Contents

400. Solve for *x*: $x=\sqrt{3\cdot\sqrt{9\cdot\sqrt{3\cdot\sqrt{9}}}}\cdot....$

This is an infinite radical expression that is repeated after two factors.
We may therefore write:

$$x=\sqrt{3\cdot\sqrt{9\cdot(\sqrt{3\cdot\sqrt{9}}}}\cdot....=\sqrt{3\cdot\sqrt{9\cdot x}}=\sqrt{3\cdot 3\sqrt{x}}=3\sqrt{\sqrt{x}}$$

So $x=3\sqrt{\sqrt{x}} \quad\Rightarrow\quad x^4=81x \quad\Leftrightarrow\quad x^3=81 \quad\Rightarrow\quad x^3=3^4 \quad\Rightarrow\quad x=3\sqrt[3]{3}$

401. Simplify the infinite exponential series of radicals: $\sqrt{2}^{\sqrt{2}^{\sqrt{2}.......}}$

$x=\sqrt{2}^{\sqrt{2}^{\sqrt{2}.......}}=\sqrt{2}^{(\sqrt{2}^{\sqrt{2}.......}}=\sqrt{2}^{x}$

$x=\sqrt{2}^{x}$ It is easily seen that the solution is $x = 2$, since: $\sqrt{2}^{2}=2$

|402. Simplify: $\dfrac{\sqrt{2}+\sqrt{3}+\sqrt{4}}{\sqrt{2}+\sqrt{3}+\sqrt{6}+\sqrt{8}+\sqrt{16}}$

$$\frac{\sqrt{2}+\sqrt{3}+\sqrt{4}}{\sqrt{2}+\sqrt{3}+\sqrt{6}+\sqrt{8}+4}=\frac{\sqrt{2}+\sqrt{3}+2}{\sqrt{2}+\sqrt{3}+\sqrt{2}\sqrt{3}+2\sqrt{2}+4}=\frac{\sqrt{2}+\sqrt{3}+2}{\sqrt{2}+\sqrt{3}+2+\sqrt{2}\sqrt{3}+2\sqrt{2}+2}=$$

$$\frac{\sqrt{2}+\sqrt{3}+2}{(\sqrt{2}+\sqrt{3}+2)+\sqrt{2}(\sqrt{3}+\sqrt{2}+2)}=\frac{1}{1+\sqrt{2}}=\sqrt{2}-1$$

403. Given: $(a+\frac{1}{b})=(b+\frac{1}{c})=(c+\frac{1}{a})$ **Determine** abc

This gives 3 equations and 3 unknowns, which in principle can be solved.

$$(a+\frac{1}{b})=(b+\frac{1}{c}) \text{ and } (b+\frac{1}{c})=(c+\frac{1}{a}) \text{ and } (a+\frac{1}{b})=(c+\frac{1}{a}) \quad\Leftrightarrow$$

$$\frac{ab+1}{b}=\frac{bc+1}{c} \quad and \quad \frac{bc+1}{c}=\frac{ac+1}{a} \quad and \quad \frac{ab+1}{b}=\frac{ac+1}{a} \quad\Leftrightarrow$$

$$abc+c=b^2c+b \quad and \quad abc+a=ac^2+c \quad and \quad abc+b=a^2b+a \quad\Leftrightarrow$$

$$abc=b^2c+b-c \quad and \quad abc=ac^2+c.-a \quad and \quad abc=a^2b+a\ -b \quad\Leftrightarrow$$

$$abc=b^2c+b-c \quad and \quad abc=ac^2+c \quad and \quad abc=a^2b+a\ -b \quad\Leftrightarrow$$

$$3abc=a^2b+b^2c+ac^2$$

Now if $a = b = c$, then any positive integer $a = b = c = n > 0$ will be a solution:

If $a = b = c = 1$ then $3abc = 3$, so: $abc = 1$
If $a = b = c = 2$ then $3abc = 8+8+8$, so: $abc = 8$
If $a = b = c = 3$ then $3abc = 27+27+27$, so: $abc = 27$,…. and so on.

404. Solve for *x*; $x^{-x^2} = \frac{1}{2}$

$x^{-x^2} = \frac{1}{2} \quad \Leftrightarrow \quad \frac{1}{x^{x^2}} = \frac{1}{2}$

As usual these exercises are solved by qualified guesswork. We find out, whether $x = \sqrt{2}$ is a candidate, and:

$x^{x^2} = \sqrt{2}^{\sqrt{2}^2} = \sqrt{2}^2 = 2$, so; $\frac{1}{x^{x^2}} = \frac{1}{2}$

405. $x = \frac{\sqrt{5}+\sqrt{3}}{\sqrt{3}-\sqrt{2}}$ **Determine** $y = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{5}-\sqrt{3}}$

$x = \frac{(\sqrt{5}+\sqrt{3})(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})} = (\sqrt{5}+\sqrt{3})(\sqrt{3}+\sqrt{2})$

We evaluate: $\frac{x}{y} = \frac{(\sqrt{5}+\sqrt{3})}{(\sqrt{3}-\sqrt{2})}\frac{(\sqrt{5}-\sqrt{3})}{(\sqrt{3}+\sqrt{2})} = \frac{5-3}{3-2} = 2$

So: $y = \frac{1}{2}x = \frac{1}{2}(\sqrt{5}+\sqrt{3})(\sqrt{3}+\sqrt{2})$

406. Do the integral: $\int_0^1 \frac{x^3 - x^2 - \frac{1}{2}}{x^2 - x - 1}dx$

We make polynomial division with the denominator:

$x^2 - x - 1 \,|\, x^3 - x^2 - \frac{1}{2} \,|\, x$

$x^3 - x^2 - x$

$x - \frac{1}{2}$

$So: \; \frac{x^3 - x^2 - \frac{1}{2}}{x^2 - x - 1} = x + \frac{x - \frac{1}{2}}{x^2 - x - 1}$

$\int_0^1 \frac{x^3 - x^2 - \frac{1}{2}}{x^2 - x - 1}dx = \int_0^1 x + \frac{x - \frac{1}{2}}{x^2 - x - 1}dx = \left[\frac{1}{2}x^2 + \frac{1}{2}\ln|x^2 - x - 1|\right]_0^1 = \frac{1}{2}$

407. Solve for *x:* $4^x + 18^x = 81^x$

$4^x + 18^x = 81^x \quad \Leftrightarrow \quad (2^x)^2 + 2^x \cdot 9^x = (9^x)^2 \quad \Leftrightarrow \quad 1 + \frac{2^x \cdot 9^x}{(2^x)^2} = \frac{(9^x)^2}{(2^x)^2} \quad \Leftrightarrow$

We put $(\frac{9}{2})^x = y$

$1 + (\frac{9}{2})^x = \left((\frac{9}{2})^x\right)^2 \quad \Leftrightarrow \quad y^2 - y - 1 = 0; \quad d = 1 + 4 = 5 \quad y = \frac{1+\sqrt{5}}{2}$

$$(\frac{9}{2})^x = \frac{1+\sqrt{5}}{2} \quad \Leftrightarrow \quad x = \frac{\ln(\frac{1+\sqrt{5}}{2})}{\ln 9 - \ln 2}$$

408. Simplify: $\left(\sqrt[3]{5+\sqrt{2}} + \sqrt[3]{5-\sqrt{2}}\right)^{2013}$

We put $y = \left(\sqrt[3]{5+\sqrt{2}} + \sqrt[3]{5-\sqrt{2}}\right)$

2013= 3 $2013 = 3 \cdot 671$, so $\left(\sqrt[3]{5+\sqrt{2}} + \sqrt[3]{5-\sqrt{2}}\right)^{2013} = \left(\left(\sqrt[3]{5+\sqrt{2}} + \sqrt[3]{5-\sqrt{2}}\right)^3\right)^{671}$

So we calculate:

$$y^3 = \left(\sqrt[3]{5+\sqrt{2}} + \sqrt[3]{5-\sqrt{2}}\right)^3 = 5+\sqrt{2}+5-\sqrt{2}+3(\sqrt[3]{5+\sqrt{2}})(\sqrt[3]{5-\sqrt{2}})(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}}) =$$

$$10 + 3\sqrt[3]{25-2} \cdot y$$

$$y^3 = 10 + 3\sqrt[3]{25-2} \cdot y \qquad \Leftrightarrow \qquad y^3 - 3\sqrt[3]{23} \cdot y - 10 = 0$$

$$(y-\sqrt[3]{23})^3 = y^3 - 23 - 3\sqrt[3]{23} \cdot y(y-\sqrt[3]{23}) = y^3 - 23 - 3\sqrt[3]{23} \cdot y^2 + 3(\sqrt[3]{23})^2 y$$

$$y^3 - 3\sqrt[3]{23} \cdot y - 10 = (y-\sqrt[3]{23})^3 - (-23 - 3\sqrt[3]{23} \cdot y(y-\sqrt[3]{23})) - 3\sqrt[3]{23} \cdot y - 10$$

409. Solve for *x*: $4^x + 4^{\frac{1}{x}} = 18$

No need for calculation, since it is obvious that $x = 2$ is the solution, but it is quite difficult to show it analytically.

410. For which *n* has $x = \sqrt[3]{n + \sqrt[3]{n + \sqrt[3]{n + \ldots}}}$ **integer solutions.**

It is a infinite series, so $x = \sqrt[3]{n + \sqrt[3]{n + \sqrt[3]{n + \ldots}}} = \sqrt[3]{n+x}$

His equation will have integer solutions for: $n = q^3 - q$ *so the solution is* $x = q$, since:

$\sqrt[3]{n+x} = \sqrt[3]{q^3 - q + q} = x = q$. For example: $n + x = 2^3 - 2 + 2 = 2^3$

411. Solve for *x*; $x^3 - 2x^2 + 1 = 0$ **(The golden section)**

We shall apply two methods, The easiest is to notice that $x = 1$ is a root, and make polynomial division with $x - 1$. Shown below:

$$x^3 - 2x^2 + 1 = (x-1)(x^2 - x - 1)$$

$$x^3 - 2x^2 + 1 = 0 \quad \Leftrightarrow \quad (x-1)(x^2 - x - 1) = 0 \quad \Leftrightarrow \quad x = 1 \quad \vee \quad x^2 - x - 1 = 0$$

$$d = 1 + 4 = 5; \quad x = \frac{1 \pm \sqrt{5}}{2}$$

$x-1 \mid x^3-2x^2+1 \mid x^2-x-1$

x^3-x^2

$-x^2 \quad +1$

$-x^2+x$

$-x+1$

$-x+1$

$(x-1)^3 = x^3-1-3x(x-1) \quad \Rightarrow \quad x^3 = (x-1)^3+3x(x-1)+1 \quad \Rightarrow$

$x^3-2x^2+1=(x-1)^3+3x(x-1)+1-2x^2+1=0 \quad \Leftrightarrow \quad (x-1)^3+3x(x-1)-2(x^2-1)=0$

$(x-1)^3+3x(x-1)-2(x-1)(x+1)=0 \quad \Leftrightarrow \quad (x-1)=0 \quad \vee \quad (x-1)^2 \; +3x-2(x+1)=0$

$x^2+1-2x+3x-2x-2=0 \quad \Leftrightarrow \quad x^2-x-1=0 \quad \Leftrightarrow \quad x=\dfrac{1\pm\sqrt{5}}{2}$

412. Solve for *x*: $\dfrac{27^x+343^x}{63^x+147^x}=\dfrac{37}{21}$

We notice that:

$37=3^3 \quad 343=7^3; \quad 63=7\cdot 3^2; \quad 147=7\cdot 3^2$

$$\frac{27^x+343^x}{63^x+147^x}=\frac{37}{21} \quad \Leftrightarrow \quad \frac{(3^x)^3+(7^x)^3}{7^x(3^x)^2+3^x(7^x)^2}=\frac{37}{21}$$

We shorten the ratio with $(7^x)^3$ to give:

$$\frac{\left(\left(\frac{3}{7}\right)^x\right)^3+1}{\left(\left(\frac{3}{7}\right)^x\right)^2+\left(\frac{3}{7}\right)^x}=\frac{37}{21}$$

We put: $y=\left(\dfrac{3}{7}\right)^x$ and so we get: $\dfrac{y^3+1}{y^2+y}=\dfrac{37}{21}$

$(y+1)^3=y^3+1+3y(y+1) \quad \Rightarrow \quad y^3+1=(y+1)^3-3y(y+1)$

So we have:

$$\frac{(y+1)^3-3y(y+1)}{y(y+1)}=\frac{37}{21} \quad \Leftrightarrow \quad \frac{(y+1)^2-3y}{y}=\frac{37}{21} \quad \Leftrightarrow$$

$21(y^2+1+2y-3y)=37y \quad \Leftrightarrow \quad 21y^2+58y+21=0; \quad d=58^2-4\cdot 21\cdot 21=1600$

$$y=\frac{58\pm 40}{42} \quad \Leftrightarrow \quad y=\frac{98}{42} \quad \vee \quad y=\frac{18}{42} \qquad y=\frac{7}{3} \quad \vee \quad y=\frac{3}{7}$$

Since; $y=\left(\dfrac{3}{7}\right)^x$ We must have: $x=1 \quad or \quad x=-1$

413. Solve for *x*: $3^x + 9^x = 27^x$

$3^x + 9^x = 27^x \quad \Leftrightarrow \quad (3^x)^3 - (3^x)^2 - 3^x = 0$

We put $y = 3^x$ and get:

$$y^3 - y^2 - y = 0 \quad \Leftrightarrow \quad y = 0 \quad \vee \quad y^2 - y - 1 = 0; \quad d = 1 + 4 = 5 \quad \Leftrightarrow \quad y = \frac{1 \pm \sqrt{5}}{2}$$

The only solution is however $y = \frac{1+\sqrt{5}}{2}$ (The golden section), since negative or zero solutions, cannot be applied.

$$y = 3^x = \frac{1 \pm \sqrt{5}}{2} \quad \Rightarrow \quad x = \frac{\ln\left(\frac{1 \pm \sqrt{5}}{2}\right)}{3}$$

414. Solve for *x,y* $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{20}}$

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{20}}.$$

Now $\sqrt{20} = 2\sqrt{5}$, and since 5 is a prime, any integer solutions $\sqrt{x}, \sqrt{y}$, must be a multiple of $\sqrt{5}$, so we write $\sqrt{x} = a\sqrt{5} \quad and \quad \sqrt{y} = b\sqrt{5}$, and we get:

$$\frac{1}{a\sqrt{5}} + \frac{1}{b\sqrt{5}} = \frac{1}{2\sqrt{5}} \quad \Leftrightarrow \quad \frac{1}{a} + \frac{1}{b} = \frac{1}{2}$$

The only integer solutions to this equation is (*x,y*) = (4,4) and (*x,y*) = (3,6), since:

$\frac{1}{a} + \frac{1}{b} = \frac{1}{2} \quad \Leftrightarrow \frac{1}{b} = \frac{1}{2} - \frac{1}{a} = \frac{a-2}{2a}$ has no integer solution for a >4

$$\sqrt{x} = a\sqrt{5} \quad and \quad \sqrt{y} = b\sqrt{5} \quad \Leftrightarrow \quad \sqrt{x} = 4\sqrt{5} \quad and \quad \sqrt{y} = 4\sqrt{5} \quad \Rightarrow \quad x = y = 80 \quad or$$

$$\sqrt{x} = 3\sqrt{5} \quad and \quad \sqrt{y} = 6\sqrt{5} \quad \Rightarrow x = 45 \quad and \quad y = 180$$

415. Solve for *x*. $\sqrt[x]{9} + \sqrt[x]{6} = \sqrt[x]{4}$

First a minor objection: The symbol; $\sqrt[n]{a}$ is only defined for integer n >1 and a non negative: $\sqrt[3]{a}$ where a is negative, should be written as: $-\sqrt[3]{-a}$, so we rewrite the exercise as:

$$9^{\frac{1}{x}} + 6^{\frac{1}{x}} = 4^{\frac{1}{x}} \quad \Leftrightarrow (3^{\frac{1}{x}})^2 + 2^{\frac{1}{x}} \cdot 3^{\frac{1}{x}} = (2^{\frac{1}{x}})^2$$

