Contents

301. Simplify: $(\sqrt{3}+3)^3-(\sqrt{3}-3)^3$

$(a+b)^3=a^3+b^3+3ab(a+b)$

$(\sqrt{3}+3)^3=\sqrt{3}^3+3^3+3\sqrt{3}\cdot 3(\sqrt{3}+3)=3^3+3^3\sqrt{3}$

$(\sqrt{3}-3)^3=\sqrt{3}^3-3^3-3\sqrt{3}\cdot 3(\sqrt{3}-3)=3^3-3^3\sqrt{3}$

$(\sqrt{3}+3)^3-(\sqrt{3}-3)^3=2\cdot 3^3+2\sqrt{3}\cdot 3^3=2\cdot 3^3(1+\sqrt{3})$

302. Determine integers (*x, y*), such that $x+y+xy=54$

There is really no analytic way to solve this, so we must resort to qualified guesswork.

$x+y+xy=54 \quad \Leftrightarrow \quad x+y=54-xy$

It seems that $xy=40$ is a reasonable assumption, 5 and 8 does not work since 5+8 =13,
But $x=4 \quad and \quad y=10$ does the trick, since: $4+10+4\cdot 10=54$

303. $x+y=2 \quad xy=3$ **Determine** x^5+y^5

This is indeed a very strange exercise, since $x+y=2 \;\; and \;\; xy=3$ has no real solution:

$xy=3 \quad \Leftrightarrow \quad y=\frac{3}{x} \quad \Rightarrow \quad x+\frac{3}{x}=2 \quad \Leftrightarrow \quad x^2-2x+3=0; \quad d=4-12<0$

Nevertheless, if we ignore this, we find;

$(x+y)^2=4 \quad \Leftrightarrow \quad x^2+y^2+2xy=4 \quad \Leftrightarrow \quad x^2+y^2+6=4 \quad \Leftrightarrow \quad x^2+y^2=-2\,!$

$(x+y)^3=8 \quad \Leftrightarrow \quad x^3+y^3+3xy(x+y)=8 \quad \Leftrightarrow \quad x^3+y^3+9\cdot 2=8 \quad \Leftrightarrow \quad x^3+y^3=-10$

$(x^3+y^3)(x^2+y^2)=(-2)(-10) \quad \Leftrightarrow \quad x^5+x^3y^2+x^2y^3+y^5=20 \quad \Leftrightarrow$

$x^5+x^3y^2+x^2y^3+y^5=20 \quad \Leftrightarrow$

$x^5+y^5+x^2y^2(x+y)=20 \quad \Leftrightarrow \quad x^5+y^5+9\cdot 2=20$

$x^5+y^5=2$

304. Solve for *x*; $\left(\sqrt{\frac{2x-1}{x-1}}\right)^2-\left(\sqrt{\frac{2x+1}{x+1}}\right)^2=4$

$\left(\sqrt{\frac{2x-1}{x-1}}\right)^2-\left(\sqrt{\frac{2x+1}{x+1}}\right)^2=4 \quad \Leftrightarrow \quad \frac{2x-1}{(x-1)^2}-\frac{2x+1}{(x+1)^2} \quad \Leftrightarrow$

$(2x-1)(x+1)^2-(2x+1)(x-1)^2=(x-1)^2(x+1)^2 \quad \Leftrightarrow$

$(2x-1)(x^2+1+2x)-(2x+1)(x^2+1-2x)=(x^2-1)^2 \quad \Leftrightarrow$

$2x^3+2x+4x^2-x^2-1-2x-(2x^3+2x-4x^2+x^2+1-2x)=x^4+1-2x^2 \quad \Leftrightarrow$

$2x^3 + 2x + 4x^2 - x^2 - 1 - 2x - 2x^3 - 2x + 4x^2 - x^2 - 1 + 2x = x^4 + 1 - 2x^2 \quad \Leftrightarrow$

$6x^2 - 2 = x^4 + 1 - 2x^2 \quad \Leftrightarrow$

$x^4 - 8x^2 + 3 = 0$

$y = x^2 \quad \Rightarrow \quad y^2 = \dfrac{8 \pm 2\sqrt{13}}{2} = 4 \pm \sqrt{13}$

$x = \sqrt{4 + \sqrt{13}} \quad \vee \quad x = \sqrt{4 - \sqrt{13}}$

305. Solve for *x*: $2^{3x} - 2^x = 120$

$2^{3x} - 2^x = 120 \quad \Leftrightarrow \quad (2^x)^3 - 2^x = 120$

We put: $y = 2^x$, and then we have: $y^3 - y - 120 = 0$

Obviously it has the solution y =5, since $125 - 5 - 120 = 0$

To find possible other solutions, we dive with $y - 5$

$y - 5 \mid y^3 - y - 120 \mid y^2 + 5y + 24$

$y^3 - 5y^2$

$5y^2 - y$

$5y^2 - 25y$

$24y - 120$

$24y - 120$

$y^2 + 5y + 24 = 0$ has no solutions, since $d = 25 - 4 \cdot 25 < 0$, so the only solution is: $2^x = 5$

$x = \dfrac{\ln 5}{\ln 2}$

306. $3^{2x} = 2^{3x} = 5184$

$5184 = 2^6 \cdot 3^4$

$3^{2x} = 2^{3x} = 5184 \quad \Leftrightarrow \quad 9^x = 8^y = 2^6 \cdot 3^4 = 8^2 \cdot 9^2$

$9^x = 8^2 \cdot 9^2 \quad \Leftrightarrow \quad 9 = (8^2 \cdot 9^2)^{\frac{1}{x}} \quad \textit{and} \quad 8^y = 8^2 \cdot 9^2 \quad \Leftrightarrow \quad 8 = (8^2 \cdot 9^2)^{\frac{1}{y}}$

$8 = (8^2 \cdot 9^2)^{\frac{1}{y}} \quad \Leftrightarrow \quad 8 = (8 \cdot 9)^{\frac{2}{y}}$

$8 \cdot 9 = (8^2 \cdot 9^2)^{\frac{1}{y}} (8^2 \cdot 9^2)^{\frac{1}{x}} \quad \Leftrightarrow \quad 8 \cdot 9 = (8 \cdot 9)^{\frac{2}{x} + \frac{2}{y}} \quad \Rightarrow \quad \dfrac{2}{x} + \dfrac{2}{y} = 1 \quad \Leftrightarrow \quad \dfrac{x + y}{xy} = \dfrac{1}{2}$

307. $2^a = 2^b = 1296$ **Determine** $\dfrac{a + b}{ab}$

$1296 = 2^4 \cdot 3^4$

$2^a = 1296 \quad \Leftrightarrow \quad 2 = (1296)^{\frac{1}{a}}$

$3^b = 1296 \quad \Leftrightarrow \quad 3 = (1296)^{\frac{1}{b}}$

$(2\cdot 3)^1 = (1296)^{\frac{1}{a}}(1296)^{\frac{1}{b}} = (1296)^{\frac{1}{a}+\frac{1}{b}} = (2^4\cdot 3^4)^{\frac{a+b}{ab}} = (2\cdot 3)^{4(\frac{a+b}{ab})} \quad \Rightarrow$

$4(\frac{a+b}{ab}) = 1 \quad \Leftrightarrow \quad \frac{a+b}{ab} = \frac{1}{4}$

308. Simplify: $2^{20} - 20^2$ **(very easy)**

$2^{20} - 20^2 = (2^{10})^2 - 20^2 = (2^{10} - 20)(2^{10} + 20) = (1024 - 20)(1024 + 20) = 1004\cdot 1044$

309. Determine positive integers *x* and *y* such that $x^3 - y^3 = xy + 61$

There is to my knowledge no analytic way to solve this problem, so we shall resort to guesswork.
It seems that x and y should be less than 6 and greater than 4.
We ca see that $x = 6$ and $y = 5$ does the trick, since $6^3 - 5^3 = 212 - 125 = 91 = 5\cdot 6 + 61$

310. Determine *a* and *b* from $a^3 + b^3 = 7 \quad and \quad a^2 + b^2 + a + b + ab = 4$

We put: $u = a + b \quad and \quad v = ab$

$(a+b)^2 = a^2 + b^2 + 2ab = 4 - ((a+b) + ab) + 2ab \quad \Leftrightarrow$

$u^2 = 4 - u - v + 2v \quad \Leftrightarrow \quad u^2 + u - v - 4 = 0$

$(a+b)^3 = a^3 + b^3 + 3ab(a+b) \quad \Leftrightarrow$

$u^3 = 7 + 3uv$

$I : u^3 - 3uv - 7 = 0$

$II : \; u^2 + u - v - 4 = 0 \quad \Leftrightarrow \quad v = u^2 + u - 4$

v inserted in I

$u^3 - 3uv - 7 = 0 \quad \Rightarrow \quad u^3 - 3u(u^2 + u - 4) - 7 = 0 \quad \Leftrightarrow$

$-2u^3 - 3u^2 + 12u - 7 = 0$

We can see that $u = 1$ is a root, since: $-2 - 3 + 12 - 7 = 0$
So we make polynomial division with; $u - 1$

$u - 1 \,|\, -2u^3 - 3u^2 + 12u - 7 \,|\, -2u^2 - 5u + 7$

$-2u^3 + 2u^2$

$-5u^2 + 12u$

$-5u^2 + 5u$

$7u - 7$

$7u - 7$

$-2u^2 - 5u + 7 = 0$; $d = 25 + 56 = 81$

$u = \frac{5 \pm 9}{-4} \Leftrightarrow u = 1 \quad or \quad u = -\frac{7}{2}$

$v = u^2 + u - 4 \Leftrightarrow v = -2 \quad or \quad v = \frac{49}{4} - \frac{14}{4} - \frac{16}{4} = \frac{19}{4}$

$u = a + b \quad and \quad v = ab$

$a + b = 1 \quad and \quad ab = -2 \Rightarrow a - \frac{2}{a} - 1 = 0 \Leftrightarrow a^2 - a - 2 = 0; \quad d = 1 + 8 = 9$

$a = \frac{1 \pm 3}{2} \Leftrightarrow a = 2 \quad or \quad a = -1 \quad and \quad b = -\frac{2}{a} \Leftrightarrow b = -1 \quad or \quad b = 2$

