Contents

Contents 1

200. Determine $\frac{x}{y}$ from the equation: $\frac{x^2}{y^2}+\frac{y^2}{z^2}=4$ 5

201. Solve: $\sqrt[3]{2-x}+\sqrt{x-1}=1$ 5

202. Solve: $5^x 16^{\frac{x-1}{x}}=100$ 5

203. Determine x from the series: $1+4+7+...+x=287$ 5

204. Determine n from the equation: $n!=n^2+11n+40$ 5

205. Determine x from: $x^{x^{11}}=11$ 5

206. Solve for x. $343^{3x-4}=\sqrt{7}$ 5

207. Solve for x: $\left(\frac{x}{8}\right)^x=8^{8^8}$ 6

208. Trivial: Find an expression for $\sqrt{20}-\sqrt{5}$ 6

209. Trivial: $2^x+2^{3x}=10$ 6

210. Solve for x. $x^{\log 25}+25^{\log x}=10$ 6

211. $x\sqrt{x}-11\sqrt{x}=10$. Determine an expression for $x-\sqrt{x}$ 6

212. Evaluate the integral: $\int_0^{\frac{3}{4}}\frac{\sin\sqrt{1-x^2}}{\sqrt{1-x^2}}dx$ 7

213. Verify that: $\int_{-1}^{1}\sqrt{1-x^2}\,dx=\int_{-1}^{1}\frac{1}{1+x^2}\,dx$ 7

214. Solve $x+\sqrt{x+\sqrt{x}}=16$ 7

215. Solve: $2^{\frac{2x-1}{x-1}}+2^{\frac{3x-2}{x-1}}=24$ 8

216. Solve: $x^4-4x-1=0$ 8

217. Solve for x: $x^2+\left(\frac{x}{1+x}\right)^2=15$ 8

218. Simplify: $\frac{5}{\sqrt[3]{25}}$ (Very easy) 9

219. Solve for x: $\sqrt{x^1}+\sqrt{x^2}=\sqrt{x^3}+\sqrt{x^4}$ 9

220. Solve for x: $(2+\sqrt{3})^x+(2-\sqrt{3})^x=4$ (Simple). 9

221. Solve for x,y: $6^x+6^y=42$, *and* $x+y=3$ 9

222. $3^{2x}=2^{3x}=5184$ 9

223. A right angle triangle has the sides $\ln x,\ \ln 2x,\ \ln 3x$ Determine x 10

224. Solve for x $\frac{27^x-3^x}{9^x-3^x}=82$ 10

225. In a triangle *A, B, C: A=B=22⁰,5* and a =25. Determine the area of the triangle. 10
226. Solve for x: $x^{\sqrt{\log x}} = 10$ 10
227. Solve for x, y: $(x+1)(y+1) = 12 \quad and \quad (x+y)(xy+1)$ 10
228. Simplify: $\sqrt{220 - 30\sqrt{35}}$ 11
229. Solve: $(x^2)^{x^6} = 3$ 11
230. Solve: $\sqrt{x} + y = 7 \quad and \quad x + \sqrt{y} = 11$ 11
231. Solve for $(x. y)$: $\sqrt{x}(x+3y) = 36 \quad and \quad \sqrt{y}(3x+y) = 28$ 11
232. Solve for x: $8^x + 4 = 4^x + 2^{x+2}$ 12
233. Simplify: $\dfrac{\sqrt{15}+\sqrt{35}+\sqrt{21}+5}{\sqrt{3+2\sqrt{5}}+\sqrt{7}}$ 12
234. Solve the system of equations: $x - y = 95 \quad and \quad \sqrt{x} + \sqrt{y} = 19$ 12
235. Solve for (x, y) $x^2 - y^2 = 12 \quad and \quad x^2y^2 = 64$ 12
236. Solve for (x, y) : $\sqrt{x} + \sqrt{y} = 13 \quad and \quad x - y = 65$ 13
237. $12^x = 18, \quad determine \quad 2^{\frac{2x-1}{x-1}}$ 13
238. Do the integral: $\int x^3 \tan^{-1}(x^2)dx$ 13
239. Solve for (x,y): $x^2 - y^2 = 9 \quad and \quad xy = 3$ 13
240. Solve for (x,y): $x + y = \sqrt{xy} \quad and \quad \sqrt{x} + \sqrt{y} = \sqrt[4]{xy}$????............ 14
241. Solve for x: $\dfrac{27^x + 343^x}{63^x + 147^x} = \dfrac{37}{21}$ 14
242. Let $27^x = 343^y = 1331^z = 231$ Show that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 3$ 15
243. Solve for (x, y): $x^3y + xy^3 = 10 \quad and \quad x^4 + 6x^2y^2 + y^4 = 41$ 15
244. $(\sqrt{3}+3)^3 - (\sqrt{3}-3)^3 = ?$ 16
245. Determine $(x$, y) where: $72^x \cdot 48^y = 6^{xy}$ 16
246. Simplify: $\sqrt[3]{77+20\sqrt{13}} + \sqrt[3]{77-20\sqrt{13}}$ 16
247. Solve for rational (x, y): $\sqrt{x} + \sqrt{y} = \sqrt{2+\sqrt{3}}$ 17
249. $x = 87647 \cdot 87649 \cdot 87654 \quad and \quad y = 87653 \cdot 87651 \cdot 87646$. Determine x- y. 17
250. Which is the biggest number. $\sqrt{5}^{\sqrt{5}} \quad or \quad \sqrt{7}^{\sqrt{3}}$ 17
251. Solve for x: $\dfrac{3^{x+1} - 5^{x+2}}{3^x - 5^x} = 1$ 18
252. Solve for (x,y): $\sqrt{x} + \sqrt{y} = \sqrt{2+\sqrt{3}}$ 18
253. Solve for x. (Easy one): $\sqrt{\dfrac{5+x}{x-1}} - \sqrt{\dfrac{x-1}{5+x}} = \dfrac{3}{2}$ 18
254. Solve for x. $x^x = 2^{3x+192}$ 19

255. Solve for x: $\frac{27^x - 8^x}{18^x - 12^x} = \frac{19}{6}$ 19
256. Solve for (x,y) : $x + y + xy = 2 + 3\sqrt{2}$ *and* $x^2 + y^2 = 6$ 20
257. Calculate the sum: $\frac{1}{7\cdot 8} + \frac{1}{8\cdot 9} + \frac{1}{9\cdot 10} + ... + \frac{1}{87\cdot 88}$ 20
258. Solve for x: $(0.01)^X = 11$ (Very easy) 21
259. Solve for x: $x^{\frac{1}{\log x}} \log x = 1$ 21
260. $a^3 + b^3 = 10$ *and* $a^2 + b^2 = 7$ $a + b = ?$ 21
261. Solve for x. $49^x - 49^{x-1} = 16464$ 22
262. Solve for x. $\frac{27^x + 343^x}{63^x + 147^x} = \frac{37}{21}$ 22
263. Solve for x: $\frac{3^{x^2}}{9^x} = 81$ 23
264. $(a+1)(b+1)(a+b) = 2022,$ $a^3 + b^3 = 1933.$ Determine $a + b$ 23
265. Determine $a^4 + b^4 + c^4$ From the three equations below 23
266. $a^4 + \frac{1}{a^4} = 47$. Determine: $a + \frac{1}{a}$ 25
267. Which number is biggest: 31^{11} *or* 17^{14} 25
268. $a^4 + \frac{1}{a^4} = 47$ Determine: $a + \frac{1}{a}$ 25
269. $2^{\frac{1}{x}} = 2^{\frac{1}{y}} = 100$. Determine $x+y$ 25
270. Solve for (a,b) $a^3 + b^3 = 2\sqrt{5}$ *and* $a^2 b + ab^2 = \sqrt{5}$ 26
271. Solve for x. $\log_2 x + \log_3 x = 1$ 26
272. Simplify: $\frac{2^{\sqrt{27}} \cdot 8^{\sqrt{75}}}{4^{\sqrt{48}}}$ 26
273. Solve: $x^{x^5} = 100$ 27
274. Solve integer solutions for (x,y): $\frac{x^3 + y^3}{x + y} = 7$ *and* $\frac{x^3 - y^3}{x - y} = 19$ 27
275. Simplify: $\sqrt{111222 - 333}$ 27
276. $100^{x-1} = 99;$ $100^{x+1} = ??$ (Strange) 27
277. Solve for x: $3^x + 4^x + 6^x = 1$ 27
278. Solve for x: $3^x \sqrt{3}^{x+4} + 20 = 0$ 27
279. Solve for x: $3^x + x^3 = 17$ 28
280. Determine (a, b) from: $\frac{1}{a} + \frac{1}{b} = 2$ *and* $3^a = 5^b$ 28
281. Calculate: $1 + (1+i) + (1+i)^2 + (1+i)^3$ 28
282. $x = a + \frac{1}{a}$ *and* $y = a - \frac{1}{a}$. Determine $\sqrt{x^4 + y^4 - 2x^2 y^2}$ 28

283. Solve for x: $x^{x^6}=\sqrt{2}^{\sqrt{2}}$ 29
284. Evaluate: $x-333^3=444^3+555^3$ 29
284. Solve for x. $\log(20-x)=\log^3 x$ (very easy) 29
285. $\frac{1}{x}-\frac{1}{y}=\frac{1}{x+y}$; $\frac{x}{y}=?$ 29
286. Solve for x: $x^{\sqrt{\log x}}=10^8$ 29
287. Determine integer solutions to: $a^2+a+34=b^2$ 30
288. $s=1^2+2\cdot 2^2+3^2+2\cdot 4^2+...+...$ 30
289. Simplify: $(3111)^2-(2889)^2$ 30
290. Solve for x: $\sqrt[3]{4}\cdot\sqrt[4]{x}-\sqrt[4]{3}\cdot\sqrt[3]{4}=0$ 30
291. Compute the sum: $s_n=\sum_{k=1}^{n}\frac{1}{k(k+1)}$ 30
292. Simplify: $\frac{\sqrt{\sqrt{5}+2}+\sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}}-\sqrt{3-2\sqrt{2}}$ 31
293. Evaluate $\left(\frac{1+i}{3+i}\right)^2$ 31
294. $8^x+\frac{1}{8^x}=9$ Determine $2^{9x}+\frac{1}{2^{9x}}$ 31
295. Determine n, such that: $(1+i)^n=(1-i)^n$, where i is the imaginary unit 32
296. Solve for (x,y) $3^{4x+5y}-200=43$ *and* $7^{8x-4y}-300=43$ 32
297. Determine a, and b, such that: $2^a-2^b=2016$ 32
298. Determine $(x+y)$ from: $x^2+y^2=7$ *and* $x^3+y^3=10$ 32
299. Solve for x: $x^4-4x-1=0$ 33
300. Solve for x: $\sqrt{x+6}-\sqrt{11-x}=3$ 33

200. Determine $\frac{x}{y}$ from the equation: $\frac{x^2}{y^2}+\frac{y^2}{z^2}=4$

$\frac{x^2}{y^2}+\frac{y^2}{z^2}=4$. If we put $z=\frac{x^2}{y^2}$, we have the equation $z+\frac{1}{z}=4$, which is the same as in 199.

So $\frac{x^2}{y^2}=2\pm\sqrt{3}$, and therefore: $\frac{x}{y}=\sqrt{2\pm\sqrt{3}}$

201. Solve: $\sqrt[3]{2-x}+\sqrt{x-1}=1$**.**

$\sqrt[3]{2-x}+\sqrt{x-1}=1$. If x is a solution, 2- x should be a cubic number, and x-1 should be a quadratic number. $2-x=0,1,8,27,64...$ and $x-1=1,4,9,16,...$ $x-1\geq 0 \quad\Rightarrow\quad x\geq 1 \quad\Rightarrow\quad 2-x\leq 1$
The only solution is therefore: $x = 1$.

202. Solve: $5^x 16^{\frac{x-1}{x}}=100$

$5^x 16^{\frac{x-1}{x}}=100$, well, 100 = 4·25, so it will be obvious to guess at: x =2, and indeed: $5^2\cdot 16^{\frac{1}{2}}=100$.