We put: $a = 2^{\frac{1}{x}} \; and \; b = 3^{\frac{1}{x}}$, and the we have: $b^2 + ab - a^2 = 0$ We divide this equation with:a^2:

$(\frac{b}{a})^2 + \frac{b}{a} - 1 = 0$ Put; $y = \frac{b}{a}$ and we have: $y^2 + y - 1 = 0; \quad d = 1 + 4 = 5; \quad y = \frac{-1 \pm \sqrt{5}}{2}$

Negative y can not be applied, so we have: $y = \frac{-1+\sqrt{5}}{2}$

$$y = \frac{b}{a} = \frac{3^{\frac{1}{x}}}{2^{\frac{1}{x}}} = (\frac{3}{2})^{\frac{1}{x}} = \frac{-1+\sqrt{5}}{2} \quad \Rightarrow \quad x = \frac{\ln 3 - \ln 2}{\ln(\frac{-1+\sqrt{5}}{2})}$$

416. Solve for *x*: $x^{x^2-5x+6} = 1$

$x^{x^2-5x+6} = 1$. Since the equation $x^a = 1$ has the solution a =0, the equation $x^{x^2-5x+6} = 1$ has the solution: $x^2 - 5x + 6 = 0;\quad d = 25 - 24 = 1;\quad x = \frac{5 \pm 1}{2} \quad \Leftrightarrow \quad x = 3 \quad or \quad x = 2$

417. Determine *x* and *y* from: $x^2 - y^2 = 9 \quad and \quad xy = 3$

$$x^2 - y^2 = 9 \quad and \quad xy = 3 \quad \Leftrightarrow \quad x^2 - y^2 = 9 \quad and \quad y = (\frac{3}{x}) \quad \Leftrightarrow \quad x^2 - (\frac{3}{x})^2 = 9 \quad \Rightarrow$$

$$x^4 - 9x^2 - 9 = 0;\quad d = 81 + 36 = 117,\quad x^2 = \frac{9 \pm \sqrt{117}}{2} \quad y^2 = x^2 - 9 = \frac{-9 \pm \sqrt{117}}{2}$$

$$x = \pm\sqrt{\frac{9+\sqrt{117}}{2}} \quad and \quad y = \pm\sqrt{\frac{-9+\sqrt{117}}{2}}$$

418. Solve for *x*: $x^2 + 252x^{-2} = 43$

$$x^2 + 252x^{-2} = 43 \quad \Leftrightarrow \quad x^4 - 43x^2 + 252;\quad d = 43^2 - 4 \cdot 252 = 1849 - 1008 = 841 = 29^2$$

$$x^2 = \frac{43 \pm 29}{2} \quad \Leftrightarrow \quad x^2 = 36 \quad or \quad x^2 = 14 \quad \Leftrightarrow$$

$$x = -6 \quad \vee \quad x = 6 \quad x = \sqrt{14} \quad \vee \quad x = -\sqrt{14}$$

419. Simplify: $\frac{\sin x \cos 3x}{\sin 3x \cos x} = \frac{\tan x}{\tan 3x}$

We shall start to look at the first expression: We need expressions for $\cos(x+y) \quad and \quad \sin(x+y)$

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$

$$\cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x = (\cos^2 x - \sin^2 x)\cos x - 2\sin x \cos x \sin x =$$

$$\cos^3 x - 3\sin^2 x \cos x = \cos^3 x - 3(1 - \cos^2 x)\cos x = 4\cos^3 x - 3\cos x$$

$$\sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x = 2\sin x \cos x \cos x + (\cos^2 x - \sin^2 x)\sin x =$$

$$3\cos^2 x \sin x - \sin^3 x$$

‘

$$\frac{\sin x \cos 3x}{\sin 3x \cos x} = \frac{\sin x(\cos^3 x - 3\sin^2 x \cos x)}{\cos x(3\cos^2 x \sin x - \sin^3 x)}$$

We divide nominator and denominator by: $\cos^2 x \sin^2 x$ to find:

$$\frac{\frac{1}{\tan x}-3\tan x}{3\frac{1}{\tan x}-\tan x}=\frac{1-3\tan^2 x}{3-\tan^2 x}=\frac{9-3\tan^2 x-8}{3-\tan^2 x}=\frac{3(3-\tan^2 x)-8}{3-\tan^2 x}=3-\frac{8}{3-\tan^2 x}$$

420. Simplify: $\frac{x^4+3x^2+4}{x^2+x+2}$

We shall make polynomial division with the denominator:

$$x^2+x+2\,|\,x^4+3x^2+4\,|\,x^2-x+2$$
$$x^4+x^3+2x^2$$
$$-x^3+x^2+4$$
$$-x^3-x^2-2x$$
$$2x^2+2x+4$$
$$2x^2+2x+4$$

So: $\frac{x^4+3x^2+4}{x^2+x+2}=x^2-x+2$

421. Determine $ab+2b$ **from** $2^a=3 \quad and \quad 12^b=8$

$$2^a=3 \quad and \quad 12^b=8 \quad \Leftrightarrow \quad 2^a=3 \quad and \quad (3\cdot 4)^b=2^3 \quad \Leftrightarrow$$

$$2=3^{\frac{1}{a}} \quad and \quad 3^b\cdot(2^b)^2=2^3 \quad \Leftrightarrow \quad 2^b=3^{\frac{b}{a}} \quad and \quad 3^b\cdot(2^b)^2=2^3$$

$$2^b=3^{\frac{b}{a}} \quad and \quad 3^b\cdot(3^{\frac{b}{a}})^2=3^{\frac{3}{a}} \quad \Leftrightarrow \quad 3^{\frac{2b+ab}{a}}=3^{\frac{3}{a}} \quad \Leftrightarrow \quad 2b+ab=3$$

422. Determine *a* and *b* such that: $7^a=3^b=441$

$441=3^2\cdot 7^2$

$$7^a=441 \quad \Leftrightarrow \quad 7=441^{\frac{1}{a}} \quad and \quad 3^b=441 \quad \Leftrightarrow \quad 3=441^{\frac{1}{b}} \quad \Rightarrow$$

$$3\cdot 7=441^{\frac{1}{b}}\cdot 441^{\frac{1}{a}}=411^{\frac{1}{a}+\frac{1}{b}}=(3^2\cdot 7^2)^{\frac{1}{a}+\frac{1}{b}} \quad \Leftrightarrow \quad \frac{(3\cdot 7)^{2(\frac{1}{a}+\frac{1}{b})}}{3\cdot 7}=1 \quad \Leftrightarrow$$

$$(3\cdot 7)^{2(\frac{1}{a}+\frac{1}{b})-1}=1 \quad \Leftrightarrow \quad 2(\frac{1}{a}+\frac{1}{b})-1=0 \quad \Leftrightarrow \quad (\frac{1}{a}+\frac{1}{b})=\frac{1}{2}$$

The last equation has the solution: $a=b=4 \quad or \quad a=3 \quad and \quad b=6$, since;

$\frac{1}{b}=\frac{1}{2}-\frac{1}{a}=\frac{a-2}{2a} \quad \Leftrightarrow \quad b=\frac{2a}{a-2}$ This equation has no integer solution for a >4.

423. Determine *x* from: $x^{x^4}=64$

These kinds of equations require some kind of guesswork. It seems that the number 4 is a key number. so we could try with 2. But: $2^{2^4}=2^{16}$. $\sqrt{2}$ gives a to small number.

We try a tentative solution: $\sqrt[4]{a}$

A middle number that involves the number 4, is $\sqrt[4]{a}$, and indeed:

$(\sqrt[4]{a})^{\sqrt[4]{a}^4} = (\sqrt[4]{a})^a$ if $(\sqrt[4]{a})^a$ =64 then a = 8, so the solution is $x = \sqrt[4]{8}$

424. Evaluate $\dfrac{\sqrt{x\sqrt{x\sqrt{x.......}}}}{x}$

We put: $y = \dfrac{\sqrt{x\sqrt{x\sqrt{x.......}}}}{x} \Leftrightarrow yx = \sqrt{x\sqrt{x\sqrt{x.......}}}$

Since $\sqrt{x\sqrt{x\sqrt{x.......}}}$ is an infinite series we must have; $yx = \sqrt{x\sqrt{x\sqrt{x.......}}} = \sqrt{xyx}$

$$yx = \sqrt{xyx} \Leftrightarrow yx = x\sqrt{y} \Leftrightarrow y = \sqrt{y} \Leftrightarrow y = 1$$

And therefore: $\dfrac{\sqrt{x\sqrt{x\sqrt{x.......}}}}{x} = 1$

425. Determine *x* in the finite series.

$$\frac{\sqrt{8}-\sqrt{7}}{(\sqrt{7}+\sqrt{8})(\sqrt{8}-\sqrt{7})} + \frac{\sqrt{9}-\sqrt{8}}{(\sqrt{9}+\sqrt{8})(\sqrt{9}-\sqrt{8})} \cdots \frac{\sqrt{x+1}-\sqrt{x}}{(\sqrt{x}+\sqrt{x+1})(\sqrt{x+1}-\sqrt{x})} = 8\sqrt{7}$$

$$\frac{\sqrt{8}-\sqrt{7}}{(8-7)} + \frac{\sqrt{9}-\sqrt{8}}{(8-7)} + \ldots + \frac{\sqrt{x+1}-\sqrt{x}}{x+1-x} = 8\sqrt{7} \Leftrightarrow$$

$$\sqrt{8}-\sqrt{7}+\sqrt{9}-\sqrt{8}+\sqrt{10}-\sqrt{9}+\ldots+\sqrt{x+1}-\sqrt{x} = 8\sqrt{7} \Leftrightarrow \sqrt{x+1}-\sqrt{7} = 8\sqrt{7} \Leftrightarrow$$

$$\sqrt{x+1}-\sqrt{7} = 8\sqrt{7} \quad \sqrt{8}-\sqrt{7}+\sqrt{9}-\sqrt{8}+\sqrt{10}-\sqrt{9}+\ldots+\sqrt{x+1}-\sqrt{x} = 8\sqrt{7} \Leftrightarrow \sqrt{x+1}-\sqrt{7} = 8\sqrt{7} \Leftrightarrow$$

$$\sqrt{x+1} = 9\sqrt{7} \Leftrightarrow x = 81\cdot 7 - 1 \Leftrightarrow x = 559$$

426. Determine *a* and *b* from: $a + 2ab + b = 22$

Actually it is easy to guess: that a = 2 and b = 4, since 2+ 16+4 = 22

$(a+1)(b+1) = a + ab + b + 1$, but $(2a+1)(2b+1) = 2a + 4ab + 2b + 1 = 45$,so

$2a + 4ab + 2b + 1 = (2a+1)(2b+1) = 2a + 4ab + 2b + 1 = 45$,

$45 = 5 \cdot 9$, so we might guess: $2a + 1 = 5 \quad and \quad 2b + 1 = 9 \Leftrightarrow a = 2 \quad and \quad b = 4$, or

$2a + 1 = 15 \quad and \quad 2b + 1 = 3 \Leftrightarrow a = 7 \quad and \quad b = 2$, or

427.Solve for *x*. $x^2 - 3 = \sqrt{x+3}$

No need to square to a fourth degree algebraic, which can be difficult to solve, since the obvious solution is x = -2. Since: $(-2)^2 - 3 = \sqrt{-2+3}$.

428. Solve for x and y: $2^{x-2} + 2^{y-2} = 2016$

$$2^{x-2} + 2^{y-2} = 2016 \Leftrightarrow 2^x + 2^y = 4 \cdot 2016 = 8064$$

To identify x, we notice that: $2^{10} = 1024 \quad and \quad 2^{13} = 8 \cdot 1024 = 8192$.
8192 – 8064 =128 = 2^7, so the solution to $2^{x-2} + 2^{y-2} = 2016$ is x = 13 and y = 7

429. Simplify: $\dfrac{\sin^2 77 - \cos^2 13}{\sin^2 13 + \sin^2 77}$

Frankly, I don't understand this exercise, since: $\cos v = \sin(90 - v) \quad and \quad \sin v = \cos(90 - v)$, so
$\sin 77 = \cos(90 - 77) = \cos 13 \quad \cos 77 = \sin(90 - 77) = \sin 13$.
We there have:

$$\frac{\sin^2 77 - \cos^2 13}{\sin^2 13 + \sin^2 77} = \frac{\cos^2 13 - \cos^2 13}{\sin^2 13 + \cos^2 13} = \frac{0}{1} = 0$$

430. Solve for *x*: $\sqrt[x]{36} + \sqrt[x]{24} = \sqrt[x]{16}$

$36 = 2^2 \cdot 3^2; \quad 24 = 2^3 \cdot 3; \quad 16 = 2^4$

$$\sqrt[x]{36} + \sqrt[x]{24} = \sqrt[x]{16} \quad \Leftrightarrow \quad (2^2 \cdot 3^2)^{\frac{1}{x}} + (2^3 \cdot 3)^{\frac{1}{x}} = (2^{\frac{1}{x}})^4 \quad \Leftrightarrow$$

$$\left(2^{\frac{1}{x}}\right)^2 \cdot \left(3^{\frac{1}{x}}\right)^2 + \left(2^{\frac{1}{x}}\right)^3 \cdot \left(3^{\frac{1}{x}}\right) = \left(2^{\frac{1}{x}}\right)^4$$

We put: $a = 2^{\frac{1}{x}} \quad and \quad b = 3^{\frac{1}{x}}$ and then we get a more friendly equation:
$a^2b^2 + a^3b = a^4$ We then divide this equation with a^2b^2.

$$1 + \frac{a}{b} - (\frac{a}{b})^2 = 0$$

We put: $y = \dfrac{a}{b}$ and find: $y^2 - y - 1 = 0 \quad ; \quad d = 1 + 4 = 5 \quad y = \dfrac{1 \pm \sqrt{5}}{2}$

$$y = \frac{2^{\frac{1}{x}}}{3^{\frac{1}{x}}} = \left(\frac{2}{3}\right)^{\frac{1}{x}} = \frac{1 + \sqrt{5}}{2} \quad \Rightarrow \quad x = \frac{2(\ln 2 - \ln 3)}{\ln(1 + \sqrt{5})}$$

431. Determine integers *a,b such that* $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{ab} = \dfrac{1}{4}$

It seems not possible to guess positive integers such that the equation is satisfied. This makes it more complicated to 'guess' a solution. So we make the following rewriting.