$u = a + b \quad and \quad v = ab$

$a + b = -\frac{7}{2} \quad and \quad ab = \frac{19}{4} \Rightarrow a + \frac{19}{4a} + \frac{7}{2} = 0 \Leftrightarrow 4a^2 + 14a + 19 = 0; \quad d = 256 - 16 \cdot 19 < 0$

$a = \frac{1 \pm 3}{2} \Leftrightarrow a = 2 \quad or \quad a = -1 \quad and \quad b = -\frac{2}{a} \Leftrightarrow b = -1 \quad or \quad b = 2$

311. Solve for *x*: $x^{\log x} - 100x = 0$

$x^{\log x} - 100x = 0 \Leftrightarrow x^{\log x} = 100x \Leftrightarrow \log(x^{\log x}) = \log(100x) \Leftrightarrow$

$(\log x)(\log x) = \log 100 + \log x \Leftrightarrow$

$(\log x)^2 - \log x - 2 = 0; \quad y = \log x$

$y^2 - y - 2 = 0; \quad d = 1 + 8 = 9$

$y = \frac{1 \pm 3}{2} \Leftrightarrow y = 2 \quad or \quad y = -1 \Rightarrow \log x = 2 \quad or \quad \log x = -1$

$x = 100 \quad or \quad x = \frac{1}{10}$

312. $100^{x-1} = 99$. **Determine** 100^{x+1}

$\frac{100^{x+1}}{100^{x-1}} = 100^2 \Rightarrow 100^{x+1} = 100^2 \cdot 100^{x-1} = 99 \cdot 10^4$

313. Given: $12^x = 18$**. Determine** $2^{\frac{2x-1}{x-2}}$

$12^x = 18 \Leftrightarrow 3^x \cdot 2^{2x} = 2 \cdot 3^2 \Leftrightarrow 3^{x-2} \cdot 2^{2x-1} = 1 \Rightarrow$

$(3^{x-2} \cdot 2^{2x-1})^{\frac{1}{x-2}} = 1^{\frac{1}{x-2}} \Rightarrow 3 \cdot 2^{\frac{2x-1}{x-2}} = 1 \Leftrightarrow 2^{\frac{2x-1}{x-2}} = \frac{1}{3}$

314. Calculate the sum: $\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+...+\frac{1}{\sqrt{8}+\sqrt{9}}$

$$\frac{\sqrt{1}-\sqrt{2}}{(\sqrt{1}+\sqrt{2})(\sqrt{1}-\sqrt{2})}+\frac{\sqrt{2}-\sqrt{3}}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})}+\frac{(\sqrt{3}-\sqrt{4})}{(\sqrt{3}+\sqrt{4})(\sqrt{3}-\sqrt{4})}+...+\frac{(\sqrt{8}-\sqrt{9})}{(\sqrt{8}+\sqrt{9})(\sqrt{8}-\sqrt{9})}=$$

$$\frac{\sqrt{1}-\sqrt{2}}{1-2}+\frac{\sqrt{2}-\sqrt{3}}{2-3}+\frac{(\sqrt{3}-\sqrt{4})}{4-3}+...+\frac{(\sqrt{8}-\sqrt{9})}{8-9}=$$

$$\frac{\sqrt{1}-\sqrt{2}}{1-2}+\frac{\sqrt{2}-\sqrt{3}}{2-3}+\frac{(\sqrt{3}-\sqrt{4})}{4-3}+...+\frac{(\sqrt{8}-\sqrt{9})}{8-9}=$$

$$\sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2}+\sqrt{4}-\sqrt{3}+...+\sqrt{9}-\sqrt{8}=\sqrt{9}-1=2$$

315. Which is greater: $\sqrt{5+5\sqrt{5}}$ **or 4**

We solve the inequality: $\sqrt{5+5\sqrt{5}}>4 \;\Leftrightarrow\; 5+5\sqrt{5}>16 \;\Leftrightarrow\; 5\sqrt{5}>11 \;\Leftrightarrow\; 125>121$

316. Given: $\frac{1}{x}+\frac{1}{y}=\frac{2}{35}$ **Determine *x* + *y***

It is rather obvious that if $x=y=35$. The equation is fulfilled, and since it is a linear equation in $\frac{1}{x}$ *and* $\frac{1}{y}$ there can be no other solutions. The solution is therefore $x+y=70$.

317. Solve for (x, y): $\frac{1}{x}+\frac{1}{y}=\frac{1}{2004}$

Her we have one equation with 2 variables, which cannot be solved by analytic methods.
However, if the variables are positive integers, we may resort to guesswork.
Since the equation is symmetric in x and y , we may assume that $x = y$.
Then it is easy, if we put x and y equal to the double of 2004, it works, since;

$$\frac{1}{4008}+\frac{1}{4008}=\frac{2}{4008}=\frac{1}{2004}$$

318. Solve for *x*: $\sin x+\sin 2x+\sin 3x=0$

We make use of the logarithmic formulas for the addition of to sine or cosine functions.

$$\sin u+\sin v=2\sin\frac{u+v}{2}\cos(\frac{u-v}{2})$$

We shall apply this formula on $\sin x+\sin 3x$

$$\sin 3x+\sin x=2\sin\frac{3x+x}{2}\cos(\frac{3x-x}{2})=2\sin 2x\cdot\cos x$$

$\sin x + \sin 2x + \sin 3x = 0 \quad \Leftrightarrow \quad \sin 2x + 2\sin 2x \cdot \cos x \quad \Leftrightarrow$

$\sin 2x(1 + 2\cos x) = 0 \quad \Leftrightarrow \quad \sin 2x = 0 \quad \Leftrightarrow \quad \sin 2x = 0 \quad \vee \quad 1 + 2\cos x = 0 \quad \Leftrightarrow$

$2x = \pi \quad \vee \quad 2x = 2\pi \quad \vee \quad \cos = -\frac{1}{2} \quad \Leftrightarrow$

$$x = \frac{\pi}{2} \quad \vee \quad x = \pi \quad \vee \quad x = \frac{2\pi}{3} \quad \vee \quad x = \frac{4\pi}{3}$$

319.Determine *a* and *b*, such that: $ab + c = 2020 \quad \textit{and} \quad a + bc = 2021$

$ab + c = 2020 \quad \textit{and} \quad a + bc = 2021$, If we add these two equations, we find;

$ab + bc + a + c = 4041 \quad \Leftrightarrow \quad b(a + c) + a + c = 4041$

$(a + c)(b + 1) = 4041$

To find the possible values of the two factors, we resolve 4041 in prime factors. Fortunately there are not so many, since: $4041 = 3 \cdot 3 \cdot 441 = 3 \cdot 1347$ (441 is a prime)
So if $(a + c)(b + 1) = 4041$, then we have the following possibilities:

$a + c = 3 \quad \textit{and} \quad b + 1 = 1347 \quad b = 1346$

$a + c = 1347 \quad \textit{and} \quad b + 1 = 3 \quad b = 2$

$a + c = 9 \quad \textit{and} \quad b + 1 = 449 \quad b = 448$

$a + c = 449 \quad \textit{and} \quad b + 1 = 9 \quad b = 8$

The first choice does not satisfy the two initial equations, but if we chose $b = 2$, we have:
$ab + c = 2020 \quad \Rightarrow \quad 2a + c = 2020$, and subtracting $a + c = 1347$, we find: $a = 2020 - 1347 = 673$
Then $c = 2020 - a = 674$. These values fulfils both initial equations, since:

$ab + c = 2 \cdot 673 + 674 = 2020 \quad \textit{and} \quad a + bc = 2 \cdot 674 + 673 = 2021$

320. Determine *a*² + *b*² from: $a + b = \frac{a}{b} + \frac{b}{a}$

It is obvious that: $a = b = 1$, is a solution, so $a^2 + b^2 = 2$, but it may be formally proven:

$$a + b = \frac{a}{b} + \frac{b}{a} \quad \Leftrightarrow \quad a + b = \frac{a^2 + b^2}{ab} \quad \Leftrightarrow \quad ab(a + b) = a^2 + b^2 \quad \Leftrightarrow$$

$ba^2 + ab^2 = a^2 + b^2 \quad \Leftrightarrow \quad a^2(b - 1) + b^2(a - 1) = 0$

Since *a* and *b* are considered non negative integers, the only solution is: $a = b = 1$, so $a^2 + b^2 = 2$.

321. Solve for *x* and *y*: $x - y = \frac{x + y}{7} = \frac{xy}{12}$

An analytic attempt leads nowhere: but an obvious guess is $x = 4 \quad \textit{and} \quad y = 3$, since:

$$x - y = \frac{x + y}{7} = \frac{xy}{12} \quad \Leftrightarrow \quad 4 - 3 = \frac{4 + 3}{7} = \frac{3 \cdot 4}{12}$$

322. Determine *m* and *n* from the equation: $2^n - 2^m = 4080$

Well: $2^{11} = 4048 \quad and \quad 4080 - 4048 = 32 = 2^5 \quad so \quad n = 11 \quad and \quad m = 5$

323. Determine *x* and *y* from: $2^x - 2^y = 1 \quad and \quad 4^x - 4^y = \frac{5}{3}$

$$4^x - 4^y = \frac{5}{3} \quad \Leftrightarrow \quad (2^x)^2 - (2^y)^2 = \frac{5}{3}$$

We put: $u = 2^x \quad and \quad v = 2^y$ then $2^x - 2^y = 1 \quad and \quad (2^x)^2 - (2^y)^2 = \frac{5}{3}$ gives:

$$u - v = 1 \quad and \quad u^2 - v^2 = \frac{5}{3} \quad \Rightarrow \quad u^2 - (u-1)^2 = \frac{5}{3}$$

$$u^2 - (u^2 + 1^2 - 2u) = \frac{5}{3} \quad \Leftrightarrow \quad 2u - 1 = \frac{5}{3} \quad \Leftrightarrow \quad 2u = \frac{8}{3} \quad \Leftrightarrow \quad u = \frac{4}{3} \quad \wedge \quad v = \frac{1}{3}$$

$$2^x = \frac{4}{3} \quad \Leftrightarrow \quad x = \frac{2\ln 2 - \ln 3}{\ln 2} \quad \wedge \quad 2^y = \frac{1}{3} \quad \Leftrightarrow \quad x = -\frac{\ln 3}{\ln 2}$$