203. Determine *x* from the series: $1+4+7+...+x=287$

The series is algebraic, where we have. $a_{k+1}-a_k=d$, and for the sum of n terms we have the formula. $S_n=\frac{n}{2}(a_1+a_n)$.

It follows that $a_k=a_1+(k-1)d$.

We thus have: $x=1+3(n-1) \quad\Leftrightarrow\quad x=3n-2 \quad\Leftrightarrow\quad n=(x+2)/3$, and thus:

$287=\frac{n}{2}(3+x) \quad\Leftrightarrow\quad 287=\frac{n}{2}(3+3n-2) \quad\Leftrightarrow\quad 574=3n^2+n \quad\Leftrightarrow\quad 3n^2+n-574='0$

$d=1+12\cdot 574=83^2;\quad n=\frac{-1\pm 83}{6}=\frac{84}{6}=14$

And therefore $x=3n-2=40$

204. Determine *n* from the equation: $n!=n^2+11n+40$

$n!=n^2+11n+40$. To my knowledge there are no analytic method to solve this equation, but $4!=24$, $5!=120$; $6!=720$. So we try with $n = 5$. $25+55+40=120=5!$

205. Determine *x* from: $x^{x^{11}}=11$

$x^{x^{11}}=11$. Well it must be qualified guesswork, but the most obvious candidate is $x=\sqrt[11]{11}$. Since: $((\sqrt[11]{11})^{\sqrt[11]{11}})^{11}=(\sqrt[11]{11})^{11}=11$

206. Solve for *x*. $343^{3x-4}=\sqrt{7}$

We notice that $343=7^3$

$343^{3x-4} = \sqrt{7} \quad \Leftrightarrow \quad (3x-4)\ln 343 = \frac{1}{2}\ln 7 \quad \Leftrightarrow \quad 3x-4 = \dfrac{\frac{1}{2}\ln 7}{3\ln 7} \quad 3x-4 = \frac{1}{6} \quad \Leftrightarrow \quad x = \dfrac{25}{18}$

207. Solve for *x*: $\left(\dfrac{x}{8}\right)^x = 8^{8^8}$

It is a rather trivial exercise, if we put $y = \dfrac{x}{8}$, we

have: $y^{8y} = 8^{8^8} \quad \Leftrightarrow \quad (y^y)^8 = 8^{8^8} \quad \Leftrightarrow \quad y = 8 \quad \Leftrightarrow \quad x = 8^2$

208. Trivial: Find an expression for $\sqrt{20} - \sqrt{5}$

$\sqrt{20} - \sqrt{5} = \sqrt{4 \cdot 5} - \sqrt{5} = 2\sqrt{5} - \sqrt{5} = \sqrt{5}$

209. Trivial: $2^x + 2^{3x} = 10$

$2^x + 2^{3x} = 10$, we put: $y = 2^x$, and we get: $y^3 + y = 10$

It requires only very little insight to find that $y = 2,$ since $2^3 + 2 = 10$. The solution is:

$2^x = 2 \quad \Leftrightarrow \quad x = 1$

210. Solve for *x*. $x^{\log 25} + 25^{\log x} = 10$

$x^{\log 25} + 25^{\log x} = 10$. The two terms on the lhs are equal, since:

$\log(x^{\log 25}) = \log 25 \log x = \log x \log 25 = \log 25^{\log x}$, so we have :

$2x^{\log 25} = 10 \quad \Leftrightarrow \quad x^{\log 25} = 5 \quad \Leftrightarrow \quad \log 25 \log x = \log 5 \quad \log x = \dfrac{\log 5}{2\log 5} = \dfrac{1}{2} \quad \Rightarrow \quad x = \sqrt{10}$

211. $x\sqrt{x} - 11\sqrt{x} = 10$**. Determine an expression for** $x - \sqrt{x}$

$x\sqrt{x} - 11\sqrt{x} = 10$. We put: $y = \sqrt{x}$, and then we have: $y^3 - 11y - 10 = 0$.

At a glance we see that $y = -1$ is a solution and we make polynomial division with: $y + 1$

$y+1 \,|\, y^3 - 11y - 10 \,|\, y^2 - y - 10$

$y^3 + y^2$

$-y^2 - 11y$

$-y^2 - y$

$-10y - 10$

$-10y - 10$

$y^2 - y - 10 = 0 \quad \Leftrightarrow \quad y^2 - y = 10 \quad \Leftrightarrow \quad x - \sqrt{x} = 10$

212. Evaluate the integral: $\int_0^{\frac{3}{4}} \frac{\sin\sqrt{1-x^2}}{\sqrt{1-x^2}}dx$

$\int_0^{\frac{3}{4}} \frac{\sin\sqrt{1-x^2}}{\sqrt{1-x^2}}dx$, we put $u=\sqrt{1-x^2} \quad \Rightarrow \quad du = \frac{1}{2\sqrt{1-x^2}}dx \quad = \frac{1}{2u}dx$

$$\int_0^{\frac{3}{4}} \frac{\sin\sqrt{1-x^2}}{\sqrt{1-x^2}}dx = \int_1^{x=\frac{3}{4}} \frac{\sin u}{u}2udu = 2\int_1^{x=\frac{3}{4}} \sin udu =$$

$$2\left[-\cos u\right]_{x=1}^{x=\frac{3}{4}} = 2\left[-\cos\sqrt{1-x^2}\right]_{x=0}^{x=\frac{3}{4}} = 2(-\cos\frac{\sqrt{7}}{4}+\cos 1)$$

This integral has no analytic solution, but (as far as I remember) $\int_1^x \frac{\sin u}{u}du$ has a name.

213. Verify that: $\int_{-1}^{1}\sqrt{1-x^2}dx = \int_{-1}^{1}\frac{1}{1+x^2}dx$

$\int_{-1}^{1}\sqrt{1-x^2}dx$. We put

$x=\cos t \quad \Rightarrow \quad \sqrt{1-x^2}=\sin t \quad dx=-\sin tdt \quad x=-1 \quad \Rightarrow \quad t=\pi \quad and \quad x=1 \quad \Rightarrow \quad t=0$

$$\int_{-1}^{1}\sqrt{1-x^2}dx = \int_0^{\pi}\sin t\sin tdt = \int_0^{\pi}\sin^2 tdt = \int_0^{\pi}\frac{1-\cos 2t}{2}dt = \left[\frac{t}{2}-\frac{\sin 2t}{4}\right]_0^{\pi} = \frac{\pi}{2}$$

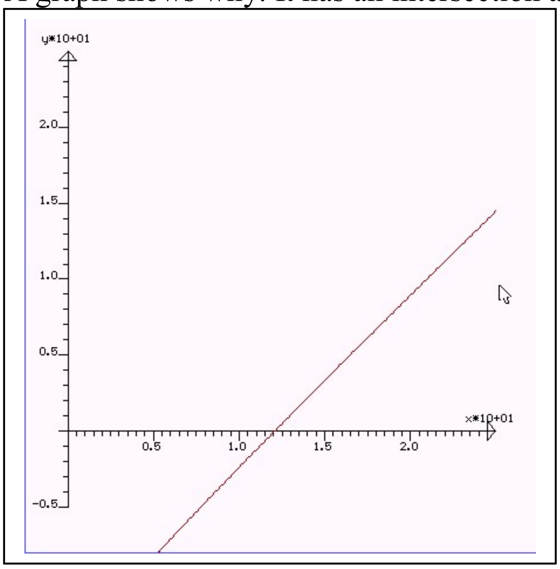
$$\int_{-1}^{1}\frac{1}{1+x^2}dx = \left[\tan^{-1}x\right]_{-1}^{1} = \frac{\pi}{4}-(-\frac{\pi}{4}) = \frac{\pi}{2}$$

214. Solve $x+\sqrt{x+\sqrt{x}}=16$

It is not difficult to verify that there are no integer solutions to this equation. Neither have $\sqrt{x+\sqrt{x}}$ any integer solution

If we put $y^2=x+\sqrt{x}$, we end up with an equation: $y^2-\sqrt{x}+y=16$, which leads nowhere:

A graph shows why. It has an intersection about 12.06



215. Solve: $2^{\frac{2x-1}{x-1}}+2^{\frac{3x-2}{x-1}}=24$

$2^{\frac{2x-1}{x-1}}+2^{\frac{3x-2}{x-1}}=24 \Leftrightarrow 2^{\frac{x+x-1}{x-1}}+2^{\frac{x+2x-2}{x-1}}=24 \Leftrightarrow 2^{\frac{x}{x-1}+\frac{x-1}{x-1}}+2^{\frac{x}{x-1}+\frac{2x-2}{x-1}}=24 \Leftrightarrow$

$2^{\frac{x}{x-1}+1}+2^{\frac{x}{x-1}+2}=24 \Leftrightarrow 2\cdot 2^{\frac{x}{x-1}}+4\cdot 2^{\frac{x}{x-1}}=24$

We put $y=\frac{x}{x-1}$.

$2\cdot 2^{y}+4\cdot 2^{y}=24 \Leftrightarrow 6\cdot 2^{y}=24 \Leftrightarrow 2^{y}=4 \Leftrightarrow y=2 \quad \frac{x}{x-1}=2 \Leftrightarrow x=2$

216. Solve: $x^4-4x-1=0$

$x^4-4x-1=0$. We notice that: $(x^2+1)^2=x^4+1+2x^2 \Leftrightarrow x^4=(x^2+1)^2-1-2x^2$

Is inserted in: $x^4-4x-1=0$ to give:

$(x^2+1)^2-1-2x^2-4x-1=0 \Leftrightarrow (x^2+1)^2-2(x^2+2x+1)=0$

$(x^2+1)^2-2(x+1)^2=0 \Leftrightarrow (x^2+1)^2=2(x+1)^2 \Leftrightarrow$

$(x^2+1)=\sqrt{2}(x+1) \Leftrightarrow x^2-\sqrt{2}x-\sqrt{2}+1=0$

$d=2+4(\sqrt{2}-1)=4\sqrt{2}-2.$

$x=\frac{\sqrt{2}\pm\sqrt{4\sqrt{2}-2}}{2}$

217. Solve for *x*: $x^2+\left(\frac{x}{1+x}\right)^2=15$

$x^2+\left(\frac{x}{1+x}\right)^2=15 \Leftrightarrow x^2\left(1+\frac{1}{(1+x)^2}\right)=15 \Leftrightarrow \left(\frac{x^2}{(1+x)^2}\right)\left((1+x)^2+\frac{1}{(1+x)^2}\right)=15$

$x^2\left(1+\frac{1}{(1+x)^2}\right)=15 \Leftrightarrow x^2\left(\frac{(1+x)^2+1}{(1+x)^2}\right)=15 \Leftrightarrow x^2\left(\frac{x^2+2x+1+1}{(1+x)^2}\right)=15 \Leftrightarrow$

$x^2\left(\frac{x^2}{(1+x)^2}+\frac{2(x+1)}{(1+x)^2}\right)=15 \Leftrightarrow x^2\left(\frac{x^2}{(1+x)^2}+\frac{2}{(1+x)}\right)=15 \Leftrightarrow$

$\frac{x^4}{(1+x)^2}+\frac{2x^2}{(1+x)}=15 \Leftrightarrow \left(\frac{x^2}{(1+x)}\right)^2+2\frac{x^2}{(1+x)}-15=0$

We put: $y=\frac{x^2}{(1+x)}$ and we thus find:

$y^2+2y-15=0; \quad d=2+4\cdot 15=64; \quad y=\frac{-2\pm 8}{2} \Leftrightarrow y=-5 \vee y=3$

$\frac{x^2}{(1+x)}=-5 \vee \frac{x^2}{(1+x)}=3 \Leftrightarrow x^2+5x+5=0 \vee x^2-3x-3=0$

$d=25-20=5 \quad or \quad d=9+12=21; \quad x=\frac{-5\pm\sqrt{5}}{2} \vee x=\frac{3\pm\sqrt{21}}{2}$

218. Simplify: $\frac{5}{\sqrt[3]{25}}$ **(Very easy)**

$\frac{5}{\sqrt[3]{25}} = \frac{\sqrt[3]{125}}{\sqrt[3]{25}} = \sqrt[3]{5}$

219. Solve for *x*: $\sqrt{x^1} + \sqrt{x^2} = \sqrt{x^3} + \sqrt{x^4}$

$\sqrt{x^1} + \sqrt{x^2} = \sqrt{x^3} + \sqrt{x^4}$.