$(\frac{1}{a} + 1)(\frac{1}{b} + 1) = \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} + 1$, so

$\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{ab} = \dfrac{1}{4} \quad \Leftrightarrow \quad (\dfrac{1}{a} + 1)(\dfrac{1}{b} + 1) = \dfrac{5}{4}$ This leaves little choice for the two integer factors:

$(\dfrac{1}{a} + 1) = \dfrac{1}{2} \quad and \quad (\dfrac{1}{b} + 1) = \dfrac{5}{2} \quad \Leftrightarrow \quad a = -2 \quad and \quad b = \dfrac{2}{3}$, and indeed:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{ab} = -\frac{1}{2} + \frac{3}{2} - \frac{3}{4} = \frac{3}{2} - \frac{5}{4} = \frac{1}{4}$$

432. Solve for *x*: $x^2-3=\sqrt{x+3}$

This equation requires a lot of calculation if we square it to a fourth degree algebraic equation, how ever, it is easily seen that x = -2 is a solution, since:

$(-2)^2-3=\sqrt{-2+3}$

433. Solve for *x*; $\left(\frac{\sqrt{2x-1}}{x-1}\right)^2-\left(\frac{\sqrt{2x+1}}{x+1}\right)^2=4$

$$\left(\frac{\sqrt{2x-1}}{x-1}\right)^2-\left(\frac{\sqrt{2x+1}}{x+1}\right)^2=4 \quad\Leftrightarrow\quad \frac{2x-1}{(x-1)^2}-\frac{2x+1}{(x+1)^2}=4 \quad\Leftrightarrow$$

$$(2x-1)(x+1)^2-(2x+1)(x-1)^2=4(x+1)^2(x-1)^2 \quad\Leftrightarrow$$

$$(2x-1)(x^2+1+2x)-(2x+1)(x^2+1-2x)=4(x^2-1)^2 \quad\Leftrightarrow$$

$$(2x^3+2x+4x^2-x^2-1-2x)-(2x^3+2x-4x^2+x^2+1-2x)=4x^4+4-8x^2$$

$$6x^2-2=4x^4+4-8x^2 \quad\Leftrightarrow\quad 4x^4-\ 14x^2+6=0 \quad\Leftrightarrow$$

$$2x^4-\ 7x^2+3=0; \quad d=49-24=25.$$

$$x^2=\frac{7\pm5}{4} \quad\Leftrightarrow\quad x^2=3 \quad or \quad x^2=\tfrac{1}{2} \quad\Leftrightarrow\quad x=\pm\sqrt{3} \quad or \quad x=\pm\frac{\sqrt{2}}{2}$$

However the negative roots do not apply: so we have the solutions:

$$x=\sqrt{3} \quad or \quad x=\frac{\sqrt{2}}{2}$$

434. Solve for *x*. $x^{\frac{1}{\sqrt{x}}}=\frac{1}{16}$

These exercises require some guesswork. A candidate could be $x=\frac{1}{4}$, and indeed, and we notice:

$$\sqrt{\tfrac{1}{4}}=\tfrac{1}{2} \quad\Rightarrow\quad \frac{1}{\sqrt{\frac{1}{4}}}=\frac{1}{\frac{1}{2}}=2$$

$$(\tfrac{1}{4})^{\frac{1}{\sqrt{\frac{1}{4}}}}=(\frac{1}{4})^2=\tfrac{1}{16}$$

435. Solve for *x*: $x^{\sqrt{x}}=\frac{1}{2}$

Guesswork, but $x=\frac{1}{4}$ seems a obvious candidate: Since: $(\frac{1}{4})^{\sqrt{\frac{1}{4}}}=(\frac{1}{4})^{\frac{1}{2}}=\sqrt{\frac{1}{4}}=\frac{1}{2}$

436. Show that: $2\sqrt{2+\sqrt{3}}=\sqrt{2}+\sqrt{6}$

We show that $(\sqrt{2}+\sqrt{6})^2=4(2+\sqrt{3})$

$(\sqrt{2}+\sqrt{6})^2=2+6+2\sqrt{2}\sqrt{6}=8+2\sqrt{2}\sqrt{2}\sqrt{3}=8+4\sqrt{3}=4(2+\sqrt{3})$, so

$(\sqrt{2}+\sqrt{6})=2\sqrt{2+\sqrt{3}}$

437. Solve for *x*: $x^{\sqrt{x}} = \sqrt{x^x}$

We take the square of both sides:

$$(x^{\sqrt{x}})^2 = (\sqrt{x^x})^2 \quad \Leftrightarrow \quad x^{2\sqrt{x}} = x^x$$

It is rather obvious that if $x = 4$, then the two exponents are equal, since: $4^{2\sqrt{4}} = 4^4$

438. Determine *b* from: $\log_{12} x = 27 \quad and \quad \log_6 x = b$

$$y = \log_a x \quad \Leftrightarrow \quad x = a^y \quad \Rightarrow \log_b x = y\log_b a = \log_a x \log_b a \quad \Rightarrow$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$b = \log_6 x = \frac{\log_{12} x}{\log_{12} 6} = \frac{27}{\log_{12} 6} = \frac{27}{\log_{12} \frac{12}{2}} = \frac{27}{1 - \log_{12} 2} ???$$

439. Solve for *x*: $\log_8 x - \log_{16} x = 2$

$$\log_8 x - \log_{16} x = 2$$

$$\log_{16} x = \frac{\log_8 x}{\log_8 16}$$

$$\log_8 16 = \log_8 (2 \cdot 8) = \log_8 2 + \log_8 8 = \tfrac{1}{3}\log_8 2^3 + 1 = \tfrac{4}{3}$$

$$\log_8 x - \log_{16} x = 2 \quad \Leftrightarrow \log_8 x - \tfrac{3}{4}\log_8 x = 2 \quad \Leftrightarrow$$

$$\tfrac{1}{4}\log_8 x = 2 \quad \Leftrightarrow \log_8 x = 8 \quad \Leftrightarrow \quad x = 8^8$$

440. Solve for *x*: $x^x = 2^{\frac{1}{x}}$

$$x^x = 2^{\frac{1}{x}} \quad \Leftrightarrow \quad (x^x)^x = (2^{\frac{1}{x}})^x \quad \Leftrightarrow \quad x^{x^2} = 2$$

From which we see that: $x = \sqrt{2}$ since: $\sqrt{2}^{\sqrt{2}^2} = \sqrt{2}^2 = 2$

441. Find the maximum value of $f(x) = \dfrac{x^2 - 4x + 7}{x^2 + 2x + 5}$

First we notice that $f(x) \to 1 \quad for \quad x \to \pm\infty$, since:

$$f(x) = \frac{x^2 - 4x + 7}{x^2 + 2x + 5} = \frac{1 - \frac{4}{x} + \frac{7}{x^2}}{1 + \frac{2}{x} + \frac{5}{x^2}} \to 1 for \quad x \to \pm\infty$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \qquad f'(x) = \frac{(2x-4)(x^2+2x+5) - (x^2-4x+7)(2x+2)}{(x^2+2x+5)^2}$$

$$\frac{2x^3 + 4x^2 + 10x - 4x^2 - 8x - 20 \quad - (\, 2x^3 + 2x^2 - 8x^2 - 8x + 14x + 14)}{(x^2+2x+5)^2} = \frac{6x^2 + 4x - 24}{(x^2+2x+5)^2}$$

$f'(x) = 0 \quad \Leftrightarrow \quad 6x^2 + 4x - 24 = 0 \quad \Leftrightarrow \quad 3x^2 + 2x - 12 = 0; \quad d = 4 + 144 = 148 = 2 \cdot 74 = 2 \cdot 2 \cdot 37$

$x = \dfrac{-2 \pm 2\sqrt{37}}{6} = \dfrac{-1 \pm \sqrt{37}}{3} \Leftrightarrow \quad x = \dfrac{-1-\sqrt{37}}{3} \approx -2.36 \quad or \quad x = \dfrac{-1+\sqrt{37}}{3} \approx 1.69$

f'(*x*) + - +
------------ -2.36------------------1.69---------------
x

We see that there is a max for *x* = -2.36 and a min for *x* = 1.69, so the maximum for *f*(*x*) is $f(\dfrac{-1-\sqrt{37}}{3})$

442. Simplify: $\left(\sqrt[3]{\sqrt{5}+2} + \sqrt[3]{\sqrt{5}-2}\right)^{2014}$

We put $y = \sqrt[3]{\sqrt{5}+2} + \sqrt[3]{\sqrt{5}-2}$ and calculate y^3

$y^3 = \left(\sqrt[3]{\sqrt{5}+2} + \sqrt[3]{\sqrt{5}-2}\right)^3 = \left(\sqrt[3]{\sqrt{5}+2}\right)^3 + \left(\sqrt[3]{\sqrt{5}-2}\right)^3 + 3\left(\sqrt[3]{\sqrt{5}+2}\right)\left(\sqrt[3]{\sqrt{5}-2}\right)\left(\left(\sqrt[3]{\sqrt{5}+2}\right) + \left(\sqrt[3]{\sqrt{5}-2}\right)\right)$

$y^3 = \left(\sqrt[3]{\sqrt{5}+2} + \sqrt[3]{\sqrt{5}-2}\right)^3 = \sqrt{5} + 2 + \sqrt{5} - 2 + 3\left(\sqrt[3]{5-4}\right)\left(\left(\sqrt[3]{\sqrt{5}+2}\right) + \left(\sqrt[3]{\sqrt{5}-2}\right)\right)$

$y^3 = \left(\sqrt[3]{\sqrt{5}+2} + \sqrt[3]{\sqrt{5}-2}\right)^3 = 2\sqrt{5} + 3\left(\left(\sqrt[3]{\sqrt{5}+2}\right) + \left(\sqrt[3]{\sqrt{5}-2}\right)\right)$

$y^3 = 2\sqrt{5} + 3y \quad \Leftrightarrow \quad y^3 - 5\sqrt{5} = 3y - 3\sqrt{5} \quad \Leftrightarrow \quad y^3 - \sqrt{5}^3 = 3(y - \sqrt{5})$

$(y-\sqrt{5})^3 + 3(y\sqrt{5})(y-\sqrt{5}) - 3(y-\sqrt{5}) \quad \Leftrightarrow \quad (y-\sqrt{5})((y-\sqrt{5})^2 + 3y\sqrt{5} - 3) = 0 \quad \Leftrightarrow$

$(y-\sqrt{5})(y^2 + 5\sqrt{5} - 2y\sqrt{5} + 3y\sqrt{5} - 3) = 0 \quad \Leftrightarrow \quad (y-\sqrt{5})(y^2 + y\sqrt{5} + 2) = 0 \quad \Leftrightarrow$

$y = \sqrt{5} \quad or \quad y^2 + y\sqrt{5} + 2 = 0; \quad d = 5 - 8 < 0 \quad no\ solution$

$y = \sqrt{5}$

$\left(\sqrt[3]{\sqrt{5}+2} + \sqrt[3]{\sqrt{5}-2}\right)^{2014} = \sqrt{5}^{2014} = 5^{1007}$

443. Solve for x. $\sqrt{\dfrac{x-6}{x+10}} + \sqrt{\dfrac{x+10}{x-6}} = \dfrac{34}{15}$

We first put $a = \dfrac{x-6}{x+10}$ and then we get: $\sqrt{a} + \sqrt{\dfrac{1}{a}} = \dfrac{34}{15}$ Then we put $y = \sqrt{a}$ and then we have:

$y + \dfrac{1}{y} = \dfrac{34}{15} \quad \Leftrightarrow \quad 15y^2 - 34y + 15 = 0; \quad d = 34^2 - 4 \cdot 15 \cdot 15 = 1156 - 900 = 256$

$y = \dfrac{34 \pm 16}{30} \quad \Leftrightarrow \quad y = \dfrac{5}{3} \quad or \quad y = \dfrac{3}{5}$

$a = y^2 \quad \Rightarrow \quad a = \dfrac{25}{9} \quad or \quad a = \dfrac{9}{25} \quad \Rightarrow \quad \dfrac{x-6}{x+10} = \dfrac{25}{9} \quad or \quad \dfrac{x-6}{x+10} = \dfrac{9}{25}$

$\frac{x-6}{x+10}=\frac{25}{9} \quad \Leftrightarrow \quad 9(x-6)=25(x+10) \quad \Leftrightarrow \quad 9x-54=25x+250 \quad \Leftrightarrow \quad 16x=-196 \quad x=-\frac{49}{4}$

This solution can not be applied, since it leads to a negative argument in the square root.

$\frac{x-6}{x+10}=\frac{9}{25} \quad \Leftrightarrow \quad 25(x-6)=9(x+10) \quad \Leftrightarrow \quad 25x-150=9x+90 \quad \Leftrightarrow \quad 16x=240 \quad \Leftrightarrow \quad x=15$

444. Determine *x* and *y*, such that: $2^x - 2^y = 496$

2^x must be larger than 496, and the first candidate is $2^9 = 512$. 512 – 496 = 16 = 2^4 , so the solution is: $x = 9$ and $y = 4$.

445. Solve for rational *x* and *y*: $\sqrt{x}+\sqrt{x}=\sqrt{2+\sqrt{3}}$

We take the square of both sides: $(\sqrt{x}+\sqrt{y})^2 = (\sqrt{2+\sqrt{3}}\,)^2$

$x + y + 2\sqrt{x}\sqrt{y} = 2 + \sqrt{3}$

Since x and y are considered rational, we must have: $x+y=2$ *and* $2\sqrt{x}\sqrt{y}=\sqrt{3}$

$2\sqrt{x}\sqrt{y}=\sqrt{3} \quad \Leftrightarrow \quad 4xy=3 \quad \Leftrightarrow \quad y=\frac{3}{4x} \quad \Rightarrow \quad x+\frac{3}{4x}=2 \quad \Leftrightarrow$

$4x^2-8x+3=0$; $d=64-48=16 \quad x=\frac{8\pm 4}{8} \quad \Leftrightarrow \quad x=\frac{3}{2}$ *or* $x=\frac{1}{2}$

$x=\frac{3}{2}$ *and* $y=\frac{1}{2}$ *or* $x=\frac{1}{2}$ *and* $y=\frac{3}{2}$

446. Solve for x. $\frac{3^{x+1}-5^{x+2}}{3^x-5^x}=1$

$\frac{3^{x+1}-5^{x+2}}{3^x-5^x}=1 \quad \Leftrightarrow \quad 3^{x+1}-5^{x+2} = 3^x-5^x \quad \Leftrightarrow \quad 3\cdot 3^x-25\cdot 5^x=3^x-5^x$

We put: $a=3^x$ *and* $b=5^x$ and we get: $3a-25-b=a-b \quad \Leftrightarrow \quad 2a-24b=0 \quad \Leftrightarrow \quad a=12b$

$3^x=12\cdot 5^x \quad \Leftrightarrow \quad x\ln 3=\ln 12+x\ln 5 \quad \Leftrightarrow \quad x=\frac{\ln 12}{\ln 3-\ln 5}$

447. Determine integers *a* and b such that: $a+2ab+b=22$

We would like to factorize the lhs, but that is not immediately possible, but if we multiply the equation by 2….