324. Solve the differential equation: $xydy = (x^2 + y^2)dx$

$$xydy = (x^2 + y^2)dx \quad \Leftrightarrow \quad \frac{dy}{dx} = \frac{x^2 + y^2}{xy} \quad \Leftrightarrow \quad \frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$$

We put:

$$y = xz \quad \Rightarrow \quad \frac{dy}{dx} = z + x\frac{dz}{dx} \quad \Rightarrow$$

$$z + x\frac{dz}{dx} = \frac{1}{z} + z \quad \Leftrightarrow \quad x\frac{dz}{dx} = \frac{1}{z} \quad \Leftrightarrow$$

$$z\frac{dz}{dx} = \frac{1}{x} \quad \Leftrightarrow \quad zdz = \frac{1}{x}dx \quad \Leftrightarrow$$

$$\int zdz = \int \frac{1}{x}dx$$

$$\tfrac{1}{2}z^2 = \ln x + c \quad \Leftrightarrow$$

$$\tfrac{1}{2}\frac{y^2}{x^2} = \ln x + c \quad \Leftrightarrow \quad \frac{y^2}{x^2} = 2\ln x + c \quad \Leftrightarrow$$

$$y^2 = x^2(2\ln x + c) \quad \Leftrightarrow$$

$$y = \pm x\sqrt{2\ln x + c}$$

325. Solve for *x*: $x^2+\left(\frac{x}{x+1}\right)^2=3$

$x^2+\left(\frac{x}{x+1}\right)^2=3 \quad \Leftrightarrow \quad x^2+\left(\frac{x+1-1}{x+1}\right)^2=3$

$x^2+\left(1-\frac{1}{x+1}\right)^2=3 \quad \Leftrightarrow$

$x^2+\frac{1}{(x+1)^2}-\frac{2}{x+1}+1=3 \quad \Leftrightarrow$

$x^2+\frac{1}{(x+1)^2}+\frac{-2}{x+1}+1=3 \quad \Leftrightarrow$

$x^2+\frac{1}{(x+1)^2}-\frac{2}{x+1}+1=3 \quad \Leftrightarrow$

$x^2+\frac{1}{(x+1)^2}+\frac{x-1}{x+1}=3 \quad \Leftrightarrow$

$x^2+\frac{1}{(x+1)^2}+\frac{x-1}{x+1}+1-1=3 \quad \Leftrightarrow$

$x^2+\frac{1}{(x+1)^2}+\frac{2x}{x+1}-1=3 \quad \Leftrightarrow$

$(x+\frac{1}{x+1})^2=4 \quad \Leftrightarrow$

$x+\frac{1}{x+1}=2 \quad \vee \quad x+\frac{1}{x+1}=-2$

$x+\frac{1}{x+1}=2 \quad \Leftrightarrow \quad x^2+x+1-2(x+1)=0 \quad \Leftrightarrow \quad x^2-x-1=0 \quad d=1+4=5$

$x=\frac{1\pm\sqrt{5}}{2}$

$x+\frac{1}{x+1}=-2 \quad \Leftrightarrow \quad x^2+x+1+2(x+1)=0 \quad \Leftrightarrow \quad x^2+3x+3=0 \quad d=9-12<0.$ No solution!

The solution is therefore: $x=\frac{1+\sqrt{5}}{2}$ *or* $x=\frac{1-\sqrt{5}}{2}$

326. Solve for *x* and *y*: $x^{99}+y^{99}=x^{100}$

There seem very little prospect in searching for an analytic solution, but tnat does not imply that we .may finfd a solution. We rewrite the equation as:

$x^{99}+y^{99}=x^{100} \quad \Leftrightarrow \quad x^{99}+y^{99}=x\cdot x^{99}$

Then it becomes obvious that $x=y=2$, since $2^{99}+2^{99}=2\cdot 2^{99}$

327. Verify that: $2\sqrt{2+\sqrt{3}}=\sqrt{2}+\sqrt{6}$

We shall show this by showing that: $(2\sqrt{2+\sqrt{3}})^2=4(2+\sqrt{3})=(\sqrt{2}+\sqrt{6})^2$ $4(2+\sqrt{3})=$

$(\sqrt{2}+\sqrt{6})^2=2+6+2\sqrt{2}\sqrt{6}=$

$2+6+2\sqrt{2}\sqrt{2}\sqrt{3}=$

$8+4\sqrt{3}=4(2+\sqrt{3})$

328. Determine integer values, such that: $x+y+xy=54$

There is really no analytic way to solve this equation, since there is one equation and two variables.

$x+y+xy=54 \quad \Leftrightarrow \quad x+y=54-xy$

$x=8 \quad and \quad y=5$ Could be a candidate, but $13=54-40=14$

$x=9 \quad and \quad y=4$ Could also be a candidate, but $13=54-36=19$

$x=10 \quad and \quad y=4$ Could also be a candidate, since $14=54-40=14$

$x=10 \quad and \quad y=4$ is a solution.

329. Simplify: $\frac{2\cdot\sqrt[3]{8}}{\sqrt[3]{2}\cdot\sqrt[3]{32}}$

$$\frac{2\cdot\sqrt[3]{8}}{\sqrt[3]{2}\cdot\sqrt[3]{32}}=\frac{2\cdot 2}{\sqrt[3]{2}\cdot\sqrt[3]{4\cdot 8}}=\frac{4}{\sqrt[3]{2}\cdot\sqrt[3]{2^2}\cdot\sqrt[3]{8}}=\frac{4}{\sqrt[3]{2^3}\cdot\sqrt[3]{8}}=\frac{4}{2\cdot 2}=1$$

330. Determine *a* + *b*, when: $a^3+b^3=2\sqrt{5} \quad and \quad a^2b+ba^2=\sqrt{5}$

$(a+b)^3=a^3+b^3+3ab(a+b) \quad and \quad a^2b+ba^2=ab(a+b)=\sqrt{5} \quad \Rightarrow$

$(a+b)^3=2\sqrt{5}+3\sqrt{5}=5\sqrt{5}=\sqrt{5}^3 \quad \Rightarrow$

$a+b=\sqrt{5}$

331. Determine integer values (*x,y*) such that. $x+xy+y=54$

There is really no way to determine integer values for this equation to fulfil , so we will resort to qualified guesswork:

$x+xy+y=54 \quad \Leftrightarrow \quad x+y=54-xy$

It seems that $x=10 \quad and \quad y=4$ does the trick: Since: 10+4=54-40

332. Solve the differential equation : $\frac{dy}{dx}=\sin(x+y)+\cos(x+y)-1$

$$\frac{dy}{dx}=\sin(x+y)+\cos(x+y)-1$$

We put $z=x+y \quad \Rightarrow \quad \frac{dz}{dx}=1+\frac{dy}{dx}$ and then we get:

$$\frac{dz}{dx}-1=\sin(x+y)+\cos(x+y)-1 \quad \Leftrightarrow$$

$$\frac{dz}{dx}=\sin(z)+\cos(z) \quad \Leftrightarrow \quad \frac{dz}{dx}=\sin(z)+\sin(\frac{\pi}{2}-z) \quad \Leftrightarrow$$

We then apply the formula:

$$\sin u+\sin v=2\sin\frac{u+v}{2}\cos\frac{u-v}{2}$$

$$\frac{dz}{dx}=\sin(z)+\sin(\frac{\pi}{2}-z) \quad \Leftrightarrow \quad \frac{dz}{dx}=2\sin(\frac{\pi}{4})\cdot\cos(z-\frac{\pi}{4})=\sqrt{2}\cos(z-\frac{\pi}{4})$$

And then we have: $\cos(z-\frac{\pi}{4})=\sin(\frac{\pi}{2}-(z-\frac{\pi}{4})=-\sin(z-\frac{3\pi}{4})$

We put: $v=\frac{3\pi}{4}-z \quad \Rightarrow \quad dv=.-dz \quad and \quad -\sin(z-\frac{3\pi}{4})=\sin v$

$$\frac{dv}{dx}=\sqrt{2}\sin v \quad \Leftrightarrow \quad \frac{dv}{\sin(v)}=\sqrt{2}dx \quad \Leftrightarrow$$

$$\frac{\left(\cos^2(\frac{v}{2})+\sin^2(\frac{v}{2})\right)dv}{2\sin(\frac{v}{2})\cos(\frac{v}{2})}=\sqrt{2}dx \quad \Leftrightarrow$$

Then we divide with $\sin(\frac{v}{2})\cos(\frac{v}{2})$

$$\frac{\left(\frac{\cos(\frac{v}{2})}{\sin(\frac{v}{2})}+\frac{\sin(\frac{v}{2})}{\cos(\frac{v}{2})}\right)dv}{2}=\sqrt{2}dx \quad \Leftrightarrow$$

$$(ctn(\frac{v}{2})+\tan(\frac{v}{2}))dv=2\sqrt{2}dx \quad \Leftrightarrow$$

$$\int(ctn(\frac{v}{2})+\tan(\frac{v}{2}))dv=2\sqrt{2}\int dx\int(ctn(\frac{v}{2})+\tan(\frac{v}{2}))dv=2\sqrt{2}\int dx \quad \Leftrightarrow$$

$$\tfrac{1}{2}\ln\sin\tfrac{v}{2}-\tfrac{1}{2}\ln\cos\tfrac{v}{2}=2\sqrt{2}x+c$$

Where have applied;

$$\int\frac{\sin x}{\cos x}dx=-\int\frac{d\cos x}{\cos x}dx=-\ln\cos x \quad and \quad \int\frac{\cos x}{\sin x}dx=\int\frac{d\sin x}{\sin x}dx=\ln\sin x$$

We then substitute back; $v=\frac{3\pi}{4}-z$

$\frac{1}{2}\ln\sin\frac{y}{2}-\frac{1}{2}\ln\cos\frac{y}{2}=2\sqrt{2}x \quad\Leftrightarrow\quad \frac{1}{2}\ln\sin(\frac{3\pi}{8}-\frac{z}{2})'-\frac{1}{2}\ln\cos(\frac{3\pi}{8}-\frac{z}{2})=2\sqrt{2}x \quad\Leftrightarrow$

$\frac{1}{2}\ln(-\sin(\frac{z}{2}-\frac{3\pi}{8}))'-\frac{1}{2}\ln(\cos(\frac{z}{2}-\frac{3\pi}{8}))=2\sqrt{2}x$

Finally we substitute back: $z=x+y$

$$\frac{1}{2}\ln(-\sin(\frac{x+y}{2}-\frac{3\pi}{8}))'-\frac{1}{2}\ln(\cos(\frac{x+y}{2}-\frac{3\pi}{8}))=2\sqrt{2}x$$

As it is obvious, that there is no way to isolate y from this equation.