We put: $y = \sqrt{x}$, and we then have:

$y + y^2 = y^{3\prime} + y^4 \quad \Leftrightarrow \quad y(y+1) = y^3(y+1) \quad \Leftrightarrow$

$y = y^3 \quad y^2 = 1 \quad \vee \quad y = 0 \quad \Leftrightarrow \quad y = 0 \quad \vee \quad y = 1.$

$\sqrt{x} = 0 \quad \vee \quad \sqrt{x} = 1 \quad x = 0 \quad \vee \quad x = 1$

220. Solve for *x*: $(2+\sqrt{3})^x + (2-\sqrt{3})^x = 4$ **(Simple).**

$(2 + \sqrt{3})^x + (2 - \sqrt{3})^x = 4$. We notice that:

$(2+\sqrt{3})^x(2-\sqrt{3})^x = ((2+\sqrt{3})(2-\sqrt{3}))^x = (4-3)^x = 1^x = 1$

We put: $a = (2+\sqrt{3})^x \quad and \quad b = (2-\sqrt{3})^x$, and we thus have:

$$a + b = 4 \quad ab = 1 \quad \Rightarrow \quad a + \frac{1}{a} = 4 \quad \Leftrightarrow \quad a^2 - 4a + 1 = 0; \quad d = 16 - 4 = 12$$

$$a = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3} \quad and \quad b = \frac{1}{a} = 2 \mp \sqrt{3}.$$

$$a = (2 \pm \sqrt{3})^x \quad \Leftrightarrow \quad 2 \pm \sqrt{3} = (2 \pm \sqrt{3})^x \quad \Leftrightarrow \quad x = 1$$

221. Solve for *x,y*: $6^x + 6^y = 42$, $and \quad x + y = 3$

$6^x + 6^y = 42$, $and \quad x + y = 3 \quad \Rightarrow \quad 6^{x+y} = 6^3 \quad \Leftrightarrow \quad 6^x \cdot 6^y = 216$.

We put: $a = 6^x \quad and \quad b = 6^y$ The two equations are then:

$$a + b = 42 \quad and \quad ab = 216 \quad \Rightarrow \quad a + \frac{216}{a} = 42 \quad \Leftrightarrow \quad a^2 - 42a + 216 = 0 \;;\; d = 42^2 - 4 \cdot 216 = 900$$

$$a = \frac{42 \pm 30}{2} \quad \Leftrightarrow \quad a = 36 \quad \vee \quad a = 6 \quad \Leftrightarrow \quad x = 2 \quad \vee \quad x = 1$$

222. $3^{2x} = 2^{3x} = 5184$

First we notice that $5184 = 72^2$.

$3^{2x} = (3^2)^x = (2^3)^x = 5184 \quad \Leftrightarrow \quad 9^x = 8^x = 72^2 \quad \Rightarrow$

$9 = (72^2)^{\frac{1}{x}} \quad 8 = (72^2)^{\frac{1}{y}} \quad \Rightarrow \quad 8 \cdot 9 = (72^2)^{\frac{1}{x}+\frac{1}{y}} \quad \Leftrightarrow \quad 72 = (72^2)^{\frac{x+y}{xy}} \quad \Leftrightarrow$

$72^{\frac{xy}{x+y}} = 72^2 \quad \Leftrightarrow \quad \frac{xy}{x+y} = 2$

223. A right angle triangle has the sides $\ln x, \ \ln 2x, \ \ln 3x$ Determine *x*

$\ln 2x = \ln 2 + \ln x; \quad \ln 3x = \ln 3 + \ln x.$ We put: $y = \ln x, \quad a = \ln 2, \quad b = \ln 3$

According to Pythagoras we have: $y^2 + (y+a)^2 = (y+b)^2$

$y^2 + y^2 + a^2 + 2ya = y^2 + b^2 + 2yb \quad \Leftrightarrow \quad y^2 + 2y(a-b) + \ a^2 - b^2 = 0$

$d = 4(a-b)^2 - 4(a^2 - b^2) = 8b^2 - 8ab = 8b(b-a)$

$y = \frac{-2(a-b) \pm \sqrt{8b(b-a)}}{2} \quad \Leftrightarrow \quad y = \frac{-2(a-b) \pm 2\sqrt{b(b-a)}}{2} \quad \Leftrightarrow \quad y = (b-a) + \sqrt{b(b-a)}$

$\ln x = \ln 3 - \ln 2 + \sqrt{\ln 3(\ln 3 - \ln 2)} \quad \Leftrightarrow \quad \ln x = \ln \tfrac{3}{2} + \sqrt{\ln 3 \ln \tfrac{3}{2}} \ x = \tfrac{3}{2} e^{\sqrt{\ln 3 \ln \frac{3}{2}}}$

224. Solve for *x* $\frac{27^x - 3^x}{9^x - 3^x} = 82$

$\frac{27^x - 3^x}{9^x - 3^x} = 82 \quad \Leftrightarrow \quad \frac{(3^3)^x - 3^x}{(3^2)^x - 3^x} = 82 \quad \Leftrightarrow \quad \frac{(3^x)^3 - 3^x}{(3^x)^2 - 3^x} = 82$

We put $y = 3^x$, and we find:

$\frac{y^3 - y}{y^2 - y} = 82 \quad \Leftrightarrow \quad \frac{y(y^2-1)}{y(y-1)} = 82 \quad \Leftrightarrow \quad \frac{y(y-1)(y+1)}{y(y-1)} = 82 \quad \Leftrightarrow \quad y + 1 = 82 \quad \Leftrightarrow$

$y = 3^x = 81 \quad \Leftrightarrow \quad 3^x = 3^4 \quad \Leftrightarrow \quad x = 4$

225. In a triangle *A, B, C: A=B=22⁰,5* and *a* =25. Determine the area of the triangle.

Since it is an isosceles triangle, we have b=25. $C = 180 - 45 = 135$,
We can then apply the formula:

226. Solve for *x*: $x^{\sqrt{\log x}} = 10$

$x^{\sqrt{\log x}} = 10$, We put; $y^2 = \log x \quad \Rightarrow \quad x = 10^{y^2}$

$(10^{y^2})^y = 10 \quad \Leftrightarrow \quad 10^{y^3} = 10^1 \quad \Leftrightarrow \quad y^3 = 1 \quad \Leftrightarrow \quad y = 1 \quad \Leftrightarrow \quad \log x = 1 \quad \Leftrightarrow \quad x = 10$

227. Solve for *x, y*: $(x+1)(y+1) = 12 \quad \textit{and} \quad (x+y)(xy+1)$

$(x+1)(y+1) = 12 \quad \Leftrightarrow \quad xy + 1 + x + y = 12 \quad \Leftrightarrow \quad \frac{35}{x+y} + x + y = 12$

We put $z = x + y$, then:

$\frac{35}{z} + z = 12 \quad \Leftrightarrow \quad z^2 - 12z + 35 = 0\,; \quad d = 144 - 4 \cdot 35 = 4$

$z = \frac{12 \pm 2}{2} \quad \Leftrightarrow \quad z = 7 \quad \vee \quad z = 5 \quad \Rightarrow \quad x + y = 7 \quad \vee \quad x + y = 5$

$xy + 1 = \frac{35}{x+y} = 5 \quad \vee \quad xy + 1 = \frac{35}{x+y} = 7 \quad \Leftrightarrow \quad xy = 4 \quad \vee \quad xy = 6$

$x + y = 7 \quad \wedge \quad xy = 4 \quad \Rightarrow \quad x(7-x) = 4 \quad \Leftrightarrow \quad -x^2 + 7x - 4 = 0; \quad d = 49 - 4 \cdot 4 = 33$

$x = \frac{7 \pm \sqrt{33}}{2} \quad \wedge \quad y = 7 - x$

$x + y = 5 \quad \wedge \quad xy = 6 \quad \Rightarrow \quad x(5-x) = 6 \quad \Leftrightarrow \quad -x^2 + 5x - 6 = 0; \quad d = 25 - 24 = 1$

$x = \frac{5 \pm 1}{2} \quad \Leftrightarrow x = 3 \quad \vee \quad x = 2 \quad \Rightarrow \quad y = 2 \quad \vee \quad y = 2$

228. Simplify: $\sqrt{220 - 30\sqrt{35}}$

Since: $\sqrt{35} = \sqrt{5 \cdot 7}$ We shall try to write $\sqrt{220 - 30\sqrt{35}}$ as $(a\sqrt{5} - b\sqrt{7})^2$

$(a\sqrt{5} - b\sqrt{7})^2 = 5a^2 + 7b^2 + 2ab\sqrt{35}$

From which we infer: $2ab = 30 \quad or \quad ab = 15$, and we try with: $a = 3 \quad and \quad b = 5$

$5a^2 + 7b^2 + 2ab\sqrt{35} = 45 + 175 + 30\sqrt{35} = 220 + 30\sqrt{35}$

So $\sqrt{220 - 30\sqrt{35}} = (3\sqrt{5} - 5\sqrt{7})$

229. Solve: $(x^2)^{x^6} = 3$

$(x^2)^{x^6} = 3$. Well. One may try: $x = \sqrt[6]{3}$, since: $(\sqrt[6]{3})^{2\sqrt[6]{3}^6} = (\sqrt[6]{3})^6 = 3$

230. Solve: $\sqrt{x} + y = 7 \quad and \quad x + \sqrt{y} = 11$

$\sqrt{x} + y = 7 \quad and \quad x + \sqrt{y} = 11$.

This is a tricky one, since a traditional algebraic procedure leads to a fourth degree equation in x or y, having a term x or y.

However by inspection, it is easily seen that $(x, y) = (9,4)$ is a solution, since:

$\sqrt{9} + 4 = 7 \quad and \quad 9 + \sqrt{4} = 11$.

231. Solve for (*x. y*): $\sqrt{x}(x + 3y) = 36 \quad and \quad \sqrt{y}(3x + y) = 28$

$\sqrt{x}(x + 3y) = 36 \quad and \quad \sqrt{y}(3x + y) = 28$

This system of equations, is similar to the previous, A traditional analytical approach leads to a fourth degree equation, that is cumbersome to solve. However if we assume an integer solution, th values of x and y, must be quadratic numbers; 1, 4, 9, 16,…

We write the equations;

$x + 3y = \frac{36}{\sqrt{x}} \quad and \quad 3x + y = \frac{28}{\sqrt{y}}$

If we put $y = 1$ we find: $3x+1=28 \Rightarrow x=9$, and indeed: $9+3=\frac{36}{3}=12$

And the solution is therefore; $(x,y) = (9,1)$

232. Solve for *x*: $8^x+4=4^x+2^{x+2}$

$8^x+4=4^x+2^{x+2} \quad \Leftrightarrow \quad (2^x)^3+4=(2^x)^2+4\cdot 2^x$

We put $y=2^x$, and then we have:

$y^3-y^2-4y+4=0 \quad \Leftrightarrow \quad y^2(y-1)-4(y-1)=0 \quad \Leftrightarrow \quad (y-1)(y^2-4)=0 \quad \Leftrightarrow$

$y=1 \quad \vee \quad y=-2 \quad \vee \quad y=2$

$2^x=1 \quad \vee \quad 2^x=-2 \quad \vee \quad 2^x=2 \quad \Leftrightarrow$

$x=0 \quad \vee \quad x=1$

233. Simplify: $\dfrac{\sqrt{15}+\sqrt{35}+\sqrt{21}+5}{\sqrt{3+2\sqrt{5}}+\sqrt{7}}$

$$\frac{\sqrt{15}+\sqrt{35}+\sqrt{21}+5}{\sqrt{3+2\sqrt{5}}+\sqrt{7}}=\frac{\sqrt{5}\sqrt{3}+\sqrt{5}\sqrt{7}+\sqrt{7}\sqrt{3}+\sqrt{5}\sqrt{5}}{\sqrt{3}+2\sqrt{5}+\sqrt{7}}$$

$$\frac{\sqrt{5}(\sqrt{3}+\sqrt{5})+\sqrt{7}(\sqrt{3}+\sqrt{5})}{(\sqrt{3}+\sqrt{5})+(\sqrt{5}+\sqrt{7})}=\frac{(\sqrt{3}+\sqrt{5})(\sqrt{5}+\sqrt{7})}{(\sqrt{3}+\sqrt{5})+(\sqrt{5}+\sqrt{7})}=\frac{1}{\frac{1}{(\sqrt{5}+\sqrt{7})}+\frac{1}{(\sqrt{3}+\sqrt{5})}}$$

234. Solve the system of equations: $x-y=95 \quad and \quad \sqrt{x}+\sqrt{y}=19$

$x-y=95 \quad and \quad \sqrt{x}+\sqrt{y}=19$

An analytic approach is not recommendable, But, we may notice that both *x* and *y*, must be quadratic numbers, and 100<*x*<200.