$a+2ab+b=22 \quad \Leftrightarrow \quad 2a+4ab+2b=44$ we may rewrite the lhs as: $(2a+1)(2b+1)-1=44$

$(2a+1)(2b+1)=45 \quad 45=5\cdot 9=15\cdot 3$, so we try with:

$2a+1=5$ *and* $2b+1=9 \quad \Leftrightarrow \quad a=2$ *and* $b=4$

$2a+1=3$ *and* $2b+1=15 \quad \Leftrightarrow \quad a=1$ *and* $b=7$

448. Determine *x* and *y* from: $x^2 - y^2 = 9 \quad and \quad xy = 3$

$$xy = 3 \quad \Leftrightarrow \quad y = \frac{3}{x} \quad \Rightarrow \quad x^2 - \left(\frac{3}{x}\right)^2 = 9 \quad \Leftrightarrow \quad x^4 - 9x^2 - 9 = 0$$

$$d = 81 + 4 \cdot 9 = 117? \quad x = \frac{9 \pm \sqrt{117}}{2} \quad and \quad y = \frac{6}{9 \pm \sqrt{117}}$$

449. Determine *x* from: $3^{(2\sin x)^2} + 3^{(2\cos x)^2} = 30$

We put: $a = 3^{(2\sin x)^2}$ and $b = 3^{(2\cos x)^2}$, so $a + b = 30$

$$ab = 3^{(2\sin x)^2} \cdot 3^{(2\cos x)^2} = 3^{(2\cos x)^2 + (2\sin x)^2} = 3^{4(\cos^2 x + \sin^2 x)} = 3^4$$

We solve: $a + b = 30$ and $ab = 81$

$$ab = 81 \quad \Rightarrow \quad b = \frac{81}{a} \quad \Rightarrow \quad a + \frac{81}{a} = 30 \quad \Leftrightarrow \quad a^2 - 30a + 81 = 0;$$

$$d = 30^2 - 4 \cdot 81 = 576 = 3 \cdot 192 = 3 \cdot 2 \cdot 96 = 3 \cdot 2 \cdot 2 \cdot 48 \cdot 6 \cdot 8 = 3^2 \cdot 2^6$$

$$a = \frac{30 \pm 24}{2} \quad \Leftrightarrow \quad a = 27 \quad or \quad a = 3 \quad and \quad b = 3 \quad or \quad b = 27$$

$$3^{(2\sin x)^2} = 3 \quad \Leftrightarrow \quad (2\sin x)^2 = 1 \quad \Leftrightarrow \quad \sin^2 x = \pm\frac{1}{4} \quad \Rightarrow \quad \sin x = \pm\frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{6} \quad or \quad x = \frac{5\pi}{6}$$

$$3^{(2\cos x)^2} = 27 \quad 4\cos^2 x = 3 \quad \Leftrightarrow \quad \cos x = \pm\frac{\sqrt{3}}{2} \quad \Leftrightarrow \quad x = \frac{\pi}{6} \quad or \quad x = \frac{7\pi}{6}$$

450. Solve for *x*. $2^{\frac{x}{3}} = 3^{\frac{2}{x}}$

$$2^{\frac{x}{3}} = 3^{\frac{2}{x}} \quad \Leftrightarrow \quad \frac{x}{3}\ln 2 = \frac{2}{x}\ln 3 \quad \Leftrightarrow \quad x^2 = 6\frac{\ln 3}{\ln 2} \quad \Leftrightarrow \quad x = \pm\sqrt{\frac{\ln 3}{\ln 2}}$$

451. Determine $x^4 + y^4 \quad from \quad x^3 + y^3 = 16 \quad and \quad x + y = 4$

$$x^3 + y^3 = (x+y)^3 - 3xy(x+y) = 16 \quad \Rightarrow \quad 64 - 12xy = 16 \quad \Rightarrow \quad xy = 4$$

$$x + y = 4 \quad and \quad xy = 4 \quad \Leftrightarrow \quad x = y = 2$$

452. Determine natural numbers $a \ and \ b \quad from: \ b^2 + 3a^2b^2 = 30a^2 + 517$

$$b^2 + 3a^2b^2 = 30a^2 + 517 \quad \Leftrightarrow b^2(1 + 3a^2) = 30a^2 + 517 \quad \Rightarrow \quad b^2 = \frac{30a^2 + 517}{1 + 3a^2}$$

$$b^2 = \frac{30a^2 + 10 + 507}{1 + 3a^2} = 10 + \frac{507}{1 + 3a^2}$$

Since b is a positive integer the rhs must also be a positive integer, and that is only the case if the fraction is an integer; $507 = 3 \cdot 13^2$ So $1 + 3a^2$ must be a divisor in 13. $a = 2$ gives in fact:

$1+3a^2$=13, so; $b^2 = \dfrac{30a^2+10+507}{1+3a^2} = 10+\dfrac{507}{1+3a^2}$=10+39 = 49.

$b=7$

453. Solve for *x*. $243^{x-1}+3^{5x+1}=\dfrac{730}{27}$

$$243^{x-1}+3^{5x+1}=\frac{730}{27} \Leftrightarrow 3^{5(x-1)}+3^{5x+1}=\frac{730}{27} \Leftrightarrow \frac{3^{5x}}{3^5}+3\cdot 3^{5x}=\frac{730}{27} \Leftrightarrow$$

$$3^{5x}(\frac{1}{3^5}+3)=\frac{730}{27} \Leftrightarrow 3^{5x}\frac{1+3\cdot 343}{343}=\frac{730}{27} \Leftrightarrow 3^{5x}\frac{730}{343}=\frac{730}{27} \Leftrightarrow 3^{5x}=\frac{343}{27}$$

$$3^{5x}=\frac{3^5}{3^3} \Leftrightarrow 3^{5x}=3^2 \Leftrightarrow 5x=2 \Leftrightarrow x=\frac{2}{5}$$

454. Solve for *x*. $\log_5 x = \log_x 25$

We notice that: $y=\log x \Leftrightarrow x=10^y$ Or more generally: $y=\log_a x \Leftrightarrow x=a^y$

We put: $y=\log_5 x$ and then: $y=\log_5 x \Leftrightarrow x=5^y$

On the other hand: $\log_5 x=\log_x 25 \Leftrightarrow 25=x^y$

$$25=x^y \text{ and } x=5^y \Leftrightarrow (5^y)^y=5^2 \Leftrightarrow 5^{y^2}=5^2 \Leftrightarrow y^2=2 \Leftrightarrow y=\sqrt{2}$$

$$\log_5 x=\sqrt{2} \Leftrightarrow x=5^{\sqrt{2}}$$

455. Solve for *x* and *y*: $2^{x-3}-2^{y-3}=504$

$$2^{x-3}-2^{y-3}=504 \Leftrightarrow \frac{2^x}{2^3}-\frac{2^y}{2^3}=504 \Leftrightarrow 2^x-2^y=504\cdot 8=4032$$

$$2^{12}=4096;\quad 4096-4032=64=2^8 \Rightarrow x=12 \text{ and } y=8$$

456. Determine *x* and *y* from: $x^3+y^3=16$ *and* $x+y=4$

Intuitively we can see that $x=y$ =2 is a solution, since $2^3+2^3=16$ *and* $2+2=4$.

This may also be proven.

$$(x+y)^3=x^3+y^3+3xy(x+y) \Rightarrow x^3+y^3=(x+y)^3-3xy(x+y) \Leftrightarrow 16=64-12xy \Rightarrow xy=4$$

$$x+y=4 \text{ and } xy=4 \Leftrightarrow x=y=2$$

457. Determine x and y from: $x^2+y^2=50$ *and* $x-y=8$

The only integer pairs *x, y* which may result in: $x^2+y^2=50$ is plus/minus (5,5) and plus/minus (7,1). plus/minus (5,5) does not work because the second condition, but: (7,-1) and (7,-1) do, since:

$$7^2+(-1)^2=50 \text{ and } 7-(-1)=8 \text{ and } 1^2+(-7)^2=50 \text{ and } 1-(-7)=8$$

458.Solve for *x*. $5^x - 4^x = 61$

We notice that $5^3 = 125 \quad and \quad 4^3 = 64 \quad and \quad 125 - 64 = 61$
So the solution is: $x = 3$.

459. Solve for *x* and *y*: $x + xy + y = 39 \quad and \quad x^2 + y^2 = 65$

$x + xy + y = 39 \quad \Leftrightarrow \quad (x+1)(y+1) - 1 = 39 \quad and \quad x^2 + y^2 = 65$

$(x+1)(y+1) = 40 \quad and \quad x^2 + y^2 = 65$

Well $5 \cdot 8 = 40 \quad and \quad 5^2 + 8^2 = 65$, so the solution is $(x,y) = (5,8)$ or $(x,y) = (8,5)$

460. Simplify: $99^3 - 48^3 - 51^3$

Well: 99 = 48 + 51, so

$99^3 = (48+51)^3 = 48^3 + 51^3 + 3 \cdot 48 \cdot 51(48+51)$

$99^3 - 48^3 - 51^3 = 3 \cdot 48 \cdot 51(48+51) = 729{,}056$

461. Solve for *x*. $4^x + 18^x = 81^x$

$4^x + 18^x = 81^x \quad \Leftrightarrow \quad (2^x)^2 + (2^x)(3^x)^2 - (3^x)^4$

We put $a = 2^x \quad and \quad b = 3^x$, and we get:

$a^2 + ab^2 - b^4 = 0 \quad \Leftrightarrow \quad b^4 - ab^2 - a^2 = 0$ We solve for b^2: $\quad d = a^2 + 4a^2 = 5a^2$

$$b^2 = \frac{a \pm a\sqrt{5}}{2} = a\frac{1 \pm \sqrt{5}}{2} \quad \Rightarrow \quad b = \sqrt{a}\sqrt{\frac{1+\sqrt{5}}{2}}$$

$$3^x = 2^{\frac{x}{2}}\sqrt{\frac{1+\sqrt{5}}{2}} \quad \Leftrightarrow \quad x\ln 3 = \tfrac{1}{2}x\ln 2 + \ln\left(\sqrt{\frac{1+\sqrt{5}}{2}}\right) \quad \Leftrightarrow \quad x = -\frac{\sqrt{\frac{1+\sqrt{5}}{2}}}{\ln 3 - \frac{1}{2}\ln 2}$$

462. Determine non negative integers *a* and *b* such that: $\sqrt{a} + \sqrt{b} = \sqrt{2009}$

Well: $2009 = 7 \cdot 287 = 7 \cdot 7 \cdot 41, \quad so \quad \sqrt{2009} = 7\sqrt{41}$

Since 41 is a prime: integer $\sqrt{a}$ and $\sqrt{b}$ must be a multiple of $\sqrt{41}$, the sum of which is $7\sqrt{41}$
This result in the possibilities:

$(\sqrt{a}, \sqrt{b}) = (0\sqrt{41}, 7\sqrt{41}), \quad \Rightarrow \quad a = 0 \quad \wedge \quad b = 2009$

$(\sqrt{a}, \sqrt{b}) = (\sqrt{41}, 6\sqrt{41}), \quad \Rightarrow \quad a = 41 \quad \wedge \quad b = 1476$

$(\sqrt{a}, \sqrt{b}) = (2\sqrt{41}, 5\sqrt{41}), \quad \Rightarrow \quad a = 164 \quad \wedge \quad b = 1025$

$(\sqrt{a}, \sqrt{b}) = (3\sqrt{41}, 4\sqrt{41}), \quad \Rightarrow \quad a = 369 \quad \wedge \quad b = 656$

463. Simplify: $(11^{\frac{1}{4}} - 1)(11^{\frac{3}{4}} + 11^{\frac{1}{2}} + 11^{\frac{1}{4}} + 1)$

We put $a = 11^{\frac{1}{4}}$, and then we have:

$(a-1)(a^3 + a^2 + a + 1) = a^4 - a^3 + a^3 - a^2 + a^2 - 1) = a^4 - 1 = (11^{\frac{1}{4}}) - 1 = 11 - 1 = 10$

464. solve for *x*: $x^{4x} = \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{x}}$

Since the left hand side grows (violently) with *x* and the right side decreases (violently) with *x*, then *x* must be less than 1. *x* = ½ gives: $\left(\frac{1}{2}\right)^2 = \frac{1}{2}$ but if we try with: $x = \frac{1}{4}$. We get;

$\left(\frac{1}{4}\right)^{4\cdot\frac{1}{4}} = \left(\frac{1}{\sqrt{2}}\right)^4 \quad \Leftrightarrow \quad \frac{1}{4} = \frac{1}{4}$, so the solution is $x = \frac{1}{4}$.

465. Solve for integer *x*. $x^3 + x^2 = 810$

x =	7	8	9
x^2	49	64	81
x^3	343	512	729

From which we can see that $x = 9$, since 81+729 = 810.

466a. Determine integer *a* and *b*, such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$

We isolate *b*: $\frac{1}{b} = \frac{1}{2} - \frac{1}{a} = \frac{a-2}{2a} \quad \Rightarrow \quad b = \frac{2a}{a-2}$

For a given *a*, *b* must be an integral number. To secure this we rewrite the expression for *b*.

$b = \frac{2a}{a-2} = \frac{2(a-2)+4}{a-2} = 2 + \frac{4}{a-2}$ The requirement to integer *a*, is then that $\frac{4}{a-2}$ is an integer.

The divisors in 4 are $\pm 1, \pm 2, \pm 4$,

$a-2=1 \Rightarrow \quad a=3 \quad and \quad b=6: \quad \frac{1}{3}+\frac{1}{6}=\frac{1}{2}$

$a-2=-1 \Rightarrow \quad a=1 \quad and \quad b=-2: \quad \frac{1}{1}-\frac{1}{2}=\frac{1}{2}$

$a-2=2 \Rightarrow \quad a=4 \quad and \quad b=4: \quad \frac{1}{4}+\frac{1}{4}=\frac{1}{2}$

$a-2=-2 \Rightarrow \quad a=0 \quad !$

$a-2=4 \Rightarrow \quad a=6 \quad and \quad b=3: \quad \frac{1}{3}+\frac{1}{6}=\frac{1}{2}$

$a-2=-4 \Rightarrow \quad a=-2 \quad and \quad b=1: \quad -\frac{1}{2}+\frac{1}{1}=\frac{1}{2}$

466. Determine integer *a* and *b*, such that $\frac{1}{a}+\frac{1}{b}=\frac{1}{5}$.

The obvious solution is $a = b = 10$, but there may be other solutions?

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{5} \quad \Leftrightarrow \quad \frac{1}{b}=\frac{1}{5}-\frac{1}{a} \quad \Leftrightarrow \quad \frac{1}{b}=\frac{a-5}{5a} \quad \Leftrightarrow \quad b=\frac{5a}{a-5}=\frac{5(a-5)+25}{a-5}=5+\frac{25}{a-5}$$

$b=5+\frac{25}{a-5}$ We must therefore require that $\frac{25}{a-5}$ is an integer.

For $a > 30$ or $a < -20$ there are no solutions since the denominator becomes larger than the nominator.

The possibilities are for $a > 0$. are $a = 10$, and $a = 30$. And for $a < 0$: $a = -20$. This corresponds to:

$$(a,b)=(10,10):\frac{1}{10}+\frac{1}{10}=\frac{1}{5}. \quad (a,b)=(30,6):\frac{1}{30}+\frac{1}{6}=\frac{1}{5}. \quad (a,b)=(-20,4):\frac{1}{-20}+\frac{1}{4}=\frac{1}{5}$$

427. Determine $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$: From the equations: $2^x=3^y=6^z$

$$2^x=3^y=6^z \quad \Leftrightarrow \quad 2^x=3^y=(3\cdot 2)^z \quad \Rightarrow \quad (2^x)^{\frac{1}{x}}=(3\cdot 2)^{\frac{z}{x}} \; and \; (3^y)^{\frac{1}{y}}=(3\cdot 2)^{\frac{z}{y}} \quad \Rightarrow$$

$$2=(3\cdot 2)^{\frac{z}{x}} \quad and \quad 3=(3\cdot 2)^{\frac{z}{y}} \quad \Leftrightarrow \quad 2\cdot 3=(3\cdot 2)^{\frac{z}{x}+\frac{z}{y}} \quad 6^1=6^{\frac{z}{x}+\frac{z}{y}} \quad \Leftrightarrow \quad \frac{z}{x}+\frac{z}{y}=1$$

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{z} \quad \Leftrightarrow \quad \frac{1}{x}+\frac{1}{y}-\frac{1}{z}=0$$

428. Determine: x^4+y^4 from: $x+y=4 \quad and \quad x^3+y^3=16$

It is rather evident that $x = y = 2$ so $x^4+y^4=32$, but we prove it below.

$$4\cdot 16=(x^3+y^3)(x+y)=x^4+y^4+x^3y+y^3x=x^4+y^4+xy(x^2+y^2) \quad \Rightarrow$$
$$4^3=(x+y)^3=x^3+y^3+3xy(x+y)=16+12xy \quad \Rightarrow \quad xy=4$$
$$16=(x+y)^2=x^2+y^2+2xy \quad \Rightarrow \quad x^2+y^2=8$$
$$4\cdot 16=(x^3+y^3)(x+y)=x^4+y^4+x^3y+y^3x=x^4+y^4+xy(x^2+y^2) \quad \Rightarrow$$
$$x^4+y^4=64-xy(x^2+y^2)=64-4\cdot 8=32$$
$$x^4+y^4=32$$

429. Determine $\frac{x}{y}$ from $\frac{1}{x}-\frac{1}{y}=\frac{1}{x+y}$

We multiply $\frac{1}{x}-\frac{1}{y}=\frac{1}{x+y}$ with x and find: $\frac{x}{x}-\frac{x}{y}=\frac{x}{x+y} \quad \Leftrightarrow \quad 1-\frac{x}{y}=\frac{\frac{x}{y}}{\frac{x}{y}+1}$