333. Simplify $\sqrt[20]{13\cdot1021\cdot79+9}$

$\sqrt[20]{13\cdot1021\cdot79+9}=?$

$13\cdot79=1027=1024+3$

$1021=1024-3$

So:

$$\sqrt[20]{13\cdot1021\cdot79+9}=\sqrt[20]{(1024+3)(1024-3)+9}=\sqrt[20]{(1024^2-3^2)+9}=\sqrt[20]{1024^2}=\sqrt[20]{2^{20}}=2$$

334. simplify $\frac{500^{1000}}{1000^{500}}$

$$\frac{500^{1000}}{1000^{500}}=\frac{500^{2\cdot500}}{1000^{500}}=\frac{500^{500}}{1000^{500}}=\frac{500^{500}}{1000^{500}}\cdot500^{500}=\left(\frac{500}{1000}\right)^{500}\cdot500^{500}=\left(\frac{1}{2}\right)^{500}\cdot500^{500}=\left(\frac{500}{2}\right)^{500}=$$
$$250^{500}$$

335. Solve for *x*: $\frac{\ln\sqrt{x}}{\sqrt{\ln x}}=2$

$\frac{\ln\sqrt{x}}{\sqrt{\ln x}}=2 \quad\Leftrightarrow\quad \frac{\frac{1}{2}\ln x}{\sqrt{\ln x}}=2 \quad\Leftrightarrow\quad \frac{\frac{1}{2}\sqrt{\ln x}^2}{\sqrt{\ln x}}=2 \quad\Leftrightarrow$

$\frac{1}{2}\sqrt{\ln x}=2 \quad\Leftrightarrow\quad \sqrt{\ln x}=4 \quad\Leftrightarrow\quad \ln x=16 \quad\Leftrightarrow\quad x=e^{16}$

336. Determine posive integers *a, b, c* such that; $2^a+4^b+8^c=328$

$2^a+4^b+8^c=328 \quad\Leftrightarrow\quad 2^a+2^{2b}+2^{3c}=328$

There is no way to solve one equations having three unknowns, so we resort to qualified guesswork $c=1,2,3$ $c=3$ is too large, so we try with: $c=2$ that gives $2^6=64$, then we try with the highest possible value of b, which is 4. And $2^8=256$ Now 256+64 = 320 and 328 – 320 = 8 = 2^3, so a =3. The solution is then: a =3, b =4 and c =2.

337. Determine *a* , *b* and *c*, such that: $1+2^a+3^b=6^c$

It is very easy to see that: a = b =c is a solution, since: 1+2+3=6

We then try c = 2. 6^2= 36 . Then we try with b = 3 $3^b=3^3$ = 36 - 27 = 9, and indeed $1+2^3=9$, So one solution is; $a = 3$, $b = 3$, $c = 2$.

338. Find integer values of *a* and *b,* such that; $\sqrt{a}+\sqrt{b}=\sqrt{2009}$

An analytic approach is rather difficult:
But we observe that $2009 = 7^2 \cdot 41$, and 41 is a prime. So the equation may be written:
$\sqrt{a}+\sqrt{b}=7\sqrt{41}$, this implies that if two integers; $x+y=7\sqrt{41}$ then they may both be a integer multiple of $\sqrt{41}$. The possibilities are: $(x,y) = (1,6), (2,5), (3,4)$,
In the first case: $a = 1\cdot 41 = 41, \quad and \quad b = 36\cdot 41$, which gives:

$$\sqrt{a}+\sqrt{b}=7\sqrt{41} \quad \Leftrightarrow \quad \sqrt{41}+\sqrt{36\cdot 41}=\sqrt{41}+6\sqrt{41} \; = 7\sqrt{41}$$

The two other cases are likewise: $a = 4\cdot 41 = 165 \quad and \quad b = 25\cdot 41 = 1025$, or
$a = 9\cdot 41 = 369 \quad and \quad b = 16\cdot 41 = 656$ which gives:

$$\sqrt{a}+\sqrt{b}=7\sqrt{41} \quad \Leftrightarrow \quad \sqrt{9\cdot 41}+\sqrt{16\cdot 41}=3\sqrt{41}+4\sqrt{41} \; = 7\sqrt{41}$$

339. Solve for *x*; $\frac{1}{x}e^{\frac{1}{x}}=e$

This is a ridiculous exercise, since it is a transcendent equation, which in general has no analytic solution.

However, it is obvious, that $x = 1$ is the only solution, since: $\frac{1}{1}e^{\frac{1}{1}}=e$

340. Solve for *x.* $x^5+x^4+x^3+x^2+x=-1$

At a glance it is obvious that $x = -1$ is a solution, since: -1 +1 -1 +1+ .1 = -1, but there may be other solutions (max 5). We do the following rewriting.

$$x^5+x^4+x^3+x^2+x+1=0 \quad \Leftrightarrow \quad x^3(x^2+x+1)+x^2+x+1=0 \quad \Leftrightarrow$$

$$(x^2+x+1)(x^3+1)=0 \quad \Leftrightarrow x^3=-1 \quad \vee \quad x^2+x+1=0 \quad (d=1-4<0) \quad \Leftrightarrow$$

$$x=-1$$

341. Solve for *x*: $x^{2x^6}=3$

$x^{2x^6}=3$ There is (to my knowledge) no analytic approach to this equation, so we shall resort to qualified guesswork. $\sqrt[6]{3}$ could be a candidate:

$$(\sqrt[6]{3})^{2\sqrt[6]{3}^6}=(\sqrt[6]{3})^{2\cdot 3}=(\sqrt[6]{3})^6=3 \text{, so}$$

$$x^{2x^6}=3 \quad \Leftrightarrow \quad x=\sqrt[6]{3}$$

342. $\frac{3^x}{4^x}=\frac{5}{2^x}$

$$\frac{3^x}{4^x}=\frac{5}{2^x} \quad \Leftrightarrow \quad 3^x=\frac{5\cdot 4^x}{2^x} \quad \Leftrightarrow \quad 3^x=\frac{5\cdot(2^x)^2}{2^x} \quad \Leftrightarrow \quad \left(\frac{3}{2}\right)^x=5 \quad \Leftrightarrow \quad x=\frac{\ln 5}{\ln 3-\ln 2}$$

343. Determine *a, b, c*, such that: $a^3+b^3+c^3=(abc)^3$

Well, to my knowledge the only numbers where $a+b+c=abc$ are 1, 2, 3, since $1+2+3=1\cdot2\cdot3$ so the solution is obviously: $a=\sqrt[3]{1}, b=\sqrt[3]{2}$, $c=\sqrt[3]{3}$, which gives:

$a^3+b^3+c^3=\sqrt[3]{1}^3+\sqrt[3]{2}^3+\sqrt[3]{3}^3=1+2+3=(\sqrt[3]{1}\cdot\sqrt[3]{2}\cdot\sqrt[3]{3})^3=1\cdot2\cdot3$

344. Find integer solutions to: $ab-cd=34$ and $ac-bd=19$

$ab-cd=34$
$ac-bd=19$

Well, we have four unknowns and two (non linear equations), so it requires a lot of guesswork: Adding the two equations gives however:

$ab-cd+ac-bd=53 \quad\Leftrightarrow\quad a(b+c)-d(b+c)=53 \quad\Leftrightarrow\quad (b+c)(a-d)=53$

Now since 53 is a prime is has only the divisors 1 and 53, so we chose: $b+c=53 \quad and\ a-d=1$

If we choose *a*, then we can find *d*, To determine how to distribute 53 on *b* and *c*. we use the firt equation, where we insert *d* = *a* – 1 and *c* = 53 – *b*.

$ab-cd=34 \quad\Rightarrow\quad ab-(53-b)(a-1)=34 \quad\Leftrightarrow\quad 2ab-53a+53-b=34$
Then we try with various values of *a* to determine *b*.

a = 1: $a=1: \Rightarrow \quad b=32,\ c=19, d=0$, Which is easily verified as a solution.
a = 2 wont work, because it results in a non integer *b*.
$a=3 \quad\Rightarrow\quad 2ab-53a+53-b=34 \quad 6b-53\cdot3+53-b=34 \quad\Rightarrow\quad 5b=140 \quad\Leftrightarrow\quad b=28$
So the solution is:

$a=3, b=28, c=25,\ d=2$

345. Solve for *x*. $x^{x^3}=729$.

It is clear that 3 is a key number to this problem. We could try with $x=\sqrt[3]{3}$. But this gives 3.