$x=10^2=100 \Rightarrow y=5$

$x=10^2=121 \Rightarrow y=26$

$x=12^2=144 \Rightarrow y=49=7^2 \; so, \; x-y=95 \quad and \quad \sqrt{x}+\sqrt{y}=12+7=19$

235. Solve for (*x, y*) $x^2-y^2=12 \quad and \quad x^2y^2=64$

$x^2-y^2=12 \quad and \quad x^2y^2=64 \quad \Leftrightarrow \quad x^2-y^2=12 \quad and \quad xy=8 \quad \Leftrightarrow$

$x^2-y^2=12 \quad and \quad y=\frac{8}{x} \quad \Leftrightarrow \quad x^2-\left(\frac{8}{x}\right)^2=12 \quad and \quad y=\frac{8}{x}$

$x^2-\left(\frac{8}{x}\right)^2=12 \quad \Leftrightarrow \quad x^4-12x^2-64=0 \quad ; \; d=144+4\cdot 64=400$

$x^2=\frac{12\pm 20}{2} \quad \Leftrightarrow \quad x^2=16 \quad \vee \quad x^2=-8 \quad \Leftrightarrow \quad x=4 \quad \vee \quad x=-4$

$x=4 \quad \wedge \quad y=2 \quad \vee \quad x=-4 \quad \wedge \quad y=-2$

236. Solve for (*x, y*) : $\sqrt{x}+\sqrt{y}=13 \quad and \quad x-y=65$

An analytic approach leads to a fourth degree equation, which is not easily solved, , but if we assume an integer solution, x must be at most 100, so we try with

$\sqrt{x}=9 \;\Rightarrow\; \sqrt{y}=4$, and indeed $81-16=65$ so the solution is: $(x,y)=(81,16)$

237. $12^x=18, \quad determine \quad 2^{\frac{2x-1}{x-1}}$

$$12^x=18 \;\Leftrightarrow\; 3^x\cdot 2^{2x}=3^2\cdot 2 \;\Leftrightarrow\; 3^{x-2}\cdot 2^{2x-1}=1 \;\Leftrightarrow$$

$12=3\cdot 2^2 \quad and \quad 18=3^2 2$ so $(x-2)\ln 3+(2x-1)\ln 2=0 \;\Leftrightarrow\; \ln 3+\dfrac{2x-1}{x-2}\ln 2=0 \;\Leftrightarrow$

$$\ln 2^{\frac{2x-1}{x-2}}=-\ln 3 \;\Rightarrow\; 2^{\frac{2x-1}{x-2}}=e^{-\ln 3}=\frac{1}{3}$$

238. Do the integral: $\int x^3\tan^{-1}(x^2)dx$

We do a partial integration:

$$\int x^3\tan^{-1}(x^2)dx=\tfrac{1}{4}\int \tan^{-1}(x^2)dx^4$$

We put: $t=x^2$: $\int \tan^{-1}(t)dt^2=\tan^{-1}(t)t^2-\int t^2 d\tan^{-1}(t)=\tan^{-1}(t)t^2-\int t^2\dfrac{1}{1+t^2}dt$

$$\int t^2\frac{1}{1+t^2}dt=\int\frac{1+t^2}{1+t^2}dt-\int\frac{1}{1+t^2}dt=\int dt-\tan^{-1}t=t-\tan^{-1}t$$

And then we substitute back

$$\int x^3\tan^{-1}(x^2)dx=\tfrac{1}{4}(\tan^{-1}(x^2)x^4-x^2+\tan^{-1}x^2)$$

239. Solve for (x,y): $x^2-y^2=9 \quad and \quad xy=3$

This exercise comes in several versions, they are easily solved, but neither of them have a "friendly" solutions.

$x^2-y^2=9 \quad and \quad xy=3 \;\Leftrightarrow\; x^2-y^2=9 \quad and \quad y=\dfrac{3}{x} \;\Rightarrow$

$x^2-y^2=9$

$x^2-\left(\dfrac{3}{x}\right)^2-9=0 \;\Leftrightarrow$

$x^4-9x^2-9=0$

We put: $z=x^2$ and find:

$z^2-9z-9=0$; $d=9^2+4\cdot 9=117=9\cdot 13 \quad z=\dfrac{9\pm 3\sqrt{13}}{2} \quad x^2=\dfrac{9+3\sqrt{13}}{2} \;\Leftrightarrow$

$x=\pm\sqrt{\dfrac{9+3\sqrt{13}}{2}}$

240. Solve for (*x,y*): $x+y=\sqrt{xy} \quad and \quad \sqrt{x}+\sqrt{y}=\sqrt[4]{xy}$ **????**

We notice that $x \; and \; y$ must be nonnegative integers, so we can take the square of both equations.

$$(x+y)^2=(\sqrt{xy})^2 \quad \Leftrightarrow \quad x^2+y^2+2xy=xy \quad \Leftrightarrow \quad x^2+y^2+xy=0$$

$$(\sqrt{x}+\sqrt{y})^2=(\sqrt[4]{xy})^2 \quad \Leftrightarrow \quad x+y+2\sqrt{xy}=\sqrt{xy} \quad \Leftrightarrow \quad x+y+\sqrt{xy}=0$$

If we put: $a=x; \;\; b=y; \;\; c=\sqrt{xy}$ we have for the three positive numbers: *a, b ,c:*

$a^2+b^2+c^2=0 \quad and \quad a+b+c=0$, which is only possible if: $a = b = c = 0$, and thus: $x = y = 0$

241. Solve for *x*: $\frac{27^x+343^x}{63^x+147^x}=\frac{37}{21}$

$$\frac{27^x+343^x}{63^x+147^x}=\frac{37}{21}.$$

We notice that:

$27=3^3 \quad 343=7^3 \quad 147=3\cdot 7^2, \quad 63=7\cdot 3^2$

For simplicity we put: $a=3^x$, $b=7^x$, and we then have:

$$\frac{a^3+b^3}{a^2b+ab^2}=\frac{37}{21} \quad \Leftrightarrow \quad \frac{a^3+b^3}{ab(a+b)}=\frac{37}{21}$$

We then make use of the formula: $a^3+b^3=(a+b)(a^2-ab+b^2)$, which can be proven by

Evaluation: $a^3-a^2b+ab^2+a^2b-ab^2+b^3=a^3+b^3$

$$\frac{a^3+b^3}{ab(a+b)}=\frac{37}{21} \quad \Leftrightarrow \quad \frac{(a+b)(a^2-ab+b^2)}{ab(a+b)}=\frac{37}{21} \quad \Leftrightarrow \quad \frac{a^2-ab+b^2}{ab}=\frac{37}{21}$$

And by division with *ab*.

$\frac{a}{b}-1+\frac{b}{a}=\frac{37}{21}$. We put: $z=\frac{a}{b}$ and we find: $z-1+\frac{1}{z}=\frac{37}{21} \quad \Leftrightarrow$

$$21z-21+\frac{21}{z}=37 \quad \Leftrightarrow \quad 21z^2-21z-21-37z=0 \quad \Leftrightarrow \quad 21z^2-58z+21=0$$

$$d=58^2-4\cdot 21\cdot 21=1600; \quad z=\frac{58\pm 40}{42} \quad \Leftrightarrow \quad z=\frac{7}{6} \quad \vee \quad z=\frac{3}{7}$$

$$(\frac{3}{7})^x=\frac{7}{6} \quad \vee \quad (\frac{3}{7})^x=\frac{3}{7} \quad \Leftrightarrow \quad x=\frac{\ln(\frac{7}{6})}{\ln(\frac{3}{7})} \quad \vee \quad x=1$$

242. Let $27^x = 343^y = 1331^z = 231$ **Show that** $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3$

$27^x = 231 \quad \Leftrightarrow \quad 27 = 231^{\frac{1}{x}}$

$343^y = 231 \quad \Leftrightarrow \quad 343 = 231^{\frac{1}{y}}$

$1331^z = 231 \quad \Leftrightarrow \quad 1331 == 231^{\frac{1}{z}}$

$27 \cdot 343 \cdot 1331 = 231^{\frac{1}{x}} \cdot 231^{\frac{1}{y}} \cdot 231^{\frac{1}{z}} \quad \Leftrightarrow$

$3^3 \cdot 7^3 \cdot 11^3 = 231^{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} \quad \Leftrightarrow \quad 231^3 = 231^{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} \quad \Leftrightarrow \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3$

243. Solve for (*x, y*): $x^3y + xy^3 = 10 \quad and \quad x^4 + 6x^2y^2 + y^4 = 41$

$x^3y + xy^3 = 10 \quad and \quad x^4 + 6x^2y^2 + y^4 = 41 \quad \Leftrightarrow$

$xy(x^2+y^2) = 10 \quad \wedge \quad (x^2+y^2)^2 + 4x^2y^2 = 41$

$xy = \frac{10}{(x^2+y^2)} \quad \Rightarrow \quad (x^2+y^2)^2 + 4(\frac{10}{(x^2+y^2)})^2 = 41$

We put: $u = x^2 + y^2 \quad and \quad v = xy$, and then we have: $u^4 + 400 - 41u^2 = 0 \quad \Leftrightarrow$
$u^4 - 41u^2 + 400 = 0 \;\; : \;\; d = 1681 - 1600 = 81$

$u^2 = \frac{41 \pm 9}{2} \quad \Leftrightarrow \quad u^2 = 25 \quad \vee \quad u^2 = 16 \quad \Leftrightarrow$

$u = 5 \quad \vee \quad u = -5 \quad \vee \quad u = 4 \quad \vee \quad u = -4$

Since u is non negative, we discard the negative solutions

$u = 5 \quad \vee \quad u = 4$

$v = \frac{10}{u} = 2 \quad \vee \quad v = \frac{10}{u} = \frac{5}{2}$

$x^2 + y^2 = 5 \;\wedge\; xy = 2 \quad \vee \quad x^2 + y^2 = 4 \;\wedge\; xy = \frac{2}{5}$

$x^2 + y^2 = 5 \quad and \quad xy = 2 \quad \Leftrightarrow \quad x^2 + y^2 = 5 \quad and \quad y = \frac{2}{x} \quad \Leftrightarrow$

$x^2 + (\frac{2}{x})^2 = 5 \quad \Leftrightarrow \quad x^4 - 5x^2 + 4 = 0 \;\; ; \quad d = 25 - 16 = 9$

$x^2 = \frac{5 \pm 3}{2} \quad \Leftrightarrow \quad x^2 = 4 \quad \vee \quad x^2 = 1 \quad \Leftrightarrow \quad x = 2 \quad \vee \quad x = 1$

$(x, y) = (2,1) \quad \vee \quad (x, y) = (1,2)$

$x^2 + y^2 = 4 \quad and \quad xy = \frac{2}{5} \quad \Leftrightarrow \quad x^2 + y^2 = 4 \quad and \quad y = \frac{2}{5x} \quad \Leftrightarrow$

$x^2 + (\frac{2}{5x})^2 = 4 \quad \Leftrightarrow \quad 25x^4 - 100x^2 + 4 = 0$

$z = 5x^2 \quad ;\Rightarrow \quad z^2 - 20z + 4 = 0 \quad ; d = 400 - 16 = 384 = 3 \cdot 128 = 3 \cdot 2 \cdot 64 = 6 \cdot 8^2$

$z = \frac{20 \pm 8\sqrt{6}}{2} \quad \Leftrightarrow \quad z = 10 \pm 4\sqrt{6} \quad \Rightarrow 5x^2 = 10 \pm 4\sqrt{6}$