We put $a = \frac{x}{y}$ and find: $1 - a = \frac{a}{1+a} \quad \Leftrightarrow \quad 1 - a^2 = a \quad \Leftrightarrow \quad a^2 + a - 1 = 0$

$a^2 + a - 1 = 0; \quad d = 1 + 4 = 5; \quad a = \frac{-1 \pm \sqrt{5}}{2}$

$\frac{x}{y} = \frac{-1 \pm \sqrt{5}}{2}$

430. Simplify: $4^{-\frac{1}{4}}\sqrt{14^4 + 114^4 + 100^4}$

We notice that: $4^{-\frac{1}{4}} = \frac{1}{\sqrt[4]{4}} = \frac{1}{\sqrt{2}} = \sqrt{\frac{1}{2}}$ and that: $114 = 100 + 14$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$114^4 = (14+100)^4 = 14^4 + 4 \cdot 14^3 \cdot 100 + 6 \cdot 14^2 \cdot 100^2 + 4 \cdot 14 \cdot 100^3 + 100^4$

$\frac{1}{2}(14^4 + 114^4 + 100^4) = \frac{1}{2}(2 \cdot 14^4 + 4 \cdot 14^3 \cdot 100 + 6 \cdot 14^2 \cdot 100^2 + 4 \cdot 14 \cdot 100^3 + 2 \cdot 100^4)$

$\frac{1}{2}(14^4 + 114^4 + 100^4) = \frac{1}{2}(2 \cdot 14^4 + 4 \cdot 14^3 \cdot 100 + 6 \cdot 14^2 \cdot 100^2 + 4 \cdot 14 \cdot 100^3 + 2 \cdot 100^4)$

$\frac{1}{2}(14^4 + 114^4 + 100^4) = 14^4 + 2 \cdot 14^3 \cdot 100 + 3 \cdot 14^2 \cdot 100^2 + 2 \cdot 14 \cdot 100^3 + 100^4$

$\frac{1}{2}(14^4 + 114^4 + 100^4) = 14^4 + 2 \cdot 14^3 \cdot 100 + 3 \cdot 14^2 \cdot 100^2 + 2 \cdot 14 \cdot 100^3 + 100^4$

$\frac{1}{2}(14^4 + 114^4 + 100^4) = 14^4 + 2 \cdot 14^3 \cdot 100 + 3 \cdot 14^2 \cdot 100^2 + 2 \cdot 14 \cdot 100^3 + 100^4$

$\frac{1}{2}(14^4 + 114^4 + 100^4) = 14^4 + 2 \cdot 14^3 \cdot 100 + 3 \cdot 14^2 \cdot 100^2 + 2 \cdot 14 \cdot 100^3 + 100^4$

For simplicity we write: a = 14 and b =100:

$\frac{1}{2}(a^4 + (a+b)^4 + b^4) = \frac{1}{2}(2a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 2b^4)$

$\frac{1}{2}(a^4 + (a+b)^4 + b^4) = a^4 + 2ab(a^2 + b^2) + 3a^2b^2 + b^4$

$\frac{1}{2}(a^4 + (a+b)^4 + b^4) = (a^2 + b^2)^2 + 2ab(a^2 + b^2) + a^2b^2$

$\frac{1}{2}(a^4 + (a+b)^4 + b^4) = ((a^2 + b^2) + ab)^2$

$4^{-\frac{1}{4}}\sqrt{14^4 + 114^4 + 100^4} = \sqrt{((14^2 + 100^2) + (14 \cdot 100))^2} = (14^2 + 100^2) + (14 \cdot 100)$

431. Determine *x*, *y*, *z*, such that: $\frac{1}{x+y} + \frac{1}{z+y} + \frac{1}{x+z} = \frac{4}{5}$

For simplicity, we shall first solve: $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{4}{5}$ for integer: a, b, c

$\frac{1}{a} = \frac{2}{5}$ has no integer solution, so we seek solutions to $\frac{1}{a} + \frac{1}{b} = \frac{3}{5}$

$\frac{1}{b} = \frac{3}{5} - \frac{1}{a} \quad \Leftrightarrow \quad b = \frac{5a}{3a-5}$, where b should be an integer: a = 2 is a candidate, since:

$b = \frac{5 \cdot 2}{3 \cdot 2 - 5} = 10$ The other candidate is (a,b) = (10,2).

So we have a solution: $a = 2, b = 10, c = 5$, since:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2}+\frac{1}{10}+\frac{1}{5}=\frac{4}{5}$$

corresponding to

$x+y=2, \quad y+z=5, \quad x+z=10$

However it is rather misleading and confusing that these equations do not have integer solutions, since:
The first minus the second equation: gives: $x-z=-3$, and added to the third equation $x+z=10$ gives $2x=7 \quad \Leftrightarrow \quad x=\frac{7}{2}$ Then we find $z=10-\frac{7}{2}=\frac{13}{2}$, and $y=2-x=2-\frac{7}{2}=-\frac{3}{2}$.

433. Solve for *x*: $x^{5x^{95}}=1444$

As it is the case of transcendent equations, there exists no general analytical method of solution, however, a small number of transcendent equations, requires a trick, namely that: $a^a=b^b \Rightarrow a=b$.
If you are not aware of this, you will never be able to solve them, but if the equation can be rewritten in this form it may be solved. First we note that:

$1444=2^2\cdot 19^2=38^2 \quad and \quad 5\cdot 19=95$

We therefore raise both sides to the power 19.

$$x^{5x^{95}}=1444 \quad \Leftrightarrow \quad (x^{5x^{95}})^{19}=(38^2)^{19} \quad \Leftrightarrow \quad x^{5\cdot 19x^{95}}=38^{38} \quad \Leftrightarrow$$
$$x^{95x^{95}}=38^{38} \quad \Leftrightarrow \quad \left(x^{95}\right)^{x^{95}}=38^{38}$$

We then apply $a^a=b^b \Rightarrow a=b$, so we have:

$$x^{95}=38 \quad \Rightarrow \quad \ln x=\frac{\ln 38}{95} \quad \Rightarrow$$
$$x=38^{\frac{1}{95}}$$

434. Solve for *x*: $9^x+6^x=2^{2x+1}$

$9^x+6^x=2^{2x+1} \quad \Leftrightarrow \quad (3^x)^2+3^x\cdot 2^x-2\cdot(2^x)^2=0$

We divide the equation by $(3^x)^2$ to get:

$$1+\frac{2^x}{3^x}-2\cdot\frac{(2^x)^2}{(3^x)^2}=0 \quad \Leftrightarrow \quad 1+\left(\frac{2}{3}\right)^x-2\cdot\left(\left(\frac{2}{3}\right)^x\right)^2=0$$

If we put $y=\left(\frac{2}{3}\right)^x$ we find:

$$2y^2-y-1=0; \quad d=1+8=9; \quad y=\frac{1\pm 3}{4} \quad \Leftrightarrow \quad y=1 \quad \vee \quad y=-\tfrac{1}{2} \quad \Leftrightarrow \quad y=1$$

$$\left(\frac{2}{3}\right)^x=1 \quad \Leftrightarrow \quad x=0.$$

435. Solve for *x*: $27^{x-1}+729x-2187=0$

As it is the case of transcendent equations, there exists no general analytical method of solution, however, a small number of transcendent equations, requires a trick, namely that: $a^a=b^b \Rightarrow a=b$. If you are not aware of this, you will never be able to solve them, but if the equation can be rewritten in this form it may be solved. First we note that:

$$27=3^3 \quad and \quad 729=3\cdot 3\cdot 81=3^6 \quad and \quad 2187=3\cdot 729=3^7$$

So we have: $3^{3x-3}+3^6x-3^7=0$ Division by 3^6: Let us put: $y=3-x$:

Then we raise both sides to $-\frac{1}{y}$

$$3^{3x-9}+x-3=0 \quad \Leftrightarrow \quad 3^{3(x-3)}=3-x \quad \Leftrightarrow$$

$$3^{-3y}=y \quad \Leftrightarrow \quad (3^{-3y})^{-\frac{1}{y}}=y^{-\frac{1}{y}} \quad \Leftrightarrow \quad 3^3=y^{-\frac{1}{y}} \quad \Leftrightarrow \quad 3^3=\left(\frac{1}{y}\right)^{\frac{1}{y}}$$

According to the rule; $a^a=b^b \Rightarrow a=b$, we must have:

$$\frac{1}{y}=3 \quad \Leftrightarrow \quad 3-x=\frac{1}{3} \quad \Leftrightarrow \quad x=\frac{8}{3}$$

And we see that it fits in $3^{3(x-3)}=3-x$, since: $3^{3\frac{8}{3}-9}=3-\frac{8}{3} \quad \Leftrightarrow \quad \frac{1}{3}=\frac{1}{3}$

436. Solve for *x*: $\frac{1}{x}+\frac{1}{x+4}=x+2$

We might guess the solution $x = -2$, but we solve it analytically.
We multiply the equation by the common denominator: $x(x+4)$ and find:

$$x+4+x=(x+2)x(x+4) \quad \Leftrightarrow \quad x^3+6x^2+6x-4=0$$

We have already guessed the root $x = -2$, so we make polynomial division with $x+2$,

$$x+2\,|\,x^3+6x^2+6x-4\,|\,x^2+4-2$$
$$x^3+2x^2$$
$$4x^2+6x$$
$$4x^2+8x$$
$$-2x-4$$
$$-2x-4$$

$$x^2+4-2=0; \quad d=16+8=24; \quad x=\frac{-4\pm 2\sqrt{6}}{2}=2\mp\sqrt{6}$$

437. Solve for *x*: $2022^{2-x}+2022^{2+x}=2\cdot 2022^2$

It is easy to see that $x = 0$ is a solution, but the equation can also be solved analytically, independently of the number 2002. We put $y = 2002$ and find:

$$y^{2-x}+y^{2+x}=2\cdot y^2 \quad \Leftrightarrow \quad y^{-x}+y^x=2 \quad \Leftrightarrow \quad (y^x)^2-2y^x+1=0 \quad \Leftrightarrow$$
$$\left(y^x-1\right)^2=0 \quad \Leftrightarrow \quad y^x=1 \quad \Leftrightarrow \quad x=0$$

438. Solve for *x*: $0.01^x=11 \quad \Leftrightarrow \quad x=\dfrac{\log 11}{\log 0.01}=\dfrac{\log 11}{-2}$

439. Simplify: $\sqrt{5-\sqrt{15}-\sqrt{16-2\sqrt{15}}}$

$$\sqrt{16-2\sqrt{15}}=(1-\sqrt{15}) \quad \text{since: } (1-\sqrt{15})^2=1+15-2\sqrt{15}=16-2\sqrt{15}$$
$$\sqrt{5-\sqrt{15}-\sqrt{16-2\sqrt{15}}}=\sqrt{5-\sqrt{15}-(1-\sqrt{15})}=\sqrt{5-1}=2$$

440. Solve for *x*: $(\sqrt{2}+1)^x+(\sqrt{2}-1)^x=6$

We put: $a=(\sqrt{2}+1)^x \quad and \quad b=(\sqrt{2}-1)^x$

Then we realize that:

$$ab=(\sqrt{2}-1)^x(\sqrt{2}+1)^x=((\sqrt{2}-1)(\sqrt{2}+1))^x=(2-1)^x=1^x=1$$

So we have two equations: $a+b=6 \quad and \quad ab=1$

$$a+b=6 \quad and \quad b=\frac{1}{a} \quad \Rightarrow \quad a+\frac{1}{a}=6 \quad \Leftrightarrow$$
$$a^2-6a+1=0; \quad d=36-4=32; \quad a=\frac{6\pm 4\sqrt{2}}{2} \quad \Leftrightarrow \quad a=\frac{6+4\sqrt{2}}{2} \ or \ a=\frac{6-4\sqrt{2}}{2}$$
$$a=3+2\sqrt{2} \quad or \quad a=3-2\sqrt{2} \quad \Leftrightarrow \quad (\sqrt{2}+1)^x=3+2\sqrt{2} \quad or \quad (\sqrt{2}+1)^x=3-2\sqrt{2} \quad \Leftrightarrow$$
$$x=\frac{\ln(3+2\sqrt{2})}{\ln(\sqrt{2}+1)} \quad or \quad x=\frac{\ln(3-2\sqrt{2})}{\ln(\sqrt{2}+1)}$$

441. Determine *x* and *y* from: $x+y+xy=80$

We rewrite the equation; $(x+1)(y+1)-1=8 \quad \Leftrightarrow \quad (x+1)(y+1)=81$

Now $81=3\cdot 27=9\cdot 9$

So we may have to two solutions:

$$x+1=3 \quad and \quad y+1=27 \quad or \quad x+1=3 \quad and \quad y+1=27 \quad or \quad x+1=9 \quad and \quad y+1=9 \quad \Leftrightarrow$$
$$x=2 \quad and \quad y=26 \quad or \quad x=8 \quad and \quad y=8$$

442. Solve for *x*. $\sqrt[3]{x}+\sqrt[6]{x}-2=0$

We put : $\sqrt[6]{x}=y$, and then we have:

$(\sqrt[6]{x})^2 + \sqrt[6]{x} - 2 = 0 \quad \Leftrightarrow \quad y^2 + y - 2 = 0$

$d = 1 + 8 = 9;$

$y = \dfrac{-1 \pm 3}{2} \quad = 1 \quad \Rightarrow \quad \sqrt[6]{x} = 1 \quad \Leftrightarrow \quad x = 1$

443. Determine *a* and *b* from: $\sqrt{a} + b = 85 \quad and \quad a - \sqrt{b} = 7$

We put: $\sqrt{a} = x \;\Leftrightarrow\; a` = x^2 \quad and \quad \sqrt{b} = y \quad \Leftrightarrow \quad b = y^2$

Then we have: $x + y^2 = 85 \quad and \quad x^2 - y = 7$

If we subtract the second equation from the first:

$y^2 - x^2 + x + y = 78 \quad \Leftrightarrow \quad (x + y)(y - x) + (x + y) = 78$

$(x + y)(y - x + 1) = 78$

Now; $78 = 6 \cdot 13$, so we put;

$x + y = 13 \quad and \quad y - x + 1 = 6$

Adding the two equations gives:

$2y = 19 - 1 \quad \Rightarrow \quad y = 9 \quad and \quad x = 4$, and we then find:

$a = 16 \quad and \quad b = 81$

444. Simplify: $\dfrac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}}{2\sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{8} + \sqrt{9} + \sqrt{10}}$

$$\frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}}{2\sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{8} + \sqrt{9} + \sqrt{10}} = \frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}}{2\sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + \sqrt{2}\sqrt{3} + 2\sqrt{2} + 3 + \sqrt{2}\sqrt{5}}$$

$$\frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}}{(\sqrt{2} + 1) + \sqrt{2}(\sqrt{2} + 1) + \sqrt{3}(\sqrt{2} + 1) + 2(\sqrt{2} + 1) + \sqrt{5}(\sqrt{2} + 1)} =$$

$$\frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}}{(\sqrt{2} + 1)(1 + \sqrt{2} + \sqrt{3} + 2 + \sqrt{5})} = \frac{1}{(\sqrt{2} + 1)} = \frac{\sqrt{2} - 1}{(\sqrt{2} + 1)(\sqrt{2} - 1)} = \sqrt{2} - 1$$

445. Determine integer *a ,b, c* such that: $2^{2a} + 2^{3b} = 2^{5c}$

It is evident that we must have; $2a = 3b = 5c - 1$.

So both 2 and 3 must be divisor in $5c - 1$. This result is $a = 12$ and $b = 8$, which gives $c = 5$.