We also notice that: $729=3^6$. Then we could try with $x=\sqrt[3]{3}^2$, and;

$(\sqrt[3]{3}^2)^{(\sqrt[3]{3}^2)^3}=(\sqrt[3]{3}^2)^9=3^6=729$

346. Solve for *x*: $\sqrt{1-\frac{x^3}{1000}}=(1-\frac{2}{3})^{-1}$

$\sqrt{1-\frac{x^3}{1000}}=(1-\frac{2}{3})^{-1}. \quad (1-\frac{2}{3})^{-1}=(\frac{1}{3})^{-1}=3$

$\sqrt{1-\frac{x^3}{1000}}=3 \quad\Leftrightarrow\quad 1-\frac{x^3}{1000}=9 \quad\Leftrightarrow\quad x^3=-8000 \quad\Leftrightarrow\quad x=-20$

347. Solve for *x*. $\sqrt[3]{2-x}+\sqrt{x-1}=1$

A good guess is $x = 10$, and indeed; $\sqrt[3]{2-10}+\sqrt{10-1}=\sqrt[3]{-8}+\sqrt{9}=-2+3=1$

348. Find integer solutions to: $\frac{1}{x}-\frac{1}{y}=\frac{2}{35}$

$$\frac{1}{x}-\frac{1}{y}=\frac{2}{35} \quad \Leftrightarrow \quad \frac{x-y}{xy}=\frac{2}{35}$$

Since 5 times 7 is 35, it is straightforward to guess: $x = 7$ and $y = 5$, and we find:

$$\frac{x-y}{xy}=\frac{7-5}{35}=\frac{2}{35}$$

359. Find the largest integer for which $\frac{n^3+100}{n+10}$ **is an integer**

The first thought is to make polynomial division with n +1 and require that the division goes up.

$$n+10 \mid n^3+100 \mid n^2-10n$$
$$n^3+10n^2$$
$$-10n^2+100$$
$$-10n^2-100n$$
$$100+100n$$

If the remainder should be 0, n should be -1, and $\frac{-1+100}{-1+10}=\frac{99}{9}=9$

If we want to find a positive solutions, we shall first resort to guesswork:

It seems that $n = 5$ is a possible candidate since: $\frac{n^3+100}{n+10}=\frac{125+100}{5+10}=\frac{225}{15}=15$

So we have two solutions: $n = -1$ or $n = 5$.

However to find possible other solutions, we rewrite n^3+100 as

$$n^3+100=n^3+1000-900$$
$$(n+10)^3=n^3+1000+3n\cdot 10(n+10)$$
$$n^3=(n+10)^3-1000-3n\cdot 10(n+10)$$
$$n^3+100=(n+10)((n+10)^2-30n)-900$$

So the possible divisors n+10, can principle be all divisors in 900.

$900 = 9 \cdot 4 \cdot 25 = 3^2 \cdot 2^2 \cdot 5^2$

We have already found n = 5. The next candidate is n = 10, and $\frac{10^3 + 100}{10 + 10} = \frac{1100}{20} = 55$

The next candidate is n = 20, and $\frac{20^3 + 100}{20 + 10} = \frac{8100}{30} = 270$

The next candidate is n = 25, but n + 10 =35 is not a divisor in 900

The next candidate is n = 40, and $\frac{40^3 + 100}{40 + 10} = \frac{64100}{50} = 1282$, but n + 10 = 35 is not a divisor in 900

The largest divisor in 900 is 900 = n + 10, So the largest n for which the division goes up is 890.

$$\frac{890^3 + 100}{900} = \frac{704969100}{900} = 783299$$

350. Determine *x* and *y* from $x + y = 2$ *and* $x^4 + y^4 = 1234$

It is obvious that two positive numbers can satisfy these two equations. We proceed as follows;

$$x + y = 2 \quad \Rightarrow \quad (x+y)^2 = 4 \quad \Leftrightarrow \quad x^2 + y^2 + 2xy = 4 \quad \Leftrightarrow x^2 + y^2 = 4 - 2xy$$

$$x + y = 2 \quad \Rightarrow \quad (x+y)^3 = 8 \quad \Leftrightarrow \quad x^3 + y^3 + 3xy(x+y) = 8 \quad \Leftrightarrow \quad x^3 + y^3 = 8 - 6xy$$

$$x + y = 2 \quad \Rightarrow \quad (x+y)^4 = 16 \quad \Leftrightarrow \quad (x+y)(x+y)^3 = 16 \quad \Leftrightarrow \quad (x+y)\,(x^3 + y^3 + 6xy) = 16 \quad \Leftrightarrow$$
$$(x+y)\,(x^3 + y^3 + 6xy) = 16 \quad \Leftrightarrow \quad x^4 + y^4 + xy^3 + yx^3 + 12xy = 16 \quad \Leftrightarrow$$
$$x^4 + y^4 + xy(x^2 + y^2) + 12xy = 16 \quad \Leftrightarrow \quad 1218 + xy(4 - 2xy) + 12xy - 16 = 0 \quad \Leftrightarrow$$
$$-2(xy)^2 + 16xy + 1218 = 0 \quad \Leftrightarrow$$
$$(xy)^2 - 8xy - 609 = 0; \quad d = 8^2 + 4 \cdot 609 = 2500 = 50^2$$
$$xy = \frac{8 \mp 50}{2} \quad \Leftrightarrow \quad xy = 29 \quad \vee \quad xy = -21$$

The positive solution can not satisfy $x + y = 2$, so we solve the two equations:
$x + y = 2$ and $xy = -21$

$xy = -21 \quad \Leftrightarrow \quad y = -\frac{21}{x}$ inserted in $x + y = 2$ gives

$$x - \frac{21}{x} = 2 \quad \Leftrightarrow \quad x^2 - 2x - 21 = 0; \quad d = 4 + 4 \cdot 21 = 4 \cdot 22$$
$$x = \frac{2 \pm 2\sqrt{22}}{2} \quad x = 1 + \sqrt{22} \quad \vee \quad x = 1 - \sqrt{22} \quad \Rightarrow$$
$$y = 2 - x \quad \Rightarrow y = 1 - \sqrt{22} \quad \vee \quad y = 1 + \sqrt{22}$$

351. Simplify: $((2-\sqrt{5})^{198}\cdot(9+4\sqrt{5})^{99}$

$(2-\sqrt{5})^2 = 4+5-4\sqrt{5} = 9-4\sqrt{5}$, so

$$((2-\sqrt{5})^{198}\cdot(9+4\sqrt{5})^{99} = ((2-\sqrt{5})^2)^{99}\cdot(9+4\sqrt{5})^{99} = (9-4\sqrt{5})^{99}\cdot(9+4\sqrt{5})^{99} =$$

$$((9-4\sqrt{5})(9+4\sqrt{5}))^{99} = (81-80)^{99} = 1^{99} = 1$$

352. Simplify: $2\sqrt[3]{8}+\sqrt[3]{2}\sqrt[3]{32}$

$$2\sqrt[3]{8}+\sqrt[3]{2}\sqrt[3]{32} = 2\cdot 2+\sqrt[3]{64} = 4+\sqrt[3]{2^6} = 4+4 = 8$$

353. Solve for *x*: $7^{\log_8 x}\cdot x^{\log_8 9} = 3969$

$3969 = 7^2\cdot 3^4$

$7^{\log_8 x}\cdot x^{\log_8 9} = 3969 \quad\Leftrightarrow\quad \log_8(7^{\log_8 x}\cdot x^{\log_8 9}) = \log_8(3969) \quad\Leftrightarrow$

$(\log_8 x)(\log_8 7)+(\log_8 9)(\log_8 x) = \log_8(7^2\cdot 3^4) \quad\Leftrightarrow$

$\log_8 x(\log_8 7+\log_8 3^2) = 2\log_8 7+4\log_8 3 \quad\Leftrightarrow$

$\log_8 x(\log_8 7+2\log_8 3) = 2(\log_8 7+2\log_8 3) \quad\Leftrightarrow$

$\log_8 x = 2 \quad\Leftrightarrow\quad x = 8^2$

354. Calculate the sum: $\frac{3}{2}+\frac{5}{8}+\frac{7}{32}+\frac{9}{128}+....$

$$S = \frac{3}{2}+\frac{5}{8}+\frac{7}{32}+\frac{9}{128}+.... = \frac{3}{2}+\frac{5}{2^3}+\frac{7}{2^5}+\frac{9}{2^7}+.... =$$

$$\frac{1+2}{2}+\frac{3+2}{2^3}+\frac{5+2}{2^5}+\frac{7+2}{2^7}+.... = (1+\frac{1}{2})+(\frac{3}{2^3}+\frac{1}{2^2})+(\frac{5}{2^5}+\frac{1}{2^4})+(\frac{7}{2^7}+\frac{1}{2^6})+... =$$

$$(1+\frac{3}{2^3}+\frac{5}{2^5}+\frac{7}{2^7}...)+(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}...) = 1+\frac{1}{2^2}(\frac{3}{2}+\frac{5}{8}+\frac{7}{32}+\frac{9}{128}+...)+\frac{1}{2}+(\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}...) =$$

$$1+\frac{1}{2^2}(\frac{3}{2}+\frac{5}{8}+\frac{7}{32}+\frac{9}{128}+...)+\frac{1}{2}+\frac{1}{2^2}(1+\frac{1}{2^2}+\frac{1}{2^3}...) = \frac{3}{2}+\frac{1}{2^2}S+\frac{1}{2^2}(\frac{1}{1-\frac{1}{2}}) =$$

$$S = \frac{3}{2}+\frac{1}{2^2}>S+\frac{1}{2^2}2 \quad\Leftrightarrow\quad S = \frac{3}{2}+\frac{1}{2^2}S+\frac{1}{2} \quad\Leftrightarrow\quad S(1-\frac{1}{2^2}) = 2 \quad\Leftrightarrow$$

$$S = \frac{8}{3}$$

355. Determine *a* and *b* from: $a^{\ln a} = b^{\ln b}$ **and** $a-b=1$

$a^{\ln a} = b^{\ln b} \quad\Leftrightarrow\quad \ln a(\ln a) = \ln b(\ln b) \quad\Leftrightarrow\quad (\ln a)^2 = (\ln b)^2 \Leftrightarrow$

$\ln a = \ln b \quad or \quad \ln a = -\ln b \quad\Leftrightarrow\quad a = b \quad or \quad \ln ab = 0$

The first solution does gives $a - b = 0$ but the second solution gives $ab = 1$.

$a-b=1 \quad and \quad ab=1 \quad \Rightarrow \quad a-\frac{1}{a}-1=0 \quad \Rightarrow$

$a^2-a-1=0$; $d=1+4=5$

$a=\frac{1\pm\sqrt{5}}{2} \quad \Rightarrow \quad a=\frac{1+\sqrt{5}}{2}$

Since *a* cannot be negative.