$x = \sqrt{2 + \frac{4}{5}\sqrt{6}} \quad and \quad y = \frac{2}{5\sqrt{2 + \frac{4}{5}\sqrt{6}}} \quad or \quad x = \sqrt{2 - \frac{4}{5}\sqrt{6}} \quad and \quad y = \frac{2}{5\sqrt{2 - \frac{4}{5}\sqrt{6}}}$

244. $(\sqrt{3} + 3)^3 - (\sqrt{3} - 3)^3 = ?$

We put $a = \sqrt{3} + 3 \quad and \quad b = \sqrt{3} - 3$ then

$ab = (\sqrt{3} + 3)(\sqrt{3} - 3) = \sqrt{3}^2 - 3^2 = -6$

$(a - b)^3 = a^3 - b^3 - 3ab(a - b)$, so

$a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)((a - b)^2 + 3ab)$

$a - b = \sqrt{3} + 3 - (\sqrt{3} - 3) = 6 \quad \Rightarrow \quad (a - b)^2 = 36$ and $3ab = -18$, so

$(\sqrt{3} + 3)^3 - (\sqrt{3} - 3)^3 = a^3 - b^3 = (a - b)((a - b)^2 + 3ab) = 6(36 - 18) = 108$

245. Determine (*x*, y) where: $72^x \cdot 48^y = 6^{xy}$

We notice that: $72 = 8 \cdot 9 = 2^3 3^2$, $\quad 48 = 3 \cdot 16 = 3 \cdot 2^4$, $\quad 6 = 2 \cdot 3$
Inserting this we find:

$(2^3 \cdot 3^2)^x (3 \cdot 2^4)^y = (2 \cdot 3)^{xy} \quad \Leftrightarrow \quad 2^{3x} \cdot 3^{2x} \cdot 3^y 2^{4y} = 2^{xy} \cdot 3^{xy} \quad \Leftrightarrow$

$2^{3x+4y} \cdot 3^{2x+y} = 2^{xy} \cdot 3^{xy} \quad \Rightarrow \quad 3x + 4y = xy \quad and \quad 2x + y = xy$

$3x + 4y = xy \quad - \quad 2x + y = xy \quad \Rightarrow \quad x + 3y = 0 \quad \Leftrightarrow \quad x = -3y$

$3x + 4y = xy \quad \Rightarrow \quad -9y + 4y = -3y^2 \quad y = 0 \quad \vee \quad y = \frac{5}{3} \quad \wedge \quad x = -5 \quad \Rightarrow \quad x + y = -\frac{10}{3}$

246. Simplify: $\sqrt[3]{77 + 20\sqrt{13}} + \sqrt[3]{77 - 20\sqrt{13}}$

We substitute: $a^3 = 77 + 20\sqrt{13} \quad and \quad b^3 = 77 - 20\sqrt{13}$
We thus have:

$a^3 + b^3 = 77 + 20\sqrt{13} + 77 - 20\sqrt{13}$

$a^3 + b^3 = 154$

$a^3 b^3 = (77 + 20\sqrt{13})(77 - 20\sqrt{13}) = 77^2 - (20\sqrt{13})^2 = 729$

We then have then two equations:

$a^3 + b^3 = 154$ and $a^3 b^3 = 729 \quad \Leftrightarrow \quad ab = 9$

They may be solved for *a* and *b*, but there solution may imply 3. roots, so we shall instead try to solve for *a* + *b*.

We use the relation: $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$ If we put $y=(a+b)$, we have the equation:

$y^3 = a^3 + b^3 + 3aby \quad \Leftrightarrow \quad y^3 - 27y - 154 = 0$

There is no easy way to solve a 3. degree equation, but we ca see, that 7 is a root, since:
$7^3 - 27 \cdot 7 - 154 = 0$ To find possible other roots, we make polynomial division, that gives:
$y^3 - 27y - 154 = (y^2 + 7y + 22)(y - 7)$
$y^2 + 7y - 22 = 0$; $d = 49 - 88 = -39 < 0$ No roots. So: $a+b = 7$

$\sqrt[3]{77+20\sqrt{13}} + \sqrt[3]{77-20\sqrt{13}} = 7$

247. Solve for rational (*x, y*): $\sqrt{x} + \sqrt{y} = \sqrt{2+\sqrt{3}}$

$\sqrt{x} + \sqrt{y} = \sqrt{2+\sqrt{3}} \quad \Leftrightarrow \quad (\sqrt{x} + \sqrt{y})^2 = (\sqrt{2+\sqrt{3}})^2 \quad \Leftrightarrow \quad x + y + 2\sqrt{xy} = 2 + \sqrt{3}$
Since x and y are considered rational numbers, we must have:

$x + y = 2 \quad and \quad +2\sqrt{xy} = \sqrt{3} \quad \Leftrightarrow \quad x + y = 2 \quad and \quad xy = \frac{3}{4}$

$x + y = 2 \quad and \quad xy = \frac{3}{4} \quad \Leftrightarrow \quad x + y = 2 \quad and \quad y = \frac{3}{4x} \quad \Rightarrow$

$x + \frac{3}{4x} = 2 \quad \Leftrightarrow \quad 4x^2 - 8x + 3 = 0; \quad d = 64 - 48 = 16$

$x = \frac{8 \pm 4}{8} \quad x = \frac{3}{2} \quad \vee \quad x = \frac{1}{2}$

Since the equation is symmetric in (x, y), we have the solution:

$(x, y) = (\frac{3}{2}, \frac{1}{2}) \quad or \quad (x, y) = (\frac{1}{2}, \frac{3}{2})$

249. $x = 87647 \cdot 87649 \cdot 87654 \ and \ y = 87653 \cdot 87651 \cdot 87646$**. Determine *x*- y.**

We put: $a = 87650$ and then we can write:
$x = (a-3)(a-1)(a+4)$
$y = (a+3)(a+1)(a-4)$
$x = (a-3)(a-1)(a+4) = (a^2 - 4a + 3)(a+4) = a^3 - 4a^2 + 3a + 4a^2 - 16a + 12 = a^3 - 13a + 12$
$y = (a+3)(a+1)(a-4) = (a^2 + 4a + 3)(a-4) = a^3 + 4a^2 + 3a - 4a^2 - 16a - 12 = a^3 - 13a - 12$

$x - y = 24$

250. Which is the biggest number. $\sqrt{5}^{\sqrt{5}} \quad or \quad \sqrt{7}^{\sqrt{3}}$

We shall decide whether the inequality $\sqrt{5}^{\sqrt{5}} < \sqrt{7}^{\sqrt{3}}$ is true or not.

We lift both number to the power $2\sqrt{3}$, and we get:

$$\sqrt{5}^{2\sqrt{5}\sqrt{3}} < \sqrt{7}^{2\sqrt{3}\sqrt{3}} \quad \Leftrightarrow \quad (\sqrt{5}^{2})^{\sqrt{15}} < (\sqrt{7}^{2})^{3} \quad \Leftrightarrow \quad 5^{\sqrt{15}} < 7^{3}$$

Now $7^3 = 343 \quad and \quad 5^4 = 625 \quad and \quad \sqrt{15} = 3.87$, So $\sqrt{5}^{\sqrt{5}} > \sqrt{7}^{\sqrt{3}}$

251. Solve for x: $\dfrac{3^{x+1}-5^{x+2}}{3^x-5^x}=1$

$$\frac{3^{x+1}-5^{x+2}}{3^x-5^x}=1 \quad \Leftrightarrow \quad 3\cdot 3^x - 5^2\cdot 5^x = 3^x - 5^x$$

$$(3-1)\cdot 3^x - (5^2-1)\cdot 5^x = 0 \quad \Leftrightarrow \quad 2\cdot 3^x - 24\cdot 5^x = 0$$

We divide the equation by 3^x.

$$2 - 24\cdot\frac{5^x}{3^x} = 0 \quad \Leftrightarrow \quad (\frac{5}{3})^x = \frac{1}{12} \quad \Leftrightarrow \quad x = -\frac{\ln 12}{\ln\frac{5}{3}}$$

252. Solve for (x,y): $\sqrt{x}+\sqrt{y}=\sqrt{2+\sqrt{3}}$

$$\sqrt{x}+\sqrt{y}=\sqrt{2+\sqrt{3}} \quad \Leftrightarrow \quad x+y+2\sqrt{xy} == 2+\sqrt{3}$$

Since *x* and *y* are considered rational numbers, we must have:

$$x+y=2 \quad and \quad 2\sqrt{xy}=\sqrt{3} \quad \Leftrightarrow \quad x+y=2 \quad and \quad xy=\frac{3}{4}$$

$$x+y=2 \quad and \quad y=\frac{3}{4x} \quad \Rightarrow \quad x+\frac{3}{4x}=2$$

$$4x^2-8x+3=0; \quad d=64-48=16$$

$$x=\frac{8\pm 4}{8} \quad \Leftrightarrow \quad x=\frac{3}{2} \quad or \quad x=\frac{1}{2}$$

Since the equation is symmetric in *x* and *y* we have:

$$(x,y)=(\frac{3}{2},\frac{1}{2}) \quad or \quad (x,y)=(\frac{1}{2},\frac{3}{2})$$

253. Solve for *x*. (Easy one): $\sqrt{\dfrac{5+x}{x-1}}-\sqrt{\dfrac{x-1}{5+x}}=\dfrac{3}{2}$

$$\sqrt{\frac{5+x}{x-1}}-\sqrt{\frac{x-1}{5+x}}=\frac{3}{2}$$

We put: $y=\dfrac{5+x}{x-1}$ And then we have:

$$\sqrt{y}-\frac{1}{\sqrt{y}}-\frac{3}{2} \quad \Leftrightarrow \quad \sqrt{y}^2-\frac{3}{2}\sqrt{y}-1=0 \quad \Leftrightarrow \quad 2\sqrt{y}^2-3\sqrt{y}-2=0; \quad d=9+16=25$$

$$\sqrt{y}=\frac{3\pm 5}{4} \quad \sqrt{y}=2 \quad \vee \quad \sqrt{y}=-\frac{1}{2} \quad \Leftrightarrow \quad y=2$$

$$y=\frac{5+x}{x-1}=2 \quad \Leftrightarrow \quad 5+x=2(x-1) \quad \Leftrightarrow \quad x=7$$

254. Solve for *x*. $x^x=2^{3x+192}$

Since it is a transcendent equation, there are no traditional analytic methods, which can be applied. Instead we shall try to write 2^{3x+192} as y^y. 192 is hardly chosen at random, so let us investigate this number. $192=3\cdot 64=3\cdot 2^6$. So we may write:

$$2^{3x+192}=2^{3x+3\cdot 2^6}=(2^3)^{x+2^6}$$

Now if we tentatively put $x=2^6$, we find: $2^{3x+192}=(2^3)^{2^6+2^6}=(2^3)^{2\cdot 2^6}=(2^{3\cdot 2})^{2^6}=(2^{6\cdot})^{2^6}$

So the solution is $x=2^6$.