Since: $2^{2a} + 2^{3b} = 2^{5c} \quad \Rightarrow \quad 2^{2 \cdot 12} + 2^{3 \cdot 8} = 2^{5 \cdot 5} \quad \Rightarrow \quad 2^{24} + 2^{24} = 2^{25}$

446. Solve for *x*: $x^3 + x^2 - 3x + 9 = 0$

Any integer roots must be divisors in 9. We can see that -3 is a root, since -27 + 9 + 9 + 9 = 0

We the make polynomials division with $x + 3$.

$x+3 \mid x^3+x^2-3x+9 \mid x^2-2x+3$

x^3+3x^2

$-2x^2-3x$

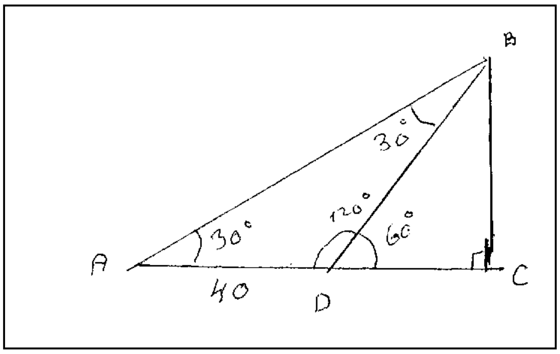
$-2x^2-6x$

$3x+9$

$3x+9$

$x^2-2x+3=0;\quad d=4-12<0$ The solution is x = -3

447. In the triangle below is given: $\angle BDC = 60^0$, $\angle A = 30,\ |AD| = 40$ **. Find** $h = |BC|$



We apply the formulas for a right angle triangle on the triangles: *DBC* and *ABC.*

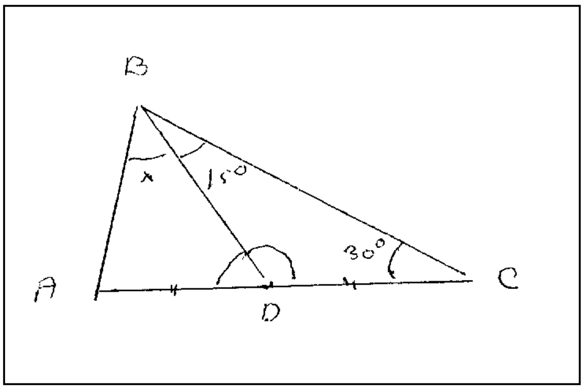
$$\tan 60 = \frac{|BC|}{|DC|} \quad and \quad \tan 30 = \frac{|BC|}{|DC+40|} \quad \Leftrightarrow$$

$$|BC| = \sqrt{3}\,|DC| \quad and \quad |BC| = \frac{\sqrt{3}}{3}|DC+40| \Rightarrow$$

$$\sqrt{3}\,|DC| = \frac{\sqrt{3}}{3}|DC+40| \quad \Rightarrow \quad |DC| = 20 \quad \Rightarrow$$

$$h = |BC| = 20\sqrt{3}$$

448. Find the angle *x* in the figure below



In the triangle to the left is given:

$C = 30^0, \angle DBC = 15^0 \ \ and \ \ AD = DC$ It follows that: $\angle CDB = 135^0 \ \ and \ \ \angle ADB = 45^0$

We apply the sine relations on triangle *DBC* and *ABD*.

$$\frac{\sin 30}{|BD|} = \frac{\sin 15}{|DC|} \quad and \quad \frac{\sin x}{|AD|} = \frac{\sin(180-(x+45)}{|BD|}$$

Now: |*AD*| = |*DC*| so:

$$\frac{\sin 30}{|BD|} = \frac{\sin 15}{|AD|} \quad and \quad \frac{\sin x}{|AD|} = \frac{\sin(180-(x+45)}{|BD|} \quad \Rightarrow$$

$$\frac{\sin x}{\sin 15} = \frac{\sin(180-(x+45))}{\sin 30} \quad \Leftrightarrow \quad \frac{\sin 30}{|BD|} = \frac{\sin 15}{|AD|} \quad and \quad \frac{\sin x}{|AD|} = \frac{\sin(180-(x+45)}{|BD|} \quad \Rightarrow$$

$$\frac{\sin x}{\sin 15} = \frac{\sin 45 \cos x + \cos 45 \sin x}{\sin 30} \quad \Leftrightarrow \sin 30 \sin x = \ \sin 15(\sin 45 \cos x + \cos 45 \sin x) \quad \Rightarrow$$

$$\frac{\sin x}{\sin 15}=\frac{\sin 45\cos x+\cos 45\sin x}{\sin 30} \Leftrightarrow \sin 30\sin x = \sin 15(\sin 45\cos x+\cos 45\sin x) \Rightarrow$$

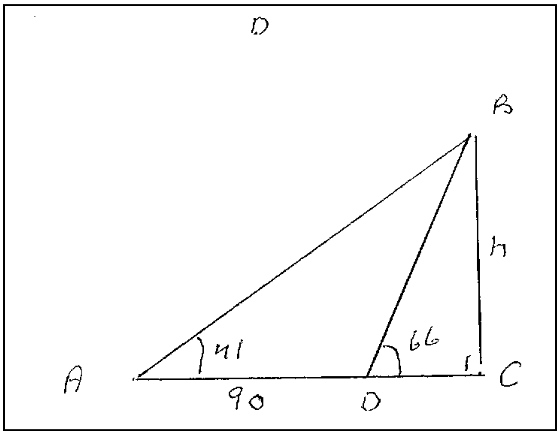
$$\sin 30\tan x=\sin 15\sin 45+\sin 15\cos 45\tan x \Rightarrow \tfrac{1}{2}\tan x=\tfrac{\sqrt{2}}{2}\sin 15(1+\tan x) \Rightarrow$$

$$(\tfrac{1}{2}-\tfrac{\sqrt{2}}{2}\sin 15)\tan x=\tfrac{\sqrt{2}}{2}\sin 15 \Rightarrow \tan x = 0.5773 \Rightarrow x=30^0$$

$$\sin 30=\tfrac{1}{2};\sin 45=\cos 45=\tfrac{\sqrt{2}}{2};\sin 15=\sqrt{\frac{1-\cos 30}{2}}=\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\sqrt{\frac{2-\sqrt{3}}{4}}$$

$$\sin(180-(x+45))=\sin(x+45)=\sin 45\cos x+\cos 45\sin x$$

449. Determine the height in the figure below.



We have:

$$\tan 66=\frac{h}{|DC|} \quad and \quad \tan 41=\frac{h}{90+|DC|} \Rightarrow$$

$$h=|DC|\tan 66=(90+|DC|)\tan 41 \Rightarrow$$

$$|DC|(\tan 66-\tan 41)=90\tan 41 \quad |DC|=56.83$$

$$h=|DC|\tan 66=127.63$$

450. Solve for *x*: $169^x-143^x=121^x$

This exercise seems (at a glance) impossible for a analytic solution.

But mainly by hazard, we find: $\frac{169}{143}=\frac{143}{121} \Leftrightarrow 169\cdot 121=143\cdot 143$

$$169^x-143^x=121^x \Leftrightarrow \left(\frac{169}{143}\right)^x-1=\frac{1}{\left(\frac{143}{121}\right)^x}$$

We put: $y=\left(\frac{169}{143}\right)^x=\left(\frac{143}{121}\right)^x$, and find;

$$y-1=\frac{1}{y} \Leftrightarrow y^2-y-1=0;\ d=1+4=5 \quad y=\frac{1\pm\sqrt{5}}{2} \Rightarrow y=\frac{1+\sqrt{5}}{2}$$

$$\left(\frac{169}{143}\right)^x=\frac{1+\sqrt{5}}{2} \Rightarrow x=\frac{\ln\left(\frac{1+\sqrt{5}}{2}\right)}{\ln 169-\ln 143}$$

452. Simplify: $\dfrac{\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}+\sqrt{6}+\sqrt{15}+\sqrt{25}+\sqrt{27}}$

453. Determine *a and b* from: $a-2ab+b=-127$

It is not possible to rewrite the left hand side as a product (factorize). What we can do is to rewrite as two products, since it is easier to guess two terms than three to form a sum.

$$a-2ab+b=-127 \quad \Leftrightarrow \quad a(1-b)+b(1-a)=-127$$

Since there is symmetry between *a* and *b* then *a* and *b* should be roughly the same:

The rest is guesswork, but -127 =-63 -64, seems to do the trick.

$$a(1-b)+b(1-a)=-127=-63-64=-7\cdot 9-8\cdot 8$$

From which we infer:

$$a(b-1)=+7\cdot 9 \quad \Rightarrow \quad a=-7 \quad and \quad 1-b=9 \quad \Leftrightarrow \quad a=-7 \quad and \quad b=-8$$
$$b(1-a)=-8\cdot 8 \quad \Rightarrow \quad b=-8 \quad and \quad 1-a=8 \quad \Leftrightarrow \quad a=-7 \quad and \quad b=-8$$

So the solution is $a=-7 \quad and \quad b=-8$, since: $a-2ab+b=-7-112-8=-127$

454. Determine *n* and *m* from: $n^2+20n+12=m^2$

To avoid reckless guessing we make the following rewriting:

$$n^2+20n+12=m^2 \quad \Leftrightarrow \quad (n+10)^2-100+12=m^2 \quad \Leftrightarrow$$
$$(n+10)^2-m^2=88 \quad \Leftrightarrow \quad (n+10-m)\,(n+10+m)=88=8\cdot 11=4\cdot 22=2\cdot 44$$

Choosing $88=8\cdot 11$ *or* $88=2\cdot 44$ gives non integer solutions but $88=4\cdot 22$ does.

$$n+10-m=4 \; and \quad n+10+m=22 \quad \Leftrightarrow \quad n-m=-6 \quad and \quad n+m=12 \quad \Leftrightarrow$$
$$2n=6 \quad \Leftrightarrow \quad n=3 \quad and \quad m=9$$

455. Solve for *x*: $4^{3^{2^x}}=2^{162}$

$$4^{3^{2^x}}=2^{162} \quad \Leftrightarrow \quad 2^{2\cdot 3^{2^x}}=2^{162} \quad \Rightarrow \quad 2\cdot 3^{2^x}=162 \quad \Leftrightarrow 3^{2^x}=81=3^4 \quad \Leftrightarrow$$
$$2^x=4 \quad x=2$$

456. Evaluate the integral: $\int_0^1 \sqrt{x(1-x)}dx$

We make the substitution $x=\cos^2 u \quad \Rightarrow \quad dx=-2\cos u\sin u du$

$$\int_0^1 \sqrt{x(1-x)}dx=-2\int_{\frac{\pi}{2}}^{0}\sqrt{\cos^2 u(1-\cos^2 u)}\cos u\sin u du=$$

$$-2\int_{\frac{\pi}{2}}^{0}\sqrt{\cos^2 u \sin^2 u)}\cos u \sin u du =$$

$$-2\int_{\frac{\pi}{2}}^{0}\cos^2 u \sin^2 u)du = -\tfrac{1}{2}\int_{\frac{\pi}{2}}^{0}4\cos^2 u \sin^2 u)du = \tfrac{1}{2}\int_{0}^{\frac{\pi}{2}}\sin^2 2udu$$

Now: $\cos 2x = 1 - 2\sin^2 x,\ so \quad \sin^2 x = \tfrac{1}{2}(1-\cos 2x),\ so \quad \sin^2 2u = \tfrac{1}{2}(1-\cos 4u)$

$$\tfrac{1}{2}\int_{0}^{\frac{\pi}{2}}\sin^2 2udu = \tfrac{1}{4}\int_{0}^{\frac{\pi}{2}}(1-\cos 4u)du =$$

$$\tfrac{1}{4}\left[u + \tfrac{1}{4}\sin 4u\right]_0^{\frac{\pi}{2}} = \frac{\pi}{8}$$

457. Determine *a* and *b* from: $4^a - 36^b = 28$

It takes only very little time to guess that $a = 3 \quad and \quad b = 1$ since $4^3 - 36 = 64 - 36 = 28$.

458. Given that: $x - \sqrt{x} = 7$ **. Determine** $x - \frac{7}{\sqrt{x}}$

$$x - \sqrt{x} = 7 \quad \Rightarrow \quad \sqrt{x} - 1 = \frac{7}{\sqrt{x}} \quad \Rightarrow \quad \sqrt{x} = \frac{7}{\sqrt{x}} + 1$$

$\sqrt{x} = \frac{7}{\sqrt{x}} + 1$ indsættes i $x - \sqrt{x} = 7$:

$$x - \frac{7}{\sqrt{x}} - 1 = 7 \Rightarrow \quad x - \frac{7}{\sqrt{x}} = 8$$

459. Solve for *x*: $x^{\ln x} = 2$

$$x^{\ln x} = 2 \quad \Leftrightarrow \quad \ln x \ln x = \ln 2 \quad \Leftrightarrow \quad \ln^2 x = \ln 2 \quad \Leftrightarrow$$

$$\ln x = \sqrt{\ln 2} \quad \Leftrightarrow \quad x = e^{\sqrt{\ln 2}}$$

460. Solve for *x*: $\sqrt[3]{x} + \sqrt[3]{x-16} + \sqrt[3]{x-8} = 0$

There is no point in trying to solve this analytically…but a qualified guess might be $x = 8$, since:
$\sqrt[3]{8} + \sqrt[3]{8-16} + \sqrt[3]{8-8} = 0$

461. Solve for *x*: $\sin x + 2\sin 2x - \sin 3x = 2$

Using the addition formulas or the logarithmic formulas for sine and cosine, do not bring us nearer to the solution, so we have to guess: $x = \frac{\pi}{2}$ seem a good candidate, since:

$$\sin\tfrac{\pi}{2} + \sin\pi - \sin\tfrac{3\pi}{2} = 1 + 0 - (-1) = 2$$

462. Solve for *x* and *y*: $x + xy + y = 39 \quad and \quad x^2 + y^2 = 65$

$x + xy + y = 39 \quad \Leftrightarrow \quad (x+1)(y+1) = 40 = 5 \cdot 8$

$x + 1 = 5 \quad and \quad y + 1 = 8 \quad \Leftrightarrow \quad x = 4 \quad and \quad y = 7$

$x^2 + y^2 = 4^2 + 7^2 = 65$

463. Determine *x + y* from: $x^2 + y^2 = 7 \quad and \quad x^3 + y^3 = 10$

$x^2 + y^2 = 7 \quad \Leftrightarrow \quad (x+y)^2 - 2xy = 7 \quad \Leftrightarrow \quad (x+y)^2 - 7 = 2xy \quad \Leftrightarrow \quad 3(x+y)^2 - 21 = 6xy$

$x^3 + y^3 = 10 \quad \Leftrightarrow \quad (x+y)^3 - 3xy(x+y) = 10 \quad \Leftrightarrow \quad 2(x+y)^3 - 6xy(x+y) = 20$

$2(x+y)^3 - (3(x+y)^2 - 21)(x+y) = 20$

We put $z = x+y$

$2z^3 - (3z^2 - 21)z = 20 \quad \Leftrightarrow \quad -z^3 + 21z - 20 = 0$

We can see that $z = 1$ is a root, To investigate possible other roots, we make polynomial division with z -1

$z - 1 \,|\, z^3 - 21z + 20 \,|\, z^2 + z - 20$

$z^3 - z^2$

$z^2 - 21z$

$z^2 - z$

$-20z + 20$

$-20z + 20$

$z^2 + z - 20 = 0; \quad d = 1 + 80 = 81 = 9^2 \qquad z = \frac{-1 \pm 9}{2} \quad \Leftrightarrow \quad z = 4 \quad or \quad z = -5$

This gives:

$x + y = 1 \quad or \quad x + y = 4 \quad or \quad x + y = -5 \quad or$

464. Solve for *x*: $\sqrt{x - \sqrt{x-2}} + \sqrt{x + \sqrt{x-2}} = 3$

We take the square of both sides:

$\left(\sqrt{x - \sqrt{x-2}}\right)^2 + \left(\sqrt{x + \sqrt{x-2}}\right)^2 + 2\sqrt{x - \sqrt{x-2}})\sqrt{x + \sqrt{x-2}}) = 9 \quad \Leftrightarrow$

$x - \sqrt{x-2} + x + \sqrt{x-2} + 2\sqrt{x^2 - (x-2)} = 9 \quad \Leftrightarrow$

$2\sqrt{x^2 - (x-2)} = 9 - 2x \quad \Leftrightarrow \quad \left(2\sqrt{x^2 - (x-2)}\right)^2 = (9 - 2x)^2 \quad \Leftrightarrow$

$4(x^2 - (x-2)) = 4x^2 + 81 - 36x \quad \Leftrightarrow \quad -4x + 8 = 81 - 36x \quad \Leftrightarrow \quad 32x = 73 \quad \Leftrightarrow \quad x = \frac{73}{32}$

465. Determine *x and* y from: $(1+x)(1+y)(x+y) = 2016 \quad and \quad x^3 + y^3 = 1216$

$(1+x)(1+y)(x+y) = 2016 \quad \Leftrightarrow \quad (x + y + xy + 1)(x+y) = 2016$

$2016 = 2^8 \cdot 11 = 2^7 \cdot 22$, so there are 8 different ways to factorize 2016.