356. Solve for *x*. $2^{\log_2 \frac{x+1}{x+2}}=4$

$2^{\log_2 \frac{x+1}{x+2}}=4 \quad \Leftrightarrow \quad \log_2 \frac{x+1}{x+2}=4 \quad \Leftrightarrow \quad \frac{x+1}{x+2}=2^4$

$\frac{x+1}{x+2}=16 \quad \Leftrightarrow \quad 15x-31=0 \quad \Leftrightarrow \quad x=\frac{31}{15}$

357. Solve for *x*; $4^x+9^x+25^x=6^x+10^x+15^x$

$4^x+9^x+25^x=6^x+10^x+15^x \quad \Leftrightarrow \quad (2^x)^2+(3^x)^2+(5^x)^2=2^x3^x+2^x5^x+3^x5^x$

We put $2^x=a$, $3^x=b$, $5^x=c$ and then we have:

$a^2+b^2+c^2=ab+ac+bc$

We then multiply this equation with 2.

$2a^2+2b^2+2c^2=2ab+2ac+2bc$

Which can be written as:

$(a-b)^2+(a-c)^2+(b-c)^2=0$

This is however only possible if $a=b=c \quad \Leftrightarrow \quad 2^x=3^x=5^x$

So the only solution is $x = 0$.

358. Solve for *x*: $(1+\frac{1}{x})^{x+1}=(1+\frac{1}{6})^6$

At a glance this seems impossible, but when you realise, that *x* might be a negative number, is it not.

$(1+\frac{1}{x})^{x+1}=(1+\frac{1}{6})^6 \quad \Leftrightarrow \quad (1+\frac{1}{x})^{x+1}=(\frac{7}{6})^6 \quad \Leftrightarrow$

$(1+\frac{1}{x})^{x+1}=(\frac{6}{7})^{-6} \quad \Leftrightarrow \quad (1+\frac{1}{x})^{x+1}=(1-\frac{1}{7})^{-6} \quad \Leftrightarrow \quad x=-7$

359. Determine *f*(*x*) from the equation: $3f(x)+4xf(\frac{1}{x})=5x-6$

1) $3f(x)+4xf(\frac{1}{x})=5x-6$ We multiply this equation with 3

We replace x with $\frac{1}{x}$:

2) $3f(\frac{1}{x})+\frac{4}{x}f(x)=\frac{5}{x}-6$ We multiply this equation with 4x

1) $9f(x)+12xf(\frac{1}{x})=15x-18$

2) $12f(\frac{1}{x})+16f(x)=20-24x$

We then subtract 1) from 2): to give:

$7f(x)=39x-38 \quad \Leftrightarrow \quad f(x)=\frac{39x-38}{7}$

360. Solve for *x*: $x^{\sqrt{x}}=\frac{1}{2}$

$x^{\sqrt{x}}=\frac{1}{2}$

To my knowledge there is no way to solve this equation analytically, so we shall resort to guesswork; $x=\frac{1}{2}$ does not work, so we try with $x=\frac{1}{4}$ and indeed:

$(\frac{1}{4})^{\sqrt{\frac{1}{4}}}=(\frac{1}{4})^{\frac{1}{2}}=\frac{1}{2}$

On the other hand:

$x^{\sqrt{x}}=\frac{1}{2} \quad \Leftrightarrow \quad x^{\sqrt{x}}=(\frac{1}{4})^{\sqrt{\frac{1}{4}}}$ so $x=\frac{1}{4}$

361. Solve for *a*: $2^{3a}+2^{a}=10$

$2^{3a}+2^{a}=10$ We put $y=2^{a}$ and we get:

$y^3+y-10=0$ We see that $y=2$ is a solution:

To find possible other solutions, we make polynomial division with y - 2;

$y-2 \mid y^3+y-10 \mid y^2+2y+5$

y^3-2y^2

$2y^2+y$

$2y^2-4y$

$5y-10$

$5y-10$

0

$y^2+2y+5=0 \quad d=4-20<0$ No solution.

362. Solve the integral: $\int \cos(x-\frac{1}{x})dx$

$\int \cos(x-\frac{1}{x})dx$ This integral has no immediate solution, but we try with the substitution

$t=\frac{1}{x} \quad \Rightarrow \quad dt=-\frac{1}{x^2}dx \quad \Rightarrow \quad dx=-x^2dt=-\frac{1}{t^2}dt$

$\int \cos(x-\frac{1}{x})dx=-\int \frac{1}{t^2}\cos(\frac{1}{t}-t)dt=-\int \frac{1}{t^2}\cos(t-\frac{1}{t})dt=-\int \frac{1}{x^2}\cos(x-\frac{1}{x})dx \quad \Rightarrow$

$\int \cos(x-\frac{1}{x})dx+\int \frac{1}{x^2}\cos(x-\frac{1}{x})dx=0 \quad \Leftrightarrow \quad \int(1+\frac{1}{x^2})\cos(x-\frac{1}{x})=0 \quad \Leftrightarrow \quad \int d\sin(x-\frac{1}{x})=0 \quad \Leftrightarrow$
$\sin(x-\frac{1}{x})=c$

363. Solve for *x*: $\frac{27^x-8^x}{18^x-12^x}=\frac{19}{6}$

$\frac{27^x-8^x}{18^x-12^x}=\frac{19}{6} \quad \Leftrightarrow \quad \frac{(3^x)^3-(2^x)^3}{(3^2\cdot 2)^x-(2^2\cdot 3)^x}=\frac{19}{6}$

We put: $a=2^x \quad and \quad b=3^x$

$\frac{(3^x)^3-(2^x)^3}{(3^2\cdot 2)^x-(2^2\cdot 3)^x}=\frac{19}{6} \quad \Leftrightarrow \quad \frac{b^3-a^3}{b^2a-ba^2}=\frac{19}{6} \quad \Leftrightarrow \quad \frac{b^3-a^3}{ab(b-a)}=\frac{19}{6}$

We notice that:

$(b-a)^3=b^3-a^3-3ba(b-a) \quad \Leftrightarrow$
$b^3-a^3=(b-a)^3+3ba(b-a)=(b-a)((b-a)^2+3ba)$

$\frac{b^3-a^3}{ab(b-a)}=\frac{19}{6} \quad \Leftrightarrow \quad \frac{(b-a)((b-a)^2+3ba)}{ab(b-a)}=\frac{19}{6} \quad \Leftrightarrow \quad \frac{(b-a)^2+3ba}{ab}=\frac{19}{6}$
$\frac{b^2+a^2-2ab+3ba}{ab}=\frac{19}{6} \quad \Leftrightarrow \quad \frac{b^2+a^2+ab}{ab}=\frac{19}{6}$

We divide the lhs by ab :

$$\frac{\frac{b}{a}+\frac{a}{b}+1}{1}=\frac{19}{6} \quad \Leftrightarrow \quad \frac{b}{a}+\frac{a}{b}+1=\frac{19}{6}$$

We put; $y=\frac{b}{a}$ and we get:

$$y+\frac{1}{y}+1=\frac{19}{6} \quad \Leftrightarrow \quad 6y^2-13y+6=0; \quad d=169-144=25$$

$$y=\frac{13\pm 5}{12} \quad \Leftrightarrow \quad y=\frac{3}{2} \quad \vee \quad y=\frac{2}{3} \quad \Rightarrow$$

$$(\frac{2}{3})^x=\frac{3}{2} \quad \vee \quad (\frac{2}{3})^x=\frac{2}{3} \quad \Leftrightarrow$$

$$x=-1 \quad \vee \quad x=1$$

364. Solve for *x*: $\sqrt[x]{9}+\sqrt[x]{6}=\sqrt[x]{4}$

First I shall argue that this exercise is not formulated in accordance with mathematical terminology.
The symbol $\sqrt[n]{a}$ is defined only for integers greater than 1, and a is a non negative real number.
The symbol a^x is defined only for positive real numbers a, and x real.
Furthermore x must be negative (which excludes the symbol $\sqrt[n]{a}$) because othjerwise both terms on the lhs are bigger than the rhs. So I assume that is meant.

$$9^{\frac{1}{x}}+6^{\frac{1}{x}}=4^{\frac{1}{x}}$$

For typographical convenience we put: $y=\frac{1}{x}$: So we have: $9^y+6^y=4^y$

We divide this equation with: 9^y to give: $1+\frac{6^y}{9^y}=\frac{4^y}{9^y} \quad \Leftrightarrow \quad 1+\left(\frac{2}{3}\right)^y=\left(\left(\frac{2}{3}\right)^y\right)^2$

We put; $z=\left(\frac{2}{3}\right)^y$ and find: $z^2-z-1=0$; $\quad d=1+4 \quad \Rightarrow \quad z=\frac{1\pm\sqrt{5}}{2}$

Since z must be positive root, we discard the negative root.

$$\left(\frac{2}{3}\right)^y=\frac{1+\sqrt{5}}{2} \quad \Rightarrow \quad y=\frac{\ln(\frac{1+\sqrt{5}}{2})}{\ln 2-\ln 3}; \quad x=\frac{1}{y} \quad \Rightarrow \quad x=-\frac{\ln 3-\ln 2}{\ln(\frac{1+\sqrt{5}}{2})}$$

365. Solve for *x*: $\frac{3^{x^2}}{9^x}=27$

$$\frac{3^{x^2}}{9^x}=27 \quad \Leftrightarrow \quad \frac{3^{x^2}}{3^{2x}}=3^3 \quad \Leftrightarrow \quad 3^{x^2-2x}=3^3 \quad \Leftrightarrow$$

$$x^2-2x=3 \quad \Leftrightarrow \quad x^2-2x-3=0; \quad d=4+12=16$$

$$x=\frac{2\pm\sqrt{16}}{2}=1\pm 2 \quad \Leftrightarrow \quad x=3 \quad \vee \quad x=-1$$

366. Determine *a, b, c* such that: $(1+\frac{1}{a})(1+\frac{1}{b})(1+\frac{1}{c}) = 2$

Well, we have one equation with three unknowns, so we will resort to guesswork.
If $a=b=c$ we can find a solution, whereas it is difficult to prove that there are other solutions.

$$a=b=c \quad \Rightarrow \quad (1+\frac{1}{a})^3 = 2 \quad \Rightarrow \quad (1+\frac{1}{a}) = \sqrt[3]{2} \quad \Rightarrow \quad a=b=c=\frac{1}{\sqrt[3]{2}-1}$$