255. Solve for *x*: $\dfrac{27^x-8^x}{18^x-12^x}=\dfrac{19}{6}$

We notice that: $27=3^3$; $8=2^3$ $18=2\cdot 3^3$ $12=3\cdot 2^2$

So we can write: $\dfrac{27^x-8^x}{18^x-12^x}=\dfrac{19}{6} \quad \Leftrightarrow \quad \dfrac{(3^x)^3-(2^x)^3}{2^x(3^x)^2-3^x(2^x)^2}=\dfrac{19}{6}$

For simplicity, we put: $a=3^x \quad and \quad b=2^x$, and the equation then becomes:

$$\frac{a^3-b^3}{ba^2-a^2b}=\frac{19}{6} \quad \Leftrightarrow \quad \frac{a^3-b^3}{ab(a-b)}=\frac{19}{6}$$

Now we recall the identity: $(a+b)^3=a^3+b^3+3ab(a+b)$

And therefore: $(a-b)^3=a^3-b^3-3ab(a-b) \quad \Leftrightarrow \quad a^3-b^3=(a-b)^3+3ab(a-b)$

$$a^3-b^3=(a-b)^3+3ab(a-b)=(a-b)((a-b)^2+3ab)=(a-b)(a^2+b^2+ab)$$

$$\frac{a^3-b^3}{ab(a-b)}=\frac{19}{6} \quad \Leftrightarrow \quad \frac{a^3-b^3}{ab(a-b)}=\frac{19}{6} \quad \Leftrightarrow \quad \frac{(a-b)(a^2+b^2+ab)}{ab(a-b)}=\frac{19}{6} \quad \Leftrightarrow$$

$$\frac{(a^2+b^2+ab)}{ab}=\frac{19}{6} \quad \Leftrightarrow \quad 6(a^2+b^2+ab)=19ab$$

We then divide the equation with: *ab*, at put $y=\dfrac{a}{b}$

$6(\frac{a}{b}+\frac{b}{a}+1)=19 \quad \Leftrightarrow \quad 6y+\frac{6}{y}+6=19 \quad \Leftrightarrow \quad 6y^2-13y+6=0; \quad d=13^2-4\cdot 36=25$

$y=\frac{13\pm 5}{12} \quad y=\frac{3}{2} \quad \vee \quad y=\frac{2}{3}$

$(\frac{3}{2})^x=\frac{3}{2} \quad \vee \quad (\frac{3}{2})^x=\frac{2}{3}=(\frac{3}{2})^{-1} \quad \Leftrightarrow \quad x=1 \quad \vee \quad x=-1$

256. Solve for (*x,y*) : $x+y+xy=2+3\sqrt{2} \quad and \quad x^2+y^2=6$

$(x+y)^2=x^2+y^2+2xy=6+2xy \quad and \quad x+y=2+3\sqrt{2}-xy$

We put: $u=x+y \quad and \quad v=xy$, and then we have the two equations:

$(x+y)^2=6+2xy \quad and \quad x+y=2+3\sqrt{2}-xy \quad \Leftrightarrow$

$u^2=6+2v \quad and \quad u=2+3\sqrt{2}-v \quad \Rightarrow$

$u^2=6+2(2+3\sqrt{2}-u)=0 \quad \Leftrightarrow \quad u^2=10+6\sqrt{2}-u=0 \quad \Leftrightarrow$

$u^2+2u-10-6\sqrt{2}=0; \quad d=4+4(10+6\sqrt{2})=44+24\sqrt{2}=4(11+6\sqrt{2})$

We shall then try to write $(11+6\sqrt{2})=(a+b\sqrt{2})^2$

$(a+b\sqrt{2})^2=a^2+b^2+2ab\sqrt{2} \quad \Rightarrow \quad ab=6$; if we put $a=3 \quad and \quad b=1$, we can see that:

$(11+6\sqrt{2})=(3+\sqrt{2})^2$

$u^2+2u-10-6\sqrt{2}=0 \quad \Leftrightarrow \quad u=\frac{-2\pm 2(3+\sqrt{2})}{2} \quad \Leftrightarrow \quad u=2+\sqrt{2} \vee \quad u=-4-\sqrt{2}$

$v=2+3\sqrt{2} \quad \Rightarrow v=2\sqrt{2} \quad \vee \quad v=-6-4\sqrt{2}$

$u=x+y=2+\sqrt{2}$ has obviously the solutions: $x=2 \ and \ y=\sqrt{2}$

But: $u=x+y \quad \Rightarrow \quad x+y=-4-\sqrt{2} \quad and \quad v=xy=6+4\sqrt{2}$ are not a solution. Since

$x^2+y^2=(x+y)^2-2xy$ is not equal to 6

257. Calculate the sum: $\frac{1}{7\cdot 8}+\frac{1}{8\cdot 9}+\frac{1}{9\cdot 10}+...+\frac{1}{87\cdot 88}$

We denote the first number in the product for *a*. And we then calculate the sum of the term and the next term:

$$\frac{1}{a(a+1)}+\frac{1}{(a+1)(a+2)}=\frac{a+2+a}{a(a+1)(a+2)}=\frac{2(a+1)}{a(a+1)(a+2)}=\frac{2}{a(a+2)}$$

Adding a third term gives:

$$\frac{2}{a(a+2)}+\frac{1}{(a+2)(a+3)}=\frac{2(a+3)+a}{a(a+2)(a+3)}=\frac{3a+6}{a(a+2)(a+3)}\frac{3(a+2)}{a(a+2)(a+3)}=\frac{3}{a(a+3)}$$

From which it is clear that the sum of *n* terms, beginning from *a* is: $S_n=\frac{n}{a(a+n)}$

and consequently:

$$S_{80} = \frac{80}{7(7+80)} = \frac{80}{609}$$

258. Solve for *x*: $(0.01)^X = 11$ **(Very easy)**

$$(0.01)^X = 11 \quad \Leftrightarrow \quad x = \frac{\ln 11}{\ln 0.01} \quad \Leftrightarrow \quad x = -\frac{\ln 11}{\ln 100}$$

259. Solve for *x*: $x^{\frac{1}{\log x}} \log x = 1$

$$x^{\frac{1}{\log x}} \log x = 1$$

We put $y = \dfrac{1}{\log x} \quad \Rightarrow \quad \log x = \dfrac{1}{y} \quad x = 10^{\frac{1}{y}}$ and get:

$$(10^{\frac{1}{y}})^y \frac{1}{y} = 1 \quad \Leftrightarrow \quad 10\frac{1}{y} = 1 \quad y = 10 \quad \log x = \frac{1}{10}$$

$$x = 10^{\frac{1}{10}}$$

260. $a^3 + b^3 = 10 \quad and \quad a^2 + b^2 = 7 \quad a + b = ?$

$$a^3 + b^3 = 10 \quad and \quad a^2 + b^2 = 7$$

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b) = 10 + 3ab(a+b)$$

$$(a+b)^2 = a^2 + b^2 + 2ab = 7 + 2ab$$

Now we put: $x = a + b \quad and \quad y = ab$, and we have the two equations:

$$x^3 = 10 + 3xy \quad and \quad x^2 = 7 + 2y$$

Multiplying the second equation by *x*, and subtracting it from the first equation gives:

$$x^3 = 10 + 3xy \quad x^3 = 7x + 2xy \quad \Rightarrow \quad 0 = 10 - 7x + xy \quad \Rightarrow \quad y = 7 - \frac{10}{x}$$

Inserted in:

$x^2 = 7 + 2y$ gives: $x^2 = 7 + 2(7 - \dfrac{10}{x}) \quad \Leftrightarrow \quad x^3 - 21x + 20 = 0$

Which we write as;

$$x^3 - x - 20x + 20 = 0$$

$$x(x^2 - 1) - 20(x-1) = 0 \quad x - 1 = 0 \quad \vee \quad x(x+1) - 20 = 0$$

$$x(x+1) - 20 = 0 \quad \Leftrightarrow \quad x^2 + x - 20 = 0; \quad d = 1 + 80 = 9^2$$

$$x = \frac{-1 \pm 3}{2} \quad \Leftrightarrow \quad x = -2 \quad \vee \quad x = 1$$

So we have: $a + b = 0 \quad \vee \quad a + b = -2 \quad a + b = 1$

261. Solve for *x*. $49^x - 49^{x-1} = 16464$

$49^x - 49^{x-1} = 16464$

First we notice that $16464 = 49 \cdot 336 = 49 \cdot 7 \cdot 48$

$$49^x - 49^{x-1} = 16464 \quad \Leftrightarrow \quad 49^x - \frac{49^x}{49} = 49 \cdot 7 \cdot 48 \quad \Leftrightarrow$$

$$49^x(1 - \frac{1}{49}) = 49 \cdot 7 \cdot 48 \quad \Leftrightarrow \quad 49^x \frac{48}{49} = 49 \cdot 7 \cdot 48 \quad \Leftrightarrow \quad 49^{x-2} = 7 \quad \Leftrightarrow$$

$$(7^2)^{x-2} = 7 \quad \Leftrightarrow \quad 7^{2\cdot(x-2)} = 7 \quad \Leftrightarrow \quad 2 \cdot (x-2) = 1 \quad \Leftrightarrow \quad x = \frac{5}{2}$$

262. Solve for *x*. $\frac{27^x + 343^x}{63^x + 147^x} = \frac{37}{21}$

We notice that: $27 = 3^3$, $343 = 7^3$, $63 = 7 \cdot 3^2$, $147 = 3 \cdot 7^2$, so we have:

$$\frac{27^x + 343^x}{63^x + 147^x} = \frac{37}{21} \quad \Leftrightarrow \quad \frac{(3^x)^3 + (7^x)^3}{7^x(3^x)^2 + 3^x(7^x)^2} = \frac{37}{21}$$

We put: $a = 3^x \quad and \quad b = 7^x$, and then the equation can be written: $\frac{a^3 + b^3}{a^2 b + ab^2} = \frac{37}{21}$

We shall use the formula: $(a+b)^3 = a^3 + b^3 + 3ab(a+b) \quad \Leftrightarrow \quad a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

$$\frac{a^3 + b^3}{a^2 b + ab^2} = \frac{37}{21} \quad \Leftrightarrow \quad \frac{(a+b)^3 - 3ab(a+b)}{ab(a+b)} = \frac{37}{21} \quad \Leftrightarrow \quad \frac{(a+b)^2 - 3ab}{ab} = \frac{37}{21} \quad \Leftrightarrow$$

$$\frac{a^2 + b^2 + 2ab - 3ab}{ab} = \frac{37}{21} \quad \Leftrightarrow \quad \frac{a^2 + b^2 - ab}{ab} = \frac{37}{21} \quad \Leftrightarrow \quad a^2 + b^2 - ab = \frac{37}{21} ab$$

We then divide the equation with *ab*.

$$\frac{a}{b} + \frac{b}{a}. - 1 = \frac{37}{21}$$

And we put $y = \frac{a}{b}$:

$$y + \frac{1}{y} - 1 = \frac{37}{21} \quad \Leftrightarrow \quad y^2 + 1 - y = \frac{37}{21} y \quad \Leftrightarrow$$

$$21y^2 - 58y + 21 = 0; \qquad d = 58^2 - 4 \cdot 21^2 = 1600 = 40^2$$

$$y = \frac{58 \pm 40}{42} \quad \Leftrightarrow \quad y = \frac{7}{3} \quad \vee \quad y = \frac{3}{7} \quad \Leftrightarrow \quad (\frac{3}{7})^x = \frac{7}{3} \quad \vee \quad (\frac{3}{7})^x = \frac{3}{7} \quad \Leftrightarrow$$

$$x = 1 \quad \vee \quad x = -1$$

263. Solve for *x*: $\frac{3^{x^2}}{9^x}=81$

$\frac{3^{x^2}}{9^x}=81 \quad \Leftrightarrow \quad \frac{3^{x^2}}{3^{2x}}=81 \quad \Leftrightarrow \quad 3^{x^2-2x}=3^4 \quad \Leftrightarrow \quad x^2-2x=4 \quad \Leftrightarrow \quad x^2-2x-4=0$

$d=4+16=20;$

$x=\frac{2\pm 2\sqrt{5}}{2} \quad \Leftrightarrow \quad x=1+\sqrt{5} \quad \vee \quad x=1-\sqrt{5}$

264. $(a+1)(b+1)(a+b)=2022, \quad a^3+b^3=1933.$ **Determine** $a+b$

$(a+1)(b+1)(a+b)=(a+b)(a+b+ab+1)$

$a^3+b^3=(a+b)^3-3ab(a+b)$

We put: $x=a+b \quad and \quad y=ab$, then we may write:

$x(x+y+1)=2022 \quad and \quad x^3-3xy=1933$

$x(x+y+1)=2022 \quad \Leftrightarrow \quad xy=2022-x^2-x$

Which we insert in: $x^3-3xy=1933$ to get:

$x^3-3(2022-x^2-x)=1933 \quad \Leftrightarrow$

$x^3+3x^2+3x-6066-1933=0 \quad \Leftrightarrow$

$x^3+3x^2+3x-7999=0 \quad \Leftrightarrow$

$x^3+1+3x^2+3x-8000=0 \quad \Leftrightarrow$

$x^3+1^3+3x^2+3x-8000=0 \quad \Leftrightarrow$

$(x+1)^3-3x^2\cdot 1-3x\cdot 1^2+3x^2+3x-8000=0 \quad \Leftrightarrow$

$(x+1)^3=8000 \quad \Leftrightarrow \quad x+1=20 \quad \Leftrightarrow \quad x=19$