Most of them have no integer solutions *x* and *y*, but $2016 = 2^7 \cdot 22$ has:

$(x+y+xy+1)(x+y)=2016=2^7\cdot 22 \quad\Rightarrow\quad x+y=22 \quad and \quad x+y+xy+1=128 \quad\Leftrightarrow$
$x+y=22 \quad and \quad xy=105$

$x+y=22 \quad and \quad y=\frac{105}{x} \quad\Rightarrow\quad x+\frac{105}{x}-22=0 \quad\Rightarrow\quad x^2-22x+105=0$
$d=484-420=64$
$x=\frac{22\pm 8}{2} \quad\Leftrightarrow\quad x=10 \quad or \quad x=7 \quad and \quad y=12 \quad or \quad y=15$

466. Solve for *x*: $x^4=4x+1=0$

$x^4=4x+1 \quad\Leftrightarrow x^4-4x-1 \quad\Leftrightarrow\quad (x^2+1)^2-2x^2-1-4x-1=0 \quad\Leftrightarrow$
$(x^2+1)^2-2(x^2+2x+1)=0 \quad\Leftrightarrow\quad (x^2+1)^2-2(x+1)^2=0 \quad\Leftrightarrow$
$(x^2+1)^2=2(x+1)^2 \quad\Leftrightarrow\quad (x^2+1)=\pm\sqrt{2}(x+1) \quad\Leftrightarrow$
$x^2-\sqrt{2}x-\sqrt{2}=0;\ d=2+4\sqrt{2} \quad or \quad x^2+\sqrt{2}x+\sqrt{2}=0;\ d=2-4\sqrt{2}<0 \quad\Leftrightarrow$
$x=\frac{\sqrt{2}\pm\sqrt{2+4\sqrt{2}}}{2}$

467. Determine *m* and *n* such that $2^m-2^n=4032$

$2^{12}=4096$, so we try with: *m* =12. 4096 - 4032= 64=2^6 so *n* = 6.

468. Determine *m* and *n* such that $(n+1)^n=2\cdot n^m+3n+1$

This kind of exercise only be solved by (qualified) guesswork.
n =3 and *m* =3 seem to do the trick, since; $4^3=64 \quad and \quad 2\cdot 3^3+3\cdot 3+1=64$

469. Determine *a* and *b* such that $\sqrt{a}+b=85 \quad and \quad a-\sqrt{b}=7$

We put $a=x^2 \quad and \quad b=y^2$ and the we have:
$x+y^2=85 \quad and \quad x^2-y=7 \quad\Rightarrow\quad y^2-x^2+x+y=78$
$(y-x)(y+x)+(x+y)=78 \quad\Leftrightarrow\quad (y+x)(y-x+1)=78=13\cdot 6$
$y+x=13 \quad and \quad y-x+1=6 \quad\Rightarrow\quad y=9 \quad and \quad x=4 \quad\Rightarrow\quad a=x^2=16 \quad and \quad b=y^2=81$
So: $\sqrt{a}+b=4+81=85 \quad and \quad a-\sqrt{b}=16-9=7$

470. Simplify: $(\sqrt[4]{5}-1)(\sqrt[4]{5}+1+\sqrt[4]{125}+\sqrt{5})$

$(\sqrt[4]{5}-1)(\sqrt[4]{5}+1+\sqrt[4]{125}+\sqrt{5})=(\sqrt[4]{5}-1)(\sqrt[4]{5}+1+\sqrt{5}\sqrt[4]{5}+\sqrt{5})=$
$(\sqrt[4]{5}-1)(\sqrt{5}+1)(\sqrt[4]{5}+1)=(\sqrt[4]{5}-1)(\sqrt[4]{5}+1)(\sqrt{5}+1)=(\sqrt{5}-1)(\sqrt{5}+1)=4$

471. Solve for *x*: $x^{x^{x^{x^5}}}=5$

The solution is $x=\sqrt[5]{5}$, $x^5=$ since $(\sqrt[5]{5})^5=5$ all the way down:

472. Determine *m* such that: $\frac{1}{mn}+\frac{1}{m}=\frac{1}{n}$

This is easy: we put: $m=n+q$: $\frac{1}{(n+q)n}+\frac{1}{n+q}=\frac{1}{n} \quad \Leftrightarrow \quad \frac{1+n}{(n+q)n}=\frac{1}{n}$

But this requires that $q=1$. So $m=n+1$.

474. Simplify $8^{888}-2^{2663}$

$8^{888}-2^{2663}=\left(2^3\right)^{888}-2^{2663}=2^{2664}-2^{2663}=2^{2663}(2-1)=2^{2663}$

475. Solve for *x*. $3^{2x}-5^{2x}=\sqrt{15^{2x}-25^{2x}}$

$$3^y-5^y=\sqrt{15^y-25^y} \quad \Rightarrow \quad (3^y-5^y)^2=15^y-25^y \quad \Rightarrow$$

We put: $y=2x$, so $3^{2y}+5^{2y}-2\cdot 3^y\cdot 5^y=15^y-25^y \quad \Rightarrow$

$$\left(3^y\right)^2+\left(5^y\right)^2-2\cdot 3^y\cdot 5^y=3^y5^y-\left(5^y\right)^2$$

We put $a=3^y$ *and* $b=5^y$ Then we have:

$a^2+b^2-2ab=ab-b^2$

$a^2+2b^2-3ab=0$ We divide with: ab

$\frac{a}{b}+2\frac{b}{a}-3=0 \qquad z+2\frac{1}{z}-3=0 \quad \Leftrightarrow$

$z^2-3z+2=0; \qquad d=9-8=1; \qquad z=\frac{3\pm 1}{2} \quad \Leftrightarrow \quad z=2 \quad or \quad z=1$

$\frac{3^{2x}}{5^{2x}}=2 \quad or \quad \frac{3^{2x}}{5^{2x}}=1 \quad \Leftrightarrow \quad \left(\frac{3}{5}\right)^{2x}=2 \quad or \quad \left(\frac{3}{5}\right)^{2x}=1$

$x=\frac{\ln 2}{2(\ln 3-\ln 5)} \quad or \quad x=0$

476. Solve for *x*. $4^{\log x}+x^{\log 4}=32$

We shall use the identity; $a^{\log b}=b^{\log a} \quad \Leftrightarrow \quad \log b\log a=\log a\log b$, so: $4^{\log x}=x^{\log 4}$

$4^{\log x}+x^{\log 4}=32 \quad \Rightarrow \quad 2\cdot 4^{\log x}=32 \quad \Leftrightarrow \quad 4^{\log x}=16 \quad \Leftrightarrow$

$\log x\log 4=\log 16 \quad \Leftrightarrow \quad \log x=\frac{2\log 4}{\log 4}=2 \quad \Leftrightarrow \quad x=100$

477. Determine *a* and *b*, such that: $3^{a-3}+3^{b-3}=242$

Well: $3^5=243$, so the solution is: $a=8$ and $b=3$, since: $3^5+3^0=243$

478. Determine *a* and *b*, such that: $a+b+c=2022 \quad and \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2022}$

We remind ourselves that $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1 \quad and \quad \frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1 \quad and \quad \frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1$.

And thre are no others.

If we multiply each of these equations by $\frac{1}{2020}$, we shall have $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2022}$, but they give $a+b+c=9\cdot 2020 \quad and \quad a+b+c=11\cdot 2020 \quad and \quad a+b+c=10\cdot 2020$.

It is a general conjecture that: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ has its maximum for constant $a+b+c$ when $a=b=c$, so there is no way that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2022}$ so that $a+b+c=2022$

479. Determine *a* and *b* from: $a^2+3ab=40 \quad and \quad b^2+2ab=11$

We rewrite the equations as:

$a(a+3b)=40 \quad and \quad b(b+2a)=11$

Now b is a prime, so the last equation has (supposedly) the solution $b = 1$ and $b + 2a = 11$

$b=1 \quad and \quad a=5$, and indeed: $a(a+3b)=5(5+3)=40$

The solution is therefore: $b=1 \quad and \quad a=5$

480. Simplify: $\sqrt{114^2-64^2-50^2}$

$$\sqrt{114^2-64^2-50^2}=\sqrt{(114-64)(114+64)-50^2}=$$
$$\sqrt{50\cdot 178-50^2}=\sqrt{50(178-50)}=\sqrt{50\cdot 128}=\sqrt{6400}=80$$

481. Solve for *x*: $\frac{x+33}{35}+\frac{x+50}{29}+\frac{x+27}{64}=6$

To solve this, we shall require that each of these three fractions must be an integer n.

$\frac{x+33}{35}=n \quad \Leftrightarrow \quad x=35n-33. \quad n=1 \Rightarrow x=2 \quad n=2 \Rightarrow x=37; \quad n=3 \Rightarrow x=72$

$\frac{x+50}{29}=n \quad \Leftrightarrow \quad x=29n-50. \quad n=1 \Rightarrow x=21 \quad n=2 \Rightarrow x=8; \quad n=3 \Rightarrow x=37$

$\frac{x+27}{64}=n \quad \Leftrightarrow \quad x=64n-27. \quad n=1 \Rightarrow x=37 \quad n=2 \Rightarrow x=101; \quad n=3 \Rightarrow x=166$

We can see that $x = 37$ result in an integer for all three fractions. Furthermore we have:

$$\frac{37+33}{35}+\frac{37+50}{29}+\frac{37+27}{64}=2+6+1=6$$

482. Simplify: $\frac{a^8-a^2}{a^6-a^2}=9 \quad \Leftrightarrow \quad \frac{a^6-1}{a^2-1}=9 \quad \Leftrightarrow \quad \frac{(a^3-1)(a^3+1)}{(a-1)(a+1)}=9 \quad \Leftrightarrow$

$$\frac{\left((a-1)^3+3a(a-1)\right)\left((a+1)^3-3a(a+1)\right)}{(a-1)(a+1)}=9 \quad \Leftrightarrow \quad \frac{\left((a-1)^2+3a\right)\left((a+1)^2-3a\right)}{1}=9 \quad \Leftrightarrow$$

$(a^2+a+1)(a^2-a+1)=9$

For integer values the only possibility is:

$(a^2+a+1)=3 \quad and \quad (a^2-a+1)=3 \quad \Leftrightarrow \quad a^2+a-2=0 \quad and \quad a^2-a-2=0$

$d=1+8=9 \quad a=\frac{-1\pm 3}{2} \quad \Leftrightarrow \quad a=1 \ or \ a=-2 \quad and \quad a=\frac{1\pm 3}{2} \Leftrightarrow \quad a=2 \quad or \ a=-1$

So there are no solution, unless we consider $(a^2+a+1)=-3 \quad and \quad (a^2-a+1)=-3$, but in that case the quadratic equations have no solution.

483. $x^2-y^2=24 \quad and \quad xy=35$ **, Determine *x + y***

$x^2-y^2=24 \quad \Leftrightarrow \quad (x-y)(x+y)=24$

If we factorize 24 then the possibilities are: $24=1\cdot 24;\ 24=2\cdot 12;\ 24=3\cdot 8;\ 24=4\cdot 6;$

We try first with: $(x-y)(x+y)=4\cdot 6$

For integer solutions, we must have;

$(x-y)=4 \quad and \quad (x+y)=6$

This gives: $x=5 \quad and \quad y=9$ But this does not comply with: $xy=35$

We try next

with: $(x-y)(x+y)=2\cdot 12 \quad \Leftarrow \quad (x-y)=2 \ and \ (x+y)=12 \quad \Leftrightarrow \quad x=7 \quad and \quad y=5$

Which is seen to comply with: $x^2-y^2=24 \quad and \quad xy=35$

Amore formal proof can be obtained:

$x^2-y^2=24 \quad \Leftrightarrow \quad (x-y)(x+y)=24$

$(x-y)^2(x+y)^2=24^2 \quad \Leftrightarrow$

$\left(x^2+y^2-2xy\right)\left(x^2+y^2+2xy\right)=24^2 \quad \Rightarrow$

$\left(\left(x^2+y^2\right)-70\right)\left((x^2+y^2)+70\right)=24^2 \quad \Rightarrow$

$(x^2+y^2)^2-70^2=24^2 \quad \Rightarrow$

$(x^2+y^2)^2=70^2+24^2=5476=74^2 \quad \Rightarrow$

$(x^2+y^2)=74$

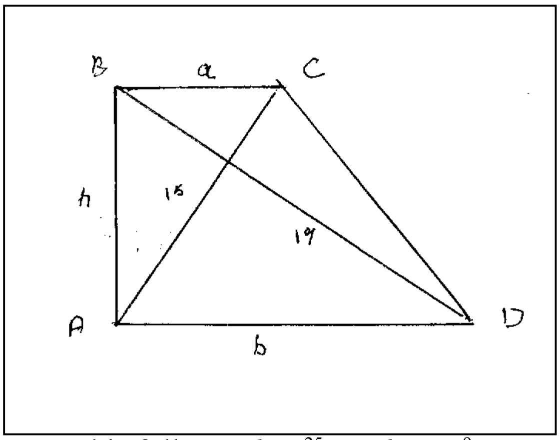
$(x+y)^2=74+70=144$

$x+y=12$

$x^2-y^2=24 \quad \Leftrightarrow \quad (x-y)(x+y)=24 \quad \Rightarrow$

$(x-y)^2(x+y)^2=24^2 \quad \Rightarrow \quad (x^2+y^2-2xy)(x^2+y^2+2xy)=24^2$

484. Determine the height in the trapezoid shown below



The trapezoid $ABCD$ has two right angles, and the diagonals are 15 and 19 respectively. The two parallel sides are denoted a and b respectively. We shall determine a value for the height. From the two right angled triangles ABC and BAD we find:

$h^2 + a^2 = 15^2 \quad and \quad h^2 + b^2 = 19^2$ from which we find:

$b^2 - a^2 = 19^2 - 15^2 = (19+15)(19-15) = 34 \cdot 4 = 17 \cdot 8$

$b^2 - a^2 = (b+a)(b-a) = = 8 \cdot 17$

There are the various possibilities for the factors, but $and \quad b - a = 8$

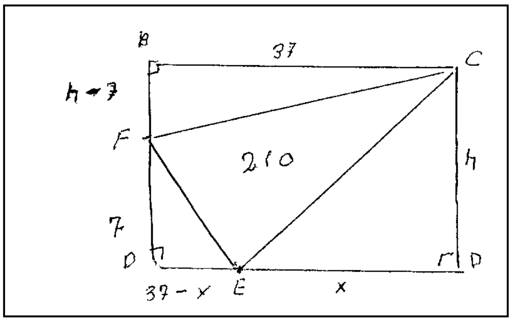
From this follows; $b = \frac{25}{2} \; and \; a = \frac{9}{2}$

We may then determine the height in two different ways:

$h^2 + a^2 = 15^2 \quad \Leftrightarrow \quad h^2 = 15^2 - \left(\frac{9}{2}\right)^2 = 204.75 \qquad and \quad h^2 = 19^2 - \left(\frac{25}{2}\right)^2 = 204.75 \;\Leftrightarrow$

The height is the $h = \sqrt{\dfrac{819}{4}} \approx 14.31$

485. Determine the height in the rectangle shown below



To the left is shown a rectangle $ABCD$, having one side 37, and where a triangle CEF having area 210 is incribed. Our aim is to find h the height in the rectangle.