367. Evaluate the integral: $\int x^2\sqrt{1-x^2}dx$

We make the substitution $t=\cos x \quad \Rightarrow \quad dt=-\sin x dx$ and we find:

$$\int x^2\sqrt{1-x^2}dx = -\int \cos^2 t\sqrt{1-\cos^2 t}\sin t\, dt =$$
$$-\int \cos^2 t \sin^2 t\, dt = -\tfrac{1}{4}\int 4\cos^2 t\sin^2 t\, dt$$
$$-\tfrac{1}{4}\int \sin^2 2t\, dt = -\tfrac{1}{8}\int (1-\cos 4t)dt =$$
$$-\tfrac{1}{8}(t-\tfrac{1}{4}\sin 4t)+c$$

We have made use of:

$$\sin 2t = 2\sin t\cos t \quad and \quad \cos 2t = 1-2\sin^2 t \quad \Rightarrow$$
$$\cos 4t = 1-2\sin^2 2t \qquad \Rightarrow = \sin^2 2t = \tfrac{1}{2}(1-\cos 4t)$$

368. Solve for *x*: $x-x^2-2x^3=\frac{1}{3}$

$$x-x^2-2x^3=\frac{1}{3} \quad \Leftrightarrow \quad 3x-3x^2-6x^3=1$$

We notice that: $(x-1)^3 = x^3-1^3-3x(x-1)$

$$3x-3x^2-6x^3-1=0 \quad \Leftrightarrow \quad x^3-1-3x(x-1) \; -7x^3=0 \quad \Leftrightarrow$$
$$(x-1)^3 = 7x^3 \quad \Leftrightarrow \quad x-1=\sqrt[3]{7}\,x \quad \Leftrightarrow$$
$$x = \frac{1}{1-\sqrt[3]{7}}$$

369. Solve for *x* and *y*; $x^3+y^3=2xy+8$ **??**

$x^3+y^3=2xy+8$

There is no reason to do calculation, since the solution evidently $x=y=2$,
since $2^3+2^3=2\cdot 2\cdot 2+8$

370. Solve for *x*: $x^{\sqrt{x}}=\sqrt{x^x}$

It is rather easy to realize that: $x=4$ is the solution, since; $x^{\sqrt{x}}=4^{\sqrt{4}}=4^2$ *and* $\sqrt{x^x}=4^{4\cdot\frac{1}{2}}=4^2$

371. Solve for *x*: $\sqrt{2x-4}-\sqrt{3x+4}=-2$

Well no need to make long calculations, since it is obvious that $x = 4$ is a solution, since:

$$\sqrt{2\cdot 4-4}-\sqrt{3\cdot 4+4}=\sqrt{4}-\sqrt{16}=2-4=-2$$

372. Determine $\frac{x}{y}$ **from** $2^{x+y}=125$ **and** $2^{x-y}=25$

$2^{x+y}=125$ and $2^{x-y}=25 \quad\Leftrightarrow\quad 2^{x+y}=5^3$ and $2^{x-y}=5^2 \quad\Leftrightarrow$

$2=5^{\frac{3}{x+y}}$ and $2=5^{\frac{2}{x-y}} \quad\Rightarrow$

$5^{\frac{3}{x+y}}=5^{\frac{2}{x-y}} \quad\Rightarrow\quad \frac{3}{x+y}=\frac{2}{x-y} \quad\Leftrightarrow\quad 3(x-y)=2(x+y) \quad\Leftrightarrow\quad x-5y=0 \quad\Leftrightarrow\quad \frac{x}{y}=5$

373. Simplify: $\left(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}}\right)^{2013}$

$2013=3\cdot 671$

$\left(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}}\right)^3=\left(\sqrt[3]{5+\sqrt{2}}\right)^3+\left(\sqrt[3]{5-\sqrt{2}}\right)^3+3\sqrt[3]{5+\sqrt{2}}\sqrt[3]{5-\sqrt{2}}(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}})=$

$5+\sqrt{2}+5-\sqrt{2}+3\cdot\sqrt[3]{25-2}(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}})=$

$10+3(125+25\sqrt{2}-10-2\sqrt{2}+125-25\sqrt{2}-10+2\sqrt{2})=$

$10+3(125-10+125-10)=10+3\cdot 230=700$

$$\left(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}}\right)^{2013}=\left(\left(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}}\right)^3\right)^{671}=700^{671}$$

374. Solve for *x*: $\log(\ln x)=\ln(\log x)$

We recall:

$y=\log x \quad\Leftrightarrow\quad x=10^y \quad\Leftrightarrow\quad \ln x=y\ln 10=\log x\ln 10 \quad\Leftrightarrow$

$\log x=\frac{\ln x}{\ln 10}$

$\log(\ln x)=\ln(\log x) \quad\Leftrightarrow\quad \frac{\ln(\ln x)}{\ln 10}=\ln(\frac{\ln x}{\ln 10})$

We put: $y=\ln x$:

$\frac{\ln y}{\ln 10}=\ln(\frac{y}{\ln 10}) \quad\Leftrightarrow\quad \ln y^{\frac{1}{\ln 10}}=\ln(\frac{y}{\ln 10}) \quad\Leftrightarrow$

$y^{\frac{1}{\ln 10}}=\frac{y}{\ln 10} \quad\Leftrightarrow\quad y^{\frac{1}{\ln 10}-1}=\frac{1}{\ln 10} \quad\Leftrightarrow$

$(\frac{1}{\ln 10}-1)\ln y=-\ln 10$

$$(\frac{1}{\ln 10}-1)\ln y=-\ln 10 \quad \Leftrightarrow \quad \ln y=\frac{\ln 10}{1-\frac{1}{\ln 10}}=\frac{(\ln 10)^2}{\ln 10-1}$$

$$\ln(\ln x)=\frac{(\ln 10)^2}{\ln 10-1} \quad \Leftrightarrow \quad \ln x=\exp(\frac{(\ln 10)^2}{\ln 10-1}) \quad \Leftrightarrow$$

$$x=\exp(\exp(\frac{(\ln 10)^2}{\ln 10-1}))$$

375. Simplify: $\sqrt{3+\sqrt{8}}$

$\sqrt{3+\sqrt{8}}=\sqrt{3+2\sqrt{2}}$

We seek to determine a and b, such that: $(a+b\sqrt{2})^2=3+2\sqrt{2}$

$(a+b\sqrt{2})^2=a^2+2b^2+2ab\sqrt{2}$

We see that this is fulfilled if; $a=b=1$, so: $(1+\sqrt{2})^2=3+\sqrt{2}$ and therefore:

$\sqrt{3+2\sqrt{2}}=\sqrt{(1+\sqrt{2})^2}=(1+\sqrt{2})$

375. Simplify: $\dfrac{\sqrt{2-\sqrt{3}}}{3-\sqrt{3}}$

We seek to determine a and b, such that: $(a-b\sqrt{3})^2=2-\sqrt{3}$

$(a-b\sqrt{3})^2=a^2+3b^2-2ab\sqrt{3}$ which gives; $a=b=1$, so

$(1-\sqrt{3})^2=2-\sqrt{3}$, so $\sqrt{2-\sqrt{3}}=\sqrt{(1-\sqrt{3})^2}=(\sqrt{3}-1)$

$$\frac{\sqrt{2-\sqrt{3}}}{3-\sqrt{3}}=\frac{\sqrt{3}-1}{3-\sqrt{3}}=\frac{(\sqrt{3}-1)(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})}=\frac{3\sqrt{3}+3-\sqrt{3}-3}{9-3}=\frac{\sqrt{3}}{3}$$

376. Solve for *x*: $2^{3^{4^x}}=512$

We notice that: $512=2^9$

$$2^{3^{4^x}}=2^9 \quad \Leftrightarrow \quad 3^{4^x}=9 \quad \Leftrightarrow \quad 3^{4^x}=3^2 \quad \Leftrightarrow \quad 4^x=2 \quad \Leftrightarrow \quad x=\frac{1}{2}$$

377. Evaluate the infinite fraction chain

$$x=1-\cfrac{1}{1-\cfrac{1}{1-....}}$$

We notice that

$$x=1-\cfrac{1}{1-\cfrac{1}{1-....}} \quad \Leftrightarrow \quad x=1-\frac{1}{x} \quad \Leftrightarrow$$

$x^2 - x + 1 = 0; \quad d = 1 - 4 = -3$

$x = \frac{1 \pm i\sqrt{3}}{2} \quad \Leftrightarrow \quad x = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

It seems utterly strange that legal operation with real numbers can actually lead to a imaginary result.

To understand it we shall look at an infinite fraction that leads to the golden ratio.

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}} \quad \Leftrightarrow \quad x = 1 + \frac{1}{x} \quad \Leftrightarrow \quad x^2 - x - 1 = 0; d = 1 + 4 = 5$$

$$x = \frac{1 + \sqrt{5}}{2}$$

The golden ratio

If we try to make a series where we cut off at the first, second,….we get

$x = 2,\ x = 1 + \frac{1}{2},\ 1.667, \ldots$ which is quite near to the golden ratio.

However, if we to the same for the first chain of fraction, we get.

$x = 0,\ x = 1 - \frac{1}{0}, \ldots.$

However it is far from transparent that this leads to a complex number, since each denominator in the infinite chain fraction is different from zero.

378. Solve for *x*: $\sqrt{x-2} + 3 = \sqrt{4x+1}$

No need to calculate, it is obvious that the solution is $x = 2$, since $\sqrt{2-2} + 3 = \sqrt{4 \cdot 2 + 1}$

379. Solve for *x*: $\sqrt{2x-4} - \sqrt{3x+4} = -2$

No need for calculation since it is obvious that the solution is $x = 4$, since:

$\sqrt{2 \cdot 4 - 4} - \sqrt{3 \cdot 4 + 4} = -2$

380. Determine *a* and *b* such that $a + ab + b = 798$

$(a+1)(b+1) = a + ab + b + 1 = 799$

$799 = 17 \cdot 47$, and both 17 and 47 are primes. So:

$a + 1 = 17 \quad and \quad b + 1 = 47 \quad \Leftrightarrow \quad a = 16 \quad b = 46$

381. Determine integer *a* such that: $a^3 + a^2 = 36$

Well $3^3 = 27 \quad and \quad 3^2 = 9 \quad and \quad 27 + 9 = 36$, so the solution is $a = 3$

382. $x=\sqrt{3+\sqrt{8}}$ **Determine** $x^5-\frac{1}{x^5}$

We begin by simplifying $\sqrt{3+\sqrt{8}}=\sqrt{3+2\sqrt{2}}$.