$a+b=19$

265. Determine $a^4+b^4+c^4$ **From the three equations below**

$a+b+c=4$

$a^2+b^2+c^2=10$

$a^3+b^3+c^3=22$

$16=(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc=10+2ab+2ac+2bc \quad \Rightarrow$

$I: \quad ab+ac+bc=3$

$$64=(a+b+c)^3=(a+(b+c))^3=a^3+(b+c)^3+3a(b+c)(a+b+c) \quad \Leftrightarrow$$
$$64=a^3+b^3+c^3+3bc(b+c)+3a(b+c)(a+b+c) \quad \Leftrightarrow$$
$$64=a^3+b^3+c^3+3bc(b+c)+3ab(a+b)+3abc+3ac(a+c)+3abc \quad \Leftrightarrow$$
$$64=22+3bc(b+c)+3ab(a+b)+3ac(a+c)+6abc \quad \Leftrightarrow$$

$$II: \quad bc(b+c)+ab(a+b)+ac(a+c)+2abc=14 \quad \Leftrightarrow$$

$$II: \ a^2b+ab^2+a^2c+ac^2+b^2c+bc^2+2abc=14$$

$$(a+b+c)(a^2+b^2+c^2)=4\cdot 10=$$
$$a^3+b^3+c^3+ab^2+ac^2+ba^2+bc^2+a^2c+b^2c$$
$$22+ab^2+ac^2+ba^2+bc^2+a^2c+b^2c=40 \quad \Rightarrow$$
$$III: \ ab^2+ac^2+ba^2+bc^2+a^2c+b^2c=18$$

If we subtract this equation (*III*) from *II*, we find: $abc=-2$

$$4\cdot 22=(a+b+c)(a^3+b^3+c^3)=a^4+b^4+c^4+ab^3+ac^3+a^3b+c^3b+a^3c+b^3c$$

$$a^4+b^4+c^4+ab^3+ac^3+a^3b+c^3b+a^3c+b^3c=88$$
$$IV: \quad a^4+b^4+c^4+a^3(b+c)+b^3(a+c)+c^3(a+b)=88$$

$$3\cdot 10=(ab+ac+bc)(a^2+b^2+c^2)=a^3b+ab^3+abc^2+a^3c+ab^2c+ac^3+a^2bc+bc^3+b^3c=$$
$$a^3(b+c)+b^3(a+c)+c^3(a+b)+abc(a+b+c)$$

$$a^3(b+c)+b^3(a+c)+c^3(a+b)-2\cdot 4=30$$
$$V: \ a^3(b+c)+b^3(a+c)+c^3(a+b)=38$$

We then subtract V from *IV*: to get:

$$a^4+b^4+c^4=88-38=50$$

266. $a^4+\frac{1}{a^4}=47$**. Determine:** $a+\frac{1}{a}$

$$(a+\frac{1}{a})^2=a^2+\frac{1}{a^2}+2$$

$$(a^2+\frac{1}{a^2})^2=a^4+\frac{1}{a^4}+2=47+2=49$$

$$a^2+\frac{1}{a^2}=7$$

$$(a+\frac{1}{a})^2=a^2+\frac{1}{a^2}+2=9$$

$$a+\frac{1}{a}=3$$

267. Which number is biggest: 31^{11} *or* 17^{14}

$31^{11}<32^{11}$ *and* $32^{11}=(2^5)^{11}=2^{55}$

$27^{14}>16^{14}$ *and* $(2^4)^{14}=2^{56}$ So we have:

$31^{11}<32^{11} \Leftrightarrow 31^{11}<2^{55}<2^{56}<17^{14}$ So:

$31^{11}<17^{14}$

268. $a^4+\frac{1}{a^4}=47$ **Determine:** $a+\frac{1}{a}$

$$(a+\frac{1}{a})^2=a^2+\frac{1}{a^2}+2 \quad\Rightarrow\quad (a+\frac{1}{a})^2-2=a^2+\frac{1}{a^2}$$

$$(a^2+\frac{1}{a^2})^2=a^4+\frac{1}{a^4}+2=47+2=49 \quad\Rightarrow\quad (a^2+\frac{1}{a^2})=7 \quad\Rightarrow$$

$$(a+\frac{1}{a})^2=2+7=9 \quad\Rightarrow\quad (a+\frac{1}{a})=3$$

269. $2^{\frac{1}{x}}=2^{\frac{1}{y}}=100$**. Determine *x+y***

$$2^{\frac{1}{x}}=100 \quad\Leftrightarrow\quad (2^{\frac{1}{x}})^x \quad 100^x \quad\Leftrightarrow\quad 2=100^x$$

$$5^{\frac{1}{y}}=100 \quad\Leftrightarrow\quad (5^{\frac{1}{y}})^y \quad 100^y \quad\Leftrightarrow\quad 5=100^y$$

$$2\cdot 5=10^1=100^x 100^y=100^{x+y}=(10^2)^{x+y}=10^{2(x+y)}$$

$$10^1=10^{2(x+y)} \quad\Rightarrow\quad 2(x+y)=1 \quad\Leftrightarrow\quad x+y=\frac{1}{2}$$

270. Solve for (*a,b*) $a^3+b^3=2\sqrt{5} \quad and \quad a^2b+ab^2=\sqrt{5}$

$$a^3+b^3=2\sqrt{5} \quad and \quad a^2b+ab^2=\sqrt{5} \quad \Leftrightarrow \quad a^3+b^3=2\sqrt{5} \quad and \quad ab(a+b)=\sqrt{5}$$

$$(a+b)^3=a^3+b^3+3ab(a+b)=2\sqrt{5}+3\sqrt{5}=5\sqrt{5}=\sqrt{5}^3 \quad \Rightarrow$$

$$a+b=\sqrt{5}.$$

$$a+b=\sqrt{5}. \quad and \quad ab(a+b)=\sqrt{5} \quad \Rightarrow \quad ab=1$$

$$a+b=\sqrt{5} \quad and \quad b=\frac{1}{a} \quad \Rightarrow \quad a+\frac{1}{a}=\sqrt{5} \quad \Leftrightarrow \quad a^2-\sqrt{5}a+1=0$$

$$d=5-4=1 \quad ; \quad a=\frac{\sqrt{5}\pm 1}{2} \quad \Leftrightarrow \quad a=\frac{\sqrt{5}+1}{2} \quad \vee \quad \frac{\sqrt{5}-1}{2} \quad \Rightarrow$$

$$b=\frac{\sqrt{5}-1}{2} \quad \vee \quad b=\frac{\sqrt{5}+1}{2}$$

271. Solve for *x*. $\log_2 x+\log_3 x=1$

$\log_2 x+\log_3 x=1$

By definition of the log function as the inverse function to the exponential function, we have:

$\log_a x=y \quad \Leftrightarrow \quad x=a^y$ since $\log_a x=\log_a a^y=y\log_a a=y$

This identity allows us to find a relation between $\log_a x$ *and* $\log_b x$

$$\log_b x=\log_b a^y=y\log_b a=\log_a x\log_b a \quad \Leftrightarrow$$

$$\log_a x=\frac{\log_b x}{\log_b a}$$

$$\log_3 x=\frac{\log_2 x}{\log_2 3}$$

$$\log_2 x+\log_3 x=\log_2 x+\frac{\log_2 x}{\log_2 3}=(1+\frac{1}{\log_2 3})\log_2 x$$

$$(1+\frac{1}{\log_2 3})\log_2 x=1 \quad \Leftrightarrow \quad (\log_2 2+\frac{1}{\log_2 3})\log_2 x=\log_2 2 \quad \Leftrightarrow$$

$$(\log_2 2\log_2 3+\log_2 2)\log_2 x=\log_2 2\log_2 3 \quad \Leftrightarrow$$

$$\log_2 x=\frac{\log_2 2\log_2 3}{\log_2 2\log_2 3+\log_2 2}=\frac{1}{1+\frac{1}{\log_2 3}}=\frac{\log_2 3}{\log_2 3+\log_2 2}=\frac{\log_2 3}{\log_2 6}$$

$$x=2^{\frac{\log_2 3}{\log_2 6}}$$

272. Simplify: $\frac{2^{\sqrt{27}}\cdot 8^{\sqrt{75}}}{4^{\sqrt{48}}}$

$$\frac{2^{\sqrt{27}}\cdot 8^{\sqrt{75}}}{4^{\sqrt{48}}}=\frac{2^{\sqrt{3^3}}\cdot 8^{\sqrt{3\cdot 5^2}}}{4^{\sqrt{3\cdot 2^4}}}=\frac{2^{3\sqrt{3}}\cdot 2^{3\cdot 5\sqrt{3}\cdot}}{2^{2\cdot 2^2\sqrt{3}\cdot}}=\frac{2^{3\sqrt{3}}\cdot 2^{15\sqrt{3}\cdot}}{2^{8\sqrt{3}\cdot}}=2^{3\sqrt{3}+15\sqrt{3}\cdot-8\sqrt{3}\cdot}=2^{10\sqrt{3}\cdot}$$

$$\left(\frac{2^{\sqrt{27}}\cdot 8^{\sqrt{75}}}{4^{\sqrt{48}}}\right)^{\frac{\sqrt{3}}{5}} = \left(2^{10\sqrt{3}\cdot}\right)^{\frac{\sqrt{3}}{5}} = 2^6$$

273. Solve: $x^{x^5} = 100$

There is no systematic way to solve this equation, so we have to resort to a qualified guess:
Our guess is: $x = \sqrt[5]{10}$, since $x^5 = (\sqrt[5]{10})^5 = 10 \quad and \quad x^{10} = (\sqrt[5]{10})^{10} = 10^2 = 100$

274. Solve integer solutions for (*x,y*): $\frac{x^3+y^3}{x+y} = 7 \quad and \quad \frac{x^3-y^3}{x-y} = 19$

$$(x+y)^3 = x^3+y^3+3xy(x+y) \quad \Rightarrow \quad x^3+y^3 = (x+y)^3 - 3xy(x+y)$$
$$(x-y)^3 = x^3-y^3-3xy(x-y) \quad \Rightarrow \quad x^3-y^3 = (x-y)^3 + 3xy(x-y)$$
$$\frac{x^3+y^3}{x+y} = 7 \quad and \quad \frac{x^3-y^3}{x-y} = 19 \quad \Leftrightarrow \quad \frac{(x+y)^3-3xy(x+y)}{x+y} = 7 \quad and \quad \frac{(x-y)^3+3xy(x-y)}{x-y} = 19 \quad \Leftrightarrow$$
$$(x+y)^2 - 3xy = 7 \quad and \quad (x-y)^2 + 3xy = 19 \quad \Leftrightarrow \quad (x+y)^2 + (x-y)^2 = 26$$

We can see that the integer solutions are $x+y=5 \quad and \quad x-y=1 \quad \Rightarrow \quad x=3 \quad and \quad y=2$

275. Simplify: $\sqrt{111222-333}$

$$\sqrt{111222-333} = \sqrt{111000+222-333} = \sqrt{3\cdot 37\cdot 1000 + 2\cdot 111 - 3\cdot 111} =$$
$$\sqrt{3\cdot 37(1000+2-3)} = \sqrt{3\cdot 37\cdot 999} = \sqrt{3\cdot 37\cdot 9\cdot 111} = \sqrt{3^4\cdot 37^2} = 3^2\cdot 37 = 333$$

276. $100^{x-1} = 99; \quad 100^{x+1} = ??$ **(Strange)**

$$100^{x-1} = 99; \quad \Leftrightarrow \quad 100^{x+1} = 9900 \quad \Leftrightarrow \quad (x+1)\log 100 = \log 99 + \log 100$$
$$2(x+1) = \log 99 + 2 \quad \Leftrightarrow \quad x = \tfrac{1}{2}\log 99 \quad \Leftrightarrow \quad x = \log\sqrt{99}$$

277. Solve for *x*: $3^x + 4^x + 6^x = 1$

$$3^x + 4^x - 6^x = 1 \quad \Leftrightarrow \quad 3^x + (2^x)^2 - 3^x 2^x = 1 \quad \Leftrightarrow \quad (2^x)^2 - 3^x 2^x + 3^x - 1 = 0$$

This is a quadratic equation in 2^x.

$$d = (3^x)^2 - 4(3^x - 1) = (3^x - 2)^2$$
$$2^x = \frac{3^x \pm 3^x - 2}{2} \quad \Leftrightarrow \quad 2^x = 3^x - 1 \quad or \quad 2^x = 1 \quad \Leftrightarrow \quad x = 1 \quad or \quad x = 0$$

278. Solve for *x*: $3^x \sqrt{3}^{x+4} + 20 = 0$

$$3^x - \sqrt{3}^{x+4} + 20 = 0 \quad \Leftrightarrow \quad 3^x - \sqrt{3}^4\sqrt{3}^x + 20 = 0 \quad \Leftrightarrow \quad 3^x - 9\sqrt{3}^x + 20 = 0 \quad \Leftrightarrow$$
$$(\sqrt{3}^x)^2 - 9\sqrt{3}^x + 20 = 0$$

This is a quadratic equation in $\sqrt{3}^x$.