To do so we express that the area of the triangle is equal to the area of the rectangle minus the area of the three right angle triangles FBC, CDE and DFE.

$37h - \frac{1}{2}37(h-7) - \frac{1}{2}hx - \frac{1}{2}7(37-x) = 210 \quad \Leftrightarrow$

$74h - 37(h-7) - hx - 7(37-x) = 420 \quad \Leftrightarrow$

$74h - 37(h-7) - hx - 7(37-x) = 420 \quad \Leftrightarrow$

$37h + 37 \cdot 7 - hx - 7(37-x) = 420 \quad \Leftrightarrow$

$h(37-x) - 7(37-x) = 420 - 37 \cdot 7 \; = 161 = 7 \cdot 23 \quad \Leftrightarrow$

$(37-x)(h-7) = 7 \cdot 23$

If we assume integer solution, (and 7 and 23 are prime numbers), we can either assume:

$(37-x) = 7 \quad and \quad (h-7) = 23 \qquad or \quad (37-x) = 23 \quad and \quad (h-7) = 7$

Which gives:

$h = 30 \quad and \quad x = 30 \quad or \qquad h = 14 \quad and \quad x = 14$

486. Determine *x* and *y* from $16^{x^2+y} + 16^{y^2+x} = 1$

Since there is complete symmetry between x and y , we try a solution where: $16^{x^2+y} = 16^{y^2+x} = \frac{1}{2}$

Since $16^{-\frac{1}{4}} = \frac{1}{2}$. We try:

$x^2 + y = -\frac{1}{4} \quad and \quad y^2 + x = -\frac{1}{4} \quad \Rightarrow \quad x^2 - y^2 + y - x = 0 \quad \Leftrightarrow$

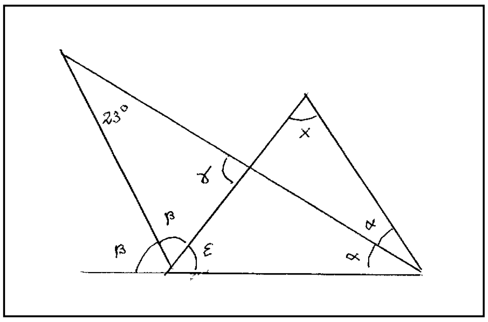
$(x-y)(x+y) - (x-y) = 0 \quad \Leftrightarrow$

$(x-y)((x+y) - 1 = 0 \quad \Leftrightarrow x = y \;\; or \quad x + y - 1 = 0$

We insert $y = x \;\; in \;\; x^2 + y = -\frac{1}{4} \;\; to\; get \;\; x^2 + x = -\frac{1}{4} \quad \Leftrightarrow$

$x^2 + x + \frac{1}{4} = 0 \quad \Leftrightarrow \quad (x + \frac{1}{2}) = 0 \quad \Leftrightarrow \quad x = y = -\frac{1}{2}$

487. Determine the angle *x* from the figure below



The angles and their relations are shown. Only one angle is given. We shall then write some equations to determine *x*.

$\beta + \gamma + 23 = 180$

$\varepsilon + x + 2\alpha = 180$

$\gamma = \varepsilon + \alpha$

$\varepsilon = 180 - 2\beta$

First we eliminate ε ftom the equations:

$\varepsilon + x + 2\alpha = 180 \quad \Rightarrow \quad 180 - 2\beta + x + 2\alpha = 180 \quad \Rightarrow$

$x + 2\alpha - 2\beta = 0$

$\gamma = \varepsilon + \alpha \quad \Rightarrow \quad \gamma = 180 - 2\beta + \alpha$

Then we eliminate γ from the equations:

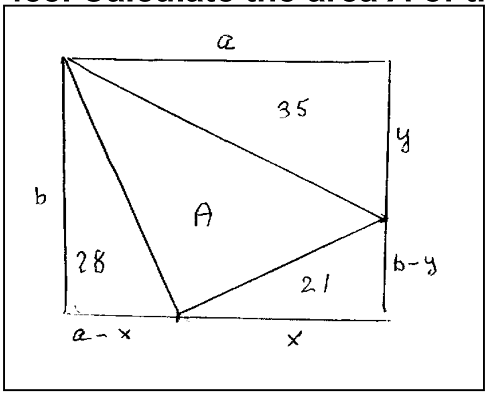
$\beta + \gamma + 23 = 180 \quad and \quad \gamma = 180 - 2\beta + \alpha \Rightarrow \beta + 180 - 2\beta + \alpha + 23 = 180 \Leftrightarrow \alpha - \beta + 23 = 0$

Then we hold the two equations together:

$x + 2\alpha - 2\beta = 0 \quad and \quad \alpha - \beta + 23 = 0 \quad \Leftrightarrow \quad x + 2(\alpha - \beta) = 0 \quad and \quad 2(\alpha - \beta) + 46 = 0 \quad \Rightarrow$

$x = 46$

488. Calculate the area *A* of the inscribed triangle, shown below



In the rectangle is inscribed a triangle having area *A*. The task is to calculate *A* from the three right angle triangles having the areas 35, 21 and 28.

We may the establish 3 equations:

$\frac{1}{2}ay = 35 \quad \frac{1}{2}x(b-y) = 21 \; and \; \frac{1}{2}b(a-x) = 28$

Which we multiply by 2.

$ay = 70 \quad x(b-y) = 42 \quad and \quad b(a-x) = 56$

We then eliminate then *x* and *y* from these equations:

$y = \dfrac{70}{a}$ is inserted in $x(b-y) = 42$ to give: $x(b - \dfrac{70}{a}) = 42$ and

$b(a-x)=56 \quad \Leftrightarrow \quad x=\frac{ba-56}{b}$ is inserted in $x(b-\frac{70}{a})=42$ to give:

$\left(\frac{ba-56}{b}\right)\left(b-\frac{70}{a}\right)=42 \quad \Leftrightarrow \quad (ba-56)(ba-70)=42ab$

We put $u = ab$

$(z-56)(z-70)=42z \quad \Leftrightarrow \quad z^2-168z+56\cdot 70=0 \quad ; \quad d=168^2-4\cdot 56\cdot 70=12544=112^2$

$z=\frac{168\pm 112}{2} \quad \Leftrightarrow \quad z=140 \quad or \quad 28$

So the area of the rectangle is 140

The area of the triangle is therefore: 140 – 35-21-28=140 – 84 = 56

489. Determine *a, b,c* from $a^2+b^2+c^2=1000$

There is no analytic solution to this equation, but a solution is: $a=30, b=8,\ c=6$, since:

$30^2+8^2+6^2=1000$

490. Simplify: $\sqrt[3]{\sqrt{5}+2}+\sqrt[3]{\sqrt{5}-2}$

We put: $y=\sqrt[3]{\sqrt{5}+2}+\sqrt[3]{\sqrt{5}-2}$ and evaluate:

$y^3=(\sqrt[3]{\sqrt{5}+2})^3+(\sqrt[3]{\sqrt{5}-2})^3+3(\sqrt[3]{\sqrt{5}+2})(\sqrt[3]{\sqrt{5}-2})((\sqrt[3]{\sqrt{5}+2})+(\sqrt[3]{\sqrt{5}-2})) \quad \Leftrightarrow$

$y^3=\sqrt{5}+2+\sqrt{5}-2+3(\sqrt[3]{5-4})y \quad \Leftrightarrow$

$y^3=2\sqrt{5}+2y \quad \Leftrightarrow \quad y^3-2y-3\sqrt{5}=0$

We can see that $y=\sqrt{5}$ is a solution. Me make polynomial division with $y-\sqrt{5}$

$y-\sqrt{5} \mid y^3-2y-3\sqrt{5} \mid y^2+\sqrt{5}y+3$

$y^3-\sqrt{5}y^2$

$\sqrt{5}y^2-5y$

$3y-3\sqrt{5}$

$3y-3\sqrt{5}$

However $y^2+\sqrt{5}y+3=0$ has no solution, so the only solution is $y=\sqrt{5}$.

490. Determine *a* + *b* and *ab* from: $a^2+ab+b^2=19$

$a^2+ab+b^2=19 \quad \Leftrightarrow \quad (a+b)^2-ab=19 \quad \Leftrightarrow$

$(a+b)^2-\sqrt{ab}^2=19 \quad \Leftrightarrow \quad ((a+b)-\sqrt{ab})((a+b)+\sqrt{ab})=19=1\cdot 19$

If we confine ourselves to integer solution, and 19 is an integer we must have:

$((a+b)-\sqrt{ab})=1 \quad and \quad ((a+b)+\sqrt{ab})=19 \quad \Rightarrow$

$2(a+b)=20 \quad \Rightarrow \quad a+b=10 \quad and \quad \sqrt{ab}=9 \quad \Rightarrow \quad ab=81$

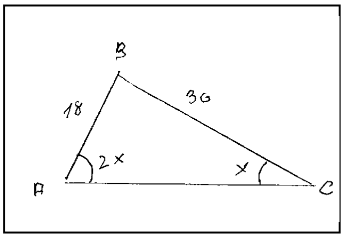
491.Determine *m* and *n* from: $2^m-2^n=2560$

Since $2^{11}=2448 \quad and \quad 2560-4048=512=2^9$ the solutoion is *m* =11 and n = 9.

492. Solve for *x*. $2^{x^2} = 1 - x^8$

I resent these kinds of transcendental exercises, which have no analytic solution, but where the solution is trivial.
In the equation above $x = 0$ is the solution.
If the equation had been $2^{x^2} = 2 - x^8$ the solution had been $x = 1$

493. Determine the angle *x* from the triangle shown below.

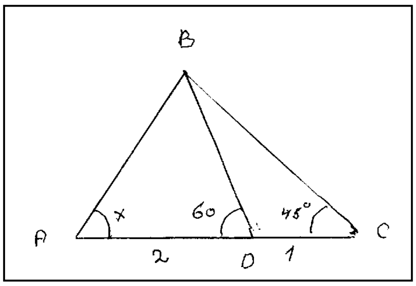


From the sine relation: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ we have:

$\frac{\sin 2x}{30} = \frac{\sin x}{18} \Leftrightarrow 18\sin 2x = 30\sin x \Leftrightarrow$

$3 \cdot 2\sin x\cos x = 5\sin x \Leftrightarrow \cos x = \frac{5}{6} \Leftarrow \quad x = 33.6$

493. Determine the angle *x* from the triangle shown below



From the sine relations : $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ we have:

In the triangle *CBD*

$\angle CBD = 180 - (120 + 45) = 15$

$\frac{\sin 15}{1} = \frac{\sin 45}{|BD|} \Leftrightarrow |BD| = \frac{\sin 45}{\sin 15}$

In the triangle *CBD* we have: $\angle DBA = 180 - x - 60 = 120 - x$

$\frac{\sin x}{|DB|} = \frac{\sin(120 - x)}{2} \Leftrightarrow 2\sin x = |DB|\sin(120 - x) = \frac{\sin 45}{\sin 15}\sin(120 - x)$

$\sin(120 - x) = \sin(180 - (120 - x)) = \sin(60 + x) = \sin 60\cos x + \cos 60\sin x = \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x$

$\sin 45 = \frac{\sqrt{2}}{2} \quad \sin\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}} \quad \sin 15 = \sqrt{\frac{1 - \cos 30}{2}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2 - \sqrt{3}}{4}}\sin 45 = \frac{\sqrt{2}}{2} \quad \sin\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$ s

$2\sin x = \frac{\frac{\sqrt{2}}{2}}{\sin 15}\left(\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right)$ And by division wit cos *x*:

$2\tan x = \frac{\frac{\sqrt{6}}{4} + \frac{1}{2}\tan x}{\sin 15} \Rightarrow \quad \tan x = \frac{\frac{\sqrt{6}}{4}}{2\sin 15 - \frac{1}{2}}$ =88.35

494.Determine *a* from the equation: $a^2 + a^3 = 392$

The answer is $a = 7$, since: $7^2 + 7^3 = 49 + 343 = 392$

495.Determine *a, b, c* from the equations: $a + 100 = b^2 \quad and \quad a + 168 = c^2$

From the two equations follows: $c^2 - b^2 = 68 \quad \Leftrightarrow \quad (c - b)(c + b) = 68 = 4 \cdot 17 = 2 \cdot 34$

If we assume integer values only: we have:

$c-b=4 \quad and \quad c+g \quad =17 \;\; or \;\; c-b=2 \quad and \quad c+b \quad =34$

The first choice does not result in integer values, so we assume:

$c-b=2 \quad and \quad c+b \quad =34 \quad \Rightarrow \quad 2c=36 \quad \Rightarrow \quad c=18 \quad and \quad b=16 \quad a=256-100=156$

496. Determine *x, y* from the equation: $x^2-y^2=29$

$x^2-y^2=29 \quad \Leftrightarrow \quad (x-y)(x+y)=29 \quad \Leftarrow$
$x-y=1 \quad and \quad x+y=29 \;\Rightarrow\; 2x=30 \;\Rightarrow\; x=15 \quad and \quad y=14$

496. Determine *a and b* from the equation: $7^{a-7}-7^{b-7}=2400$
We put $x = a - 7$ and $y = b - 7$ so we have:
$7^x-7^y=2400$. Now $7^4=2001$, so the solution is $a = 11$ and $b = 7$

496. Determine *x, y* from the equations: $y-\sqrt{x}=43 \quad and \quad x-\sqrt{y}=29$

We put $a=\sqrt{x} \quad and \quad b=\sqrt{y}$ and we find:
$b^2-a=43 \quad and \quad a^2-b=29 \quad \Rightarrow \quad b^2-a^2-(b-a)=14 \quad \Rightarrow$
$(b-a)((b+a)-1)=14 \quad \Leftarrow \quad (b-a)=2 \quad and \quad (b+a)-1) \quad \Leftrightarrow \quad b=4 \;\; and \;\; a=2 \;\; \Rightarrow$
$x=4 \quad and \quad y=16$

497. solve for *x*: $x^{x^5}=5$

It is guesswork, but $\sqrt[5]{5}$ does the trick, since: $\sqrt[5]{5}^{\sqrt[5]{5}^5}=\sqrt[5]{5}^5=5$

498. solve for *x*: $x^{3x^{15}}=5$

It is guesswork, but $\sqrt[15]{10}$ does the trick, since: $\sqrt[15]{10}^{3\sqrt[15]{10}^{15}}=(\sqrt[15]{10}^{15})^2=10^2=100$

499. solve for *x*: $x^{x^5}=10$

It is guesswork, but $\sqrt[5]{10}$ does the trick, since: $\sqrt[5]{10}^{\sqrt[5]{10}^5}=\sqrt[5]{10}^{10}=100$

500. solve for *x*: $\dfrac{27^x-3^x}{9^x-3^x}=82$

$$\frac{27^x-3^x}{9^x-3^x}=82 \quad \Leftrightarrow \quad \frac{(3^x)^3-3^x}{(3^x)^2-3^x}=82 \quad \Leftrightarrow \quad \frac{(3^x)^2-1}{3^x-1}=82$$

We put $y=3^x$ to get:

$$\frac{y^2-1}{y-1}=82 \quad \Leftrightarrow \quad \frac{(y-1)(y+1)}{y-1}=82 \quad \Leftrightarrow \quad y+1=82 \quad \Leftrightarrow \quad 3^x=81 \quad \Leftrightarrow \quad x=4$$