$\sqrt{3+2\sqrt{2}}=(1+\sqrt{2})$, since $(1+\sqrt{2})^2=1+2+2\sqrt{2}=3+2\sqrt{2}$

$$x-\frac{1}{x}=1+\sqrt{2}-\frac{1}{1+\sqrt{2}}=\frac{(1+\sqrt{2})^2-1}{1+\sqrt{2}}=\frac{1+2+2\sqrt{2}-1}{1+\sqrt{2}}=\frac{2+2\sqrt{2}}{1+\sqrt{2}}=\frac{2(1+\sqrt{2})}{1+\sqrt{2}}=2$$

$$(x-\frac{1}{x})^2=4=x^2+\frac{1}{x^2}-2 \quad \Rightarrow x^2+\frac{1}{x^2}=6$$

$$(x^2+\frac{1}{x^2})(x-\frac{1}{x})=2\cdot 6=x^3-\frac{1}{x^3}-x+\frac{1}{x}=x^3-\frac{1}{x^3}-(x-\frac{1}{x})=x^3-\frac{1}{x^3}-2 \quad \Rightarrow$$

$$x^3-\frac{1}{x^3}=14.$$

$$(x^3-\frac{1}{x^3})(x^2+\frac{1}{x^2})=14\cdot 6=x^5-\frac{1}{x^5}+x-\frac{1}{x}=x^5-\frac{1}{x^5}-2 \quad \Rightarrow$$

$$x^5-\frac{1}{x^5}=86$$

383. $147^x=189$ **Determine** $7^{\frac{2x-1}{x-3}}$

We notice that: $147=3\cdot 7^2$ *and* $189=7\cdot 3^3$ so

$$147^x=189 \quad \Leftrightarrow \quad (3\cdot 7^2)^x=3^3\cdot 7 \quad \Rightarrow$$

$$3^{x-3}\cdot 7^{2x-1}=1 \quad \Leftrightarrow \quad 3\cdot 7^{\frac{2x-1}{x-3}}=1^{\frac{1}{x-3}} \quad \Leftrightarrow \quad 3\cdot 7^{\frac{2x-1}{x-3}}=1 \quad \Leftrightarrow$$

$$7^{\frac{2x-1}{x-3}}=\frac{1}{3}$$

384. Solve for *x*: $(\frac{1}{3})^x+(\frac{1}{5})^x=34$

$$(\frac{1}{3})^x+(\frac{1}{5})^x=34 \quad \Leftrightarrow \quad 3^{-x}+5^{-x}=34$$

Now; $3^2=9$ *and* $5^2=25$ *and* $9+25=34$,
So the solution is x = -2

385. Determine *x* from

$$2-\cfrac{1}{\cfrac{1}{3}-\cfrac{1}{\cfrac{1}{4}-x}}=5$$

$$2-\frac{1}{\frac{1}{3}-\frac{1}{\frac{1}{4}-x}}=5 \Leftrightarrow 2-\frac{1}{\frac{1}{3}+\frac{1}{\frac{4x-1}{4}}}=5 \quad \Leftrightarrow \quad 2-\frac{1}{\frac{1}{3}+\frac{4}{4x-1}}=5 \quad \Leftrightarrow$$

$$2-\frac{1}{\frac{4x+11}{12x-3}}=5 \quad \Leftrightarrow \quad 2-\frac{12x-3}{4x+11}=5$$

$$-3(4x+11)=12x-3 \quad \Leftrightarrow \quad 24x=-30 \quad \Leftrightarrow$$

$$x=-\frac{4}{5}$$

386. Given $a^2-a-1=0$ **Determine** a^6

$$a^2-a-1=0 \quad \Leftrightarrow \quad a=\frac{1\pm\sqrt{5}}{2}$$

$$a^2=a+1 \quad \Rightarrow \quad a^4=(a+1)^2$$

$$a^6=a^4a^2=(a+1)^2(a+1)=\text{Solve}$$

$$(\frac{1+\sqrt{5}}{2}+1)^2(\frac{1+\sqrt{5}}{2}+1)=(\frac{3+\sqrt{5}}{2})^2(\frac{3+\sqrt{5}}{2})=$$

$$(\frac{9+5+6\sqrt{5}}{2})(\frac{3+\sqrt{5}}{2})=2(\frac{7+3\sqrt{5}}{2})(\frac{3+\sqrt{5}}{2})=(7+3\sqrt{5})(\frac{3+\sqrt{5}}{2})=+1)$$

$$\frac{21+7\sqrt{5}+9\sqrt{5}+15}{2}=\frac{36+16\sqrt{5}}{2}=18+8\sqrt{5}$$

387. Solve for *x*: $x^x=2^{1-x^2}$

There is (to my knowledge) no analytic way to solve this equation, but $x = 1$ is seen to be a solution, since $1^1=2^{1-1}$

388. Solve for (*x,y*) $2^{\frac{1}{x}}=5^{\frac{1}{y}}=100$

$$2^{\frac{1}{x}}=100 \quad \Leftrightarrow \quad 2=100^x \text{ and } \quad 5^{\frac{1}{y}}=100 \quad 5=100^y$$

$$2\cdot 5=100^x100^y \quad \Rightarrow \quad 10=100^{x+y} \quad \Rightarrow \quad 10=(10^2)^{x+y} \quad \Rightarrow \quad 10^1=10^{2(x+y)} \quad \Rightarrow$$

$$2(x+y)=1 \quad \Rightarrow \quad x+y=\frac{1}{2}$$

$$2=100^x \quad \Rightarrow \quad \log 2=x\log 100 \quad \Rightarrow \quad x=\frac{\log 2}{2}$$

$$5=100^y \quad \Rightarrow \quad \log 5=y\log 100 \quad \Rightarrow \quad y=\frac{\log 5}{2}$$

389. Solve for *x*: $\log_2 x+\log_3 x=1$

We notice that:

$\log_2 x = y \quad \Leftrightarrow \quad x = 2^y \quad and \quad \log_3 x = y\log_3 2 = \log_2 x \log_3 2$

And by the same token:

$$\log_2 x = y\log_2 3 = \log_3 x \log_2 3 \quad \Rightarrow \quad \log_3 x = \frac{\log_2 x}{\log_2 3}$$

$$\log_2 x + \log_3 x = 1 \quad \Leftrightarrow \quad \log_2 x + \frac{\log_2 x}{\log_2 3} = 1 \quad \Leftrightarrow \quad (1 + \frac{1}{\log_2 3})\log_2 x = 1 \quad \Leftrightarrow$$

$$(\frac{\log_2 3 + 1}{\log_2 3})\log_2 x = 1 \quad \Leftrightarrow \quad (\frac{\log_2 3 + \log_2 2}{\log_2 3})\log_2 x = 1 \quad \Leftrightarrow \quad (\frac{\log_2 6}{\log_2 3})\log_2 x = 1 \quad \Leftrightarrow$$

$$\log_2 x = \frac{\log_2 3}{\log_2 6} \quad \Leftrightarrow \quad x = 2^{\frac{\log_2 3}{\log_2 6}}$$

390. $7^{x-1} = 5$ **Determine** 7^{x+1}

$$7^{x-1} = 5 \quad \Leftrightarrow \quad \frac{7^x}{7} = 5 \quad \Leftrightarrow \quad 7^x = 35$$

$$7^{x+1} = 7 \cdot 7^x = 7 \cdot 35 = 245$$

391. Solve for *x*: $x^{x^5} = 100$

To my knowledge there is no analytic way to solve such an equation, so we shall resort to qualified guesswork.

One candidate might be $\sqrt[5]{10}$, and indeed: $\sqrt[5]{10}^{\sqrt[5]{10}^5} = \sqrt[5]{10}^{10} = 10^2 = 100$

392. Solve for *x*: $x^{-x^2} = \frac{1}{2}$

$x^{-x^2} = \frac{1}{2}$ These exercises are based on qualified guesswork, and an obvious candidate is $x = \sqrt{2}$,

since; $x^{-x^2} = \frac{1}{2} \quad \Leftrightarrow \quad \frac{1}{x^{x^2}} = \frac{1}{\sqrt{2}^{\sqrt{2}^2}} = \frac{1}{\sqrt{2}^2} = \frac{1}{2}$

393. $x^2 - x + 1 = 0$ **Show that** $x^{2020} + x^{1010} + 1 = 0$

$$x^2 - x + 1 = 0 \quad \Leftrightarrow \quad x^2 = x - 1 \quad x = 1 - \frac{1}{x} \quad \Leftrightarrow \quad x - 1 = -\frac{1}{x} \quad \Rightarrow \quad -\frac{1}{x} = x^2 \quad \Rightarrow \quad x^3 = -1$$

This can also be obtained much easier:

$$x^2 - x + 1 = 0 \quad \Leftrightarrow \quad x^3 - x^2 + x = 0 \quad \Leftrightarrow \quad x^3 - (x-1) + x = 0 \quad \Leftrightarrow \quad \Leftrightarrow \quad x^3 + 1 = 0 \quad \Leftrightarrow \quad x^3 = -1$$

So we could hope that 1010 was a multiply besides a remainder less than 3 (and it is of course)

$$1010 = 336 \cdot 3 + 2$$

$x^{2020}+x^{1010}+1=0 \quad \Leftrightarrow \quad (x^{1010})^2+x^{1010}+1=0$
$x^{1010}=x^{3\cdot 336+2}=(x^3)^{336}x^2=(-1)^{336}=x^2$
$(x^{1010})^2=x^4=x^3x=-x$, So
$x^{2020}+x^{1010}+1=0 \quad -x+x^2+1=0 \quad \Leftrightarrow \quad x^2-x+1=0$

Which was what we should show.