$d = 81 - 4 \cdot 20 = 1$

$$(\sqrt{3})^x = \frac{9 \pm 1}{2} \quad \Leftrightarrow \quad (\sqrt{3})^x = 5 \quad or \quad (\sqrt{3})^x = 4 \quad \Leftrightarrow \quad x = \frac{2\ln 5}{\ln 3} \quad or \quad x = \frac{2\ln 4}{\ln 3}$$

279. Solve for *x*: $3^x + x^3 = 17$

Since there is no analytic way to solve a transcendent equation, we must resort to guessing. It is actually quite easy, since $x = 2$ is a solution: $3^2 + 2^3 = 17$

280. Determine (*a, b)* from: $\frac{1}{a} + \frac{1}{b} = 2 \quad and \quad 3^a = 5^b$

We put: $x = \frac{1}{a} \quad and \quad y = \frac{1}{b}$

We then have: $\frac{1}{a} + \frac{1}{b} = 2 \quad and \quad 3^a = 5^b \quad \Leftrightarrow \quad x + y = 2 \quad and \quad 3^{\frac{1}{x}} = 5^{\frac{1}{y}}$

$$3^{\frac{1}{x}} = 5^{\frac{1}{y}} \quad \Leftrightarrow \quad (3^{\frac{1}{x}})^x = (5^{\frac{1}{y}})^x \quad \Leftrightarrow \quad 3 = 5^{\frac{x}{y}} \quad \Rightarrow \quad \frac{x}{y} = \frac{\ln 3}{\ln 5}$$

We then solve: $x + y = 2 \quad and \quad \frac{x}{y} = \frac{\ln 3}{\ln 5} \quad \Leftrightarrow \quad x + y = 2 \quad and \quad x = \frac{\ln 3}{\ln 5} y$

$$\frac{\ln 3}{\ln 5} y + y = 2 \quad \Leftrightarrow \quad y\left(\frac{\ln 3}{\ln 5} + 1\right) = 2 \quad \Leftrightarrow \quad y = \frac{2\ln 5}{\ln 3 + \ln 5} = \frac{\ln 25}{\ln 15}$$

$$x = 2 - y = 2 - \frac{\ln 25}{\ln 15} = \frac{2\ln 15 - \ln 25}{\ln 15} = \frac{\ln 9}{\ln 15}$$

$$a = \frac{1}{x} = \frac{\ln 15}{\ln 9} \quad and \quad b = \frac{1}{y} = \frac{\ln 15}{\ln 25}$$

281. Calculate: $1 + (1+i) + (1+i)^2 + (1+i)^3$

$(1+i)^2 = 1 - 1 + 2i = 2i$

$(1+i)^3 = 2i(1+i) = 2i - 2$

$1 + (1+i) + (1+i)^2 + (1+i)^3 = 1 + 1 + i + 2i + 2i - 2 = 5i$

282. $x = a + \frac{1}{a} \quad and \quad y = a - \frac{1}{a}$. **Determine** $\sqrt{x^4 + y^4 - 2x^2y^2}$

$$\sqrt{x^4 + y^4 - 2x^2y^2} = \sqrt{(x^2 - y^2)^2} = (x^2 - y^2)$$

$$(x^2 - y^2) = \left(a + \frac{1}{a}\right)^2 - \left(a - \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2 - (a^2 + \frac{1}{a^2} - 2) = 4$$

$$\sqrt{x^4 + y^4 - 2x^2y^2} = 4$$

283. Solve for *x*: $x^{x^6} = \sqrt{2}^{\sqrt{2}}$

There is no analytic way to solve this kind of equation, so we make some rewriting:

$$x^{x^6} = \sqrt{2}^{\sqrt{2}} \quad \Leftrightarrow \quad \left(x^{x^6}\right)^{\sqrt{2}} = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \quad \Leftrightarrow \quad \left(x^{\sqrt{2}x^6}\right) = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} \quad \Leftrightarrow$$

$$x^{\sqrt{2}x^6} = \sqrt{2}^2 \quad \Leftrightarrow \quad x^{\sqrt{2}x^6} = 2$$

Here we have to guess, and we try with: $x = \sqrt[4]{2}$ so $\sqrt{2}x^6 = \sqrt{2}(\sqrt[4]{2})^6 = 2^{\frac{1}{2}}2^{\frac{3}{2}} = 2^2 = 4$

$\sqrt[4]{2}^{\sqrt{2}x^6} = \sqrt[4]{2}^4 = 2$ So that: $x^{x^6} = \sqrt{2}^{\sqrt{2}} \quad \Leftrightarrow \quad x = \sqrt[4]{2}$

284. Evaluate: $x - 333^3 = 444^3 + 555^3$

$333^3 + 444^3 + 555^3 = 3^3 111^3 + 4^3 111^3 + 5^3 111^3$

$111 = 3 \cdot 37$

$3^3 3^3 \cdot 37^3 + 4^3 3^3 37^3 + 5^3 3^3 37^3 = 3^3 37^3 (3^3 + 4^3 + 3^3) = 3^3 37^3 \cdot 216 = 3^3 \cdot 37^3 \cdot 3^3 \cdot 2^3 = (18 \cdot 37) = 666^3$

284. Solve for *x*. $\log(20 - x) = \log^3 x$ **(very easy)**

$\log(20 - x) = \log^3 x$. O get the same argument, we solve: $20 - x = x \quad x = 10$

$\log(10) = \log^3 10 \quad \Leftrightarrow \quad 1 = 1$

285. $\frac{1}{x} - \frac{1}{y} = \frac{1}{x+y}$; $\frac{x}{y} = ?$

We multiply the equation by *x*.

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{x+y} \quad \Leftrightarrow \quad 1 - \frac{x}{y} = \frac{x}{x+y} \quad \Leftrightarrow \quad 1 - \frac{x}{y} = \frac{1}{1 + \frac{y}{x}} \quad \Leftrightarrow$$

$$(1 - \frac{x}{y})(1 + \frac{y}{x}) = 1 \quad \Leftrightarrow \quad 1 + \frac{y}{x} - \frac{x}{y} - 1 = 1 \quad \Leftrightarrow \quad \frac{1}{\frac{x}{y}} - \frac{x}{y} - 1 = 0$$

$$-z + \frac{1}{z} - 1 = 0 \quad \Leftrightarrow \quad -z^2 - z + 1 = 0 \quad \Leftrightarrow \quad z^2 + z - 1 = 0$$

We put $z = \frac{x}{y} : d = 1 + 4 = 5; \quad z = \frac{-1 \pm \sqrt{5}}{2} \quad \Leftrightarrow$

$$\frac{x}{y} = \frac{-1 + \sqrt{5}}{2} \quad or \quad \frac{x}{y} = \frac{-1 - \sqrt{5}}{2}$$

286. Solve for *x*: $x^{\sqrt{\log x}} = 10^8$

We put: $x = 10^n$. $x^{\sqrt{\log x}} = 10^8 \quad \Leftrightarrow \quad (10^n)^{\sqrt{n}}$

It is easy to see, that the equation is fulfilled for $n = 4$, since $(10^4)^{\sqrt{4}} = (10^4)^2 = 10^8$

287. Determine integer solutions to: $a^2 + a + 34 = b^2$

We make a systematic try: If a = 1, then $b^2 = 36$, so the solution is: $a = 1$ and b = 6.

288. $s = 1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + ... + ...$

$s = .1^2 + 2^2 + 3^2 + 4^2 + 5^2 ... + n^2 + 2^2(1 + 2^2 + 3^2 + n^2 ..)$

$s = 1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + ... = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 ... + n^2 + (2^2 + 4^2 + 6^2 + (2n)^2 ..)$

$s = 1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + ... = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 ... + n^2 + (2^2 + 4^2 + 6^2 + (2n)^2 ..) =$

$s = 1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + ... = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 ... + n^2 + (2^2 + 4^2 + 6^2 + (2n)^2 ..) =$

$s = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 ... + n^2 + 2^2(1 + 2^2 + 3^2 + (n-2)^2)$

The formula for the series: $s = .1^2 + 2^2 + 3^2 + 4^2 + 5^2 ... + n^2$ is: $s_2(n) = \frac{1}{6} n(n+1)(2n+1)$, so we have:

$s = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 ... + n^2 + 2^2(1 + 2^2 + 3^2 + (n-2)^2) =$

$\frac{1}{6} n(n+1)(2n+1) + \frac{1}{6}\cdot 2^2 (n-2)(n-1)(2n-3)$

289. Simplify: $(3111)^2 - (2889)^2$

$(3111)^2 - (2889)^2 = (3000 + 111)^2 + (3000 - 111)^2$

$3000^2 + (111)^2 + 2\cdot 3000\cdot 111 - (3000^2 + (111)^2 - 2\cdot 3000\cdot 111) =$

$4\cdot 3000\cdot 111 = 2^2\cdot 3\cdot 1000\cdot 3\cdot 37 = 2^2\cdot 3^2\cdot 10^3\cdot 37$

290. Solve for *x*: $\sqrt[3]{4}\cdot\sqrt[4]{x} - \sqrt[4]{3}\cdot\sqrt[3]{4} = 0$

$\sqrt[3]{4}\cdot\sqrt[4]{x} - \sqrt[4]{3}\cdot\sqrt[3]{x} = 0 \quad\Leftrightarrow\quad 4^{\frac{1}{3}}\cdot x^{\frac{1}{4}} = \left(3^{\frac{1}{4}}\cdot x^{\frac{1}{3}}\right) \quad\Leftrightarrow\quad \left(4^{\frac{1}{3}}\cdot x^{\frac{1}{4}}\right)^{12} = \left(3^{\frac{1}{4}}\cdot x^{\frac{1}{3}}\right)^{12} \quad\Leftrightarrow$

$4^4\cdot x^3 = 3^3\cdot x^4 \quad\Leftrightarrow\quad x = \frac{4^4}{3^3}$

291. Compute the sum: $s_n = \sum_{k=1}^{n}\frac{1}{k(k+1)}$

$$\frac{1}{k} - \frac{1}{k+1} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{1}{k(k+1)} =$$

$$s_n = \sum_{k=1}^{n}\frac{1}{k(k+1)} = \sum_{k=1}^{n}\frac{1}{k} - \frac{1}{k+1} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} ... - \frac{1}{n-1} + \frac{1}{n-1} - \frac{1}{n} = 1 - \frac{1}{n}$$

$$s_n = \sum_{k=1}^{n}\frac{1}{k(k+1)} = 1 - \frac{1}{n}$$

292. Simplify: $\dfrac{\sqrt{\sqrt{5}+2}+\sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}}-\sqrt{3-2\sqrt{2}}$

We put: $a=\sqrt{5}+2,\quad b=\sqrt{5}-2 \qquad a-1=\sqrt{5}+1$

$ab=(\sqrt{5}+2)(\sqrt{5}-2)=(\sqrt{5})^2-2^2=1\ ,\ so\quad b=\dfrac{1}{a}$

$\dfrac{\sqrt{\sqrt{5}+2}+\sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}}-\sqrt{3-2\sqrt{2}}=\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a-1}}-\sqrt{3-2\sqrt{