars pass, when the speed is zero. At the same time

$f(x) = \frac{1}{a + \frac{b}{x}} \to \frac{1}{a} = \frac{1}{t_r}$ *for* $x \to \infty$ (At large speed one car passes in the reaction time)

We shall then investigate, how the frequency depends on speed. We assume therefore that the cars drive in a queue with speed *v*, and we shall find the change of the frequency, as the speed of the queue changes. So we differentiate the frequency with respect to the speed.

$$f'(v) = (\rho \cdot v)' = \rho' v + \rho \cdot 1 = -\frac{vt_r}{(vt_r + w)^2} + \frac{1}{vt_r + w} \,,$$

Reduced to:

$$f'(v) = \frac{w}{(vt_r + w)^2}$$

Since $f'(v) > 0$ the frequency is not surprisingly an increasing function of speed.
It is perhaps a bit more surprising that the frequency does not reduce linearly with increasing speed, but much less. Let us first write a general expression for the change in frequency, when the speed is increased/decreased from v_1 to v_2.

$$f(v_2) - f(v_1) = \frac{v_2}{v_2 t_r + w} - \frac{v_1}{v_1 t_r + w} = \frac{w(v_2 - v_1)}{(v_2 t_r + w)(v_1 t_r + w)}$$

Example
As an example we shall find out how much the frequency changes, if the speed decreases from 72 *km/h* = 20 *m/s* to 36 *km/h* = 10 *m/s*. We put the reaction time to 1.5 *s*, and we find from the formula above $\Delta f = 0.071$ cars per second. The initial frequency was 0.57 cars per second. It is surprising how small the change in frequency is.

3. Settlement of traffic queues before traffic lights

The time a traffic light shows red is denoted t_{red} and correspondingly for t_{green}.
We shall assume that cars with speed v are approaching the traffic light with constant frequency $f(v)$

In this approximation there will be $n_{queue} = f(v)\cdot t_{red}$ cars in the queue towards the stop light, and the queue will have the length $l_{queue} = f(v)\cdot t_{red}\cdot w$, where w is the length of a car.

Even for morerately lon queues this cannot be uphold, since cars entering the rear of the queue drives a shorter distance l_{queue} before entering the queue. So the queue will grow faster.
(This corresponds to the Doppler effect for sound).

Corresponding to the frequency f, we have the period T, the time interval with which a car passes a given position. The frequency is the reciprocal of the period.

(3.1) $$f = \frac{1}{T} \Leftrightarrow T = \frac{1}{f}$$

Since the queue is prolonged by w (the length of a car) in each period, then the queue grows opposite to the direction of motion of the cars with the speed:

(3.2) $$v_{queue} = \frac{w}{T} = wf$$

The following car, approaching the queue, must in the period driver a shorter distance w with the velocity v, and it will therefore arrive at the rear end of the queue the time interval $\Delta T = \frac{w}{v}$ earlier.

The true frequency which the queue grows is therefore: $T' = T - \frac{w}{v}$. Since $w = v_{queue}T$, we have:

(3.3) $$T' = T - \frac{v_{queue}}{v}T \quad \Leftrightarrow \quad T' = T(1 - \frac{v_{queue}}{v}),$$

Corresponding to the frequency:

(3.4) $$f' = \frac{f}{(1 - \frac{v_{queue}}{v})}$$

In general it is easier to use the formula using the periods: $T' = T - \frac{w}{v}$

Using these premises we shall establish a formula for how long there must be green light to settle a queue that has been accumulated before a red stoplight. This should be a sufficient condition that there should not accumulate a growing queue before a traffic light.

When the light turns to green, the cars in the queue begin to move, delayed by the time interval t_r, the reaction time (from the driver in the following car register that the car in front of him starts to move until he puts the car in gear and release the clutch). We further assume that all cars accelerate with the same constant acceleration a.
We shall then establish an equation for the time when the *nth* car crosses the stop line and passes the traffic intersection.

The first car in the queue moves at the time t_r after the lights have changed, the next to the time $2t_r$, and so on. The second car must however accelerate the length w to reach the stop line.

The third car must accelerate the length $2w$ to reach the stop line, and the *nth* car must accelerate the length $(n\text{-}1)w$ to reach the stop line. The time Δt_n it takes to move this length can be found from the equation: $s = ½a\Delta t_n^{\,2} = (n-1)w$, which we then solve for Δt_n to give:

(3.5) $$\Delta t_n = \sqrt{\frac{2(n-1)w}{a}}$$

The time t_n , it takes for the *nth* car before crossing the stop line is therefore:

(3.6) $$t_n = nt_r + \sqrt{\frac{2(n-1)w}{a}}$$

If there are n cars in the queue, then $t_{green} > t_n$ for the settlement of the queue before the light changes to red again. If the frequency of cars coming to the traffic light is $f(v)$, then $n = f(v)t_{red}$. So to avoid an accumulation of cars in front of a traffic light, we must have:

(3.7) $$t_{green} = f(v)t_{red}t_r + \sqrt{\frac{2(f(v)t_{red}-1)w}{a}}$$

3.8 Numerical example

Let us assume that the speed towards the traffic light is 60 *km/h* = 16.7 *m/s*, and that the frequency $f(v)$ = 0.5 cars per second. The adjusted period: $T' = T(1-\frac{v_{queue}}{v})$= 1.70 *s*. and the adjusted frequency $f'(v)$ = 0.59 s^{-1}.

When the queue is filled up, there will be $n = f'(v)\cdot t_{red}$ = 17.7 cars in the queue, and it will have the length 88.5 *m*. The acceleration is put to 2.0 m/s^2 and the length of the cars is put to 5.0 *m*. We put the reaction time t_r = 2.0 *s* , and t_{red} = 30 *s*. Inserting in (2.6) we then find t_{green} = 44.4 *s*.

However if the frequency of incoming cars to the traffic light drops to 0.1 per second, we find t_{green} = 9.2 *s*.

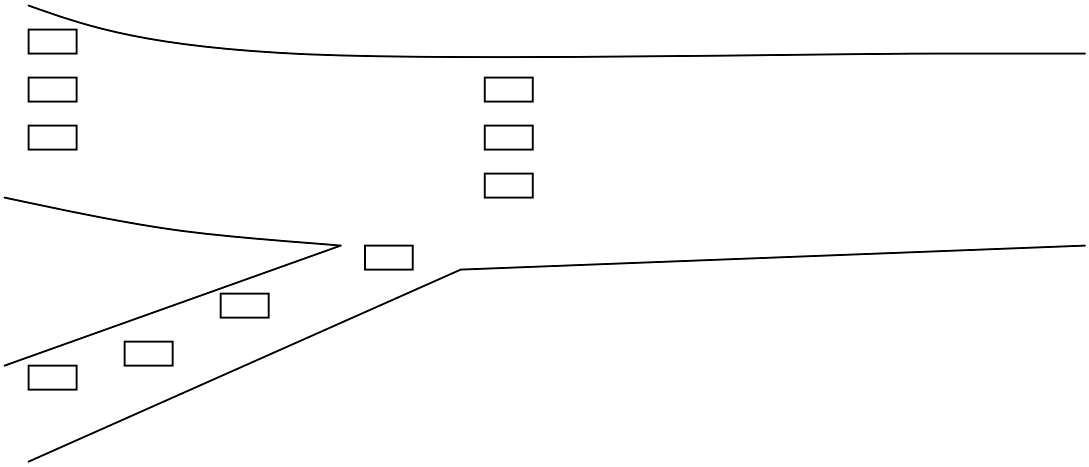
Not surprisingly, the time for the green light to be on to avoid queues depends heavily on the frequency of the incoming cars, but it is always satisfactory to be able to put this trivial observation into concrete numbers.

4. Traffic queues on highways

In contrast to the psychological sensation you sometimes get, the traffic usually does not entirely stops, if the speed is slowed down for some reason, so that a queue is formed. In general it just means that the cars move on, but with a lower speed.

If the speed is lowered or raised from v_1 to v_2, in a queue, then all cars (in the model we use), during acceleration or deceleration move the same distance $s_2 - s_1$ given by $2a(s_2 - s_1) = v_2^{\,2} - v_1^{\,2}$.
In a mathematical queue, the car which follow will start breaking after the reaction time t_r , and in this time have moved vt_r. The braking distance is the same for the two cars, and the speed v_2 is reached after the time interval t_r. with respect to the car ahead.
In this time interval the preceding car has moved the distance v_2t_r and the distance between the fronts of the two cars will be v_2t_r+w, corresponding to a new queue density: $\rho(v) = \frac{1}{vt_r + w}$

But if more cars are added to a queue (only for a shorter time), because of a highway access or because of a blocking of a lane, then the mathematical model (and reality shows) that it can result in a complete standstill of the queue. This situation is illustrated below.



v_a , f_a , v, ρ, f

We consider a situation, where we have a highway with several lanes and an highway access with one lane.(The number of lanes is, however insignificant for the conclusion)
On the highway, prior to the highway access, the cars are assumed to drive with speed v_a and frequency f_a, and the frequency is supposed to be less than the queue frequency. So there is not a queue on the highway (yet). If m is the number of lanes on the highway the frequency is.

(4.1) $$f_a = \frac{m \cdot v_a}{v_a t_r + w} .$$

In the access to the highway we have similarly the speed v_b and frequency f_b, assuming that there is only one lane. After entering the highway the cars are assumed to drive with speed v, density ρ and frequency f. The frequency after entering the highway is therefore:

(4.2) $$f = f_a + f_b$$

If this frequency is less than the queue frequency

(4.3) $$f_{queue} = \frac{m \cdot v}{v t_r + w} ,$$

Corresponding to the speed v, then the speed after the access will be unchanged, and there will not be formed a queue.

We shall then look at some mathematical models on how the speed on the highway will change, if both f_a and f_b are less than the queue frequency, while $f = f_a + f_b$ is bigger than the queue frequency, corresponding to the speed v.

In the first and most simple model, we assume that the frequency at the highway access is constant. We set up the following equations, which has a stepwise progression: n = 1, 2, 3,…in time.
The speed, density and frequency at the "time" n are denoted: $v(n)$, $\rho(n)$ and $f(n)$ and the frequency is given by: $f = f_a + f_b$.

The density after the access is: $\rho(n) = \dfrac{f}{v(n)}$ and this formula is valid, queue or not queue.
If the calculated $\rho(n)$ is less than the queue density at the speed v, then the speed is unaltered otherwise the speed must be calculated from the density: From the formula for queue density.

(4.4) $$\rho = \frac{1}{vt_r + w}$$

it follows:

(4.5) $$v = \frac{1}{t_r}\left(\frac{1}{\rho} - w\right) \quad \text{and} \quad v(n+1) = \frac{1}{t_r}\left(\frac{1}{\rho(n)} - w\right)$$

By inserting in the expression for ρ, we get:

(4.6) $$v(n+1) = \frac{1}{t_r}\left(\frac{v(n)}{f} - w\right) \quad \Rightarrow \quad v(n+1) = \frac{1}{t_r f}v(n) - \frac{w}{t_r}$$

(4.6) is a so called difference equation. Difference equations can in general only be solved with numerical methods, but in some cases they have, (as is the case with differential equations), an analytical solution.

To make a guess on an analytical solution the trick is to consider *n* as an continuous variable, and Use the approximate value for $v(n+1)$: $v(n+1) = v(n)+v'(n)$.

The solution to the resulting differential equation:

(4.7) $$v'(n) = \left(\frac{1}{t_r f} - 1\right)v(n) - \frac{w}{t_r}$$

Appear to be an exponential function plus a constant. We therefore try with: $v(n) = c_1 a^{k \cdot n} + c_2$.
Inserting this in (4.7) gives:

$$c_1 a^{k\cdot(n+1)} + c_2 = \frac{1}{t_r f}(c_1 a^{k\cdot n} + c_2) - \frac{w}{t_r}$$

Putting $a^{k \cdot n}$ outside a parenthesis gives:

(4.8) $$(a^k - \frac{1}{t_r f})c_1 a^{k\cdot n} + c_2 - c_2\frac{1}{t_r f} + \frac{w}{t_r} = 0$$

From which we see, that $v(n) = c_1 a^{k \cdot n} + c_2$ is a solution of the differential equation if and only if

(4.9) $$(a^k - \frac{1}{t_r f}) = 0 \Leftrightarrow a^k = \frac{1}{t_r f} \quad \text{and} \quad c_2 - c_2\frac{1}{t_r f} + \frac{w}{t_r} = 0 \Leftrightarrow c_2 = \frac{wf}{1 - t_r f}$$

Hereafter the solution is:

(4.10) $$v(n) = c_1\left(\frac{1}{t_r f}\right)^n + \frac{wf}{1-t_r f}$$

c_1 is determined by the initial conditions: $v(0) = v_0$., which gives: $c_1 = v_0 - \frac{wf}{1-t_r f}$.

Thus we find the solution:

(4.11) $$v(n) = (v_0 - \frac{wf}{1-t_r f})\left(\frac{1}{t_r f}\right)^n + \frac{wf}{1-t_r f}$$

As you can see the solution is totally dependent on the quantity $t_r f$, that is, the product of the reaction time and the frequency of cars.

If we look at the frequency of a queue, valid for one lane:

(4.12) $$f_{queue} = \frac{v}{vt_r + w} = \frac{1}{t_r + \frac{w}{v}} < \frac{1}{t_r}$$

From the last inequality it follows: $t_r f_{queue} < 1$.

So as long as the frequency fulfils the condition of driving in a queue, then $\frac{1}{t_r f} > 1$, and the two denominators $(1 - t_r f)$ in (4.11) will both be positive. In that case the velocity $v(n)$ will grow exponentially with n.

The formula is however derived from the opposite assumption that $f > f_{queue}$ and in that case the velocity decrease exponentially with n, until it becomes negative. The latter is of course a mathematical fact, since the traffic will stop.

But the model predicts that a continuous increase of the density of cars will eventually bring the traffic to a stop! (Simply because the safety distance requires it)

We shall now look at an modified model, since the assumption that $f = f_a + f_b$, independently of n is not realistic. We therefore assume:

(4.13) $f(n) = f_a(n) + f_b(n)$ and $\rho(n) = \frac{1}{m}\frac{f(n)}{v(n)}$,

where m is the number of lanes, and ρ is the density in each lane. n is the *nth* time step, the size of which we shall not fix. In any case the equations must be solved numerically by iteration

The formulas (,) apply, (queue or no queue). As stated earlier we have:

(4.14) $$\rho_{queue} = \frac{1}{vt_r + w} \quad \Leftrightarrow \quad v_{queue} = \frac{1}{t_r}\left(\frac{1}{\rho} - w\right) = \frac{1}{t_r}\left(\frac{v}{f} - w\right).$$

Given that: $f(n) = f_a(n) + f_b(n)$, and $v(n)$ is calculated from:

(4.15) $$\rho(n) = \frac{1}{m}\frac{f(n)}{v(n)}.$$

If $\rho(v) < \rho_{queue}(v)$, then the speed will be unaltered. In the opposite case, the new queue speed can be calculated from:

(4.16) $$v(n+1) = \frac{1}{t_r}\left(\frac{m \cdot v(n)}{f(n)} - w\right)$$

And the new queue frequency as:

(4.17) $$f(n+1) = m\frac{v(n+1)}{v(n+1)t_r + w}$$

The frequency before the interlacing cannot be greater than after the interlacing, so if $f_a > f$, we put $f_a = f$. Correspondingly the cars coming from the highway access cannot have greater speed than after the interlacing. So if $f(n+1)/m < f_b$, so we set $f_b = f(n+1)/m$. After which the iteration is repeated.

If ρ becomes less than $\frac{1}{w}$ (w is the length of a car), then we put $\rho = \frac{1}{w}$, (the cars are in a row, bumper against bumper), and the speed is zero. The traffic will stand still.

Below is shown the result of applying the model described above.
The iterative solution is done with Excel. The time (iteration step) is on the first axis and the speed on the second axis. The unit is *m/s*.
As it appears then a continuous increase of the frequency ultimately cause the traffic to a stand still. The conclusion is independent of the reaction time. The only thing that matters is the increase of density, caused by a highway access, blocking of a lane or otherwise will increase towards the critical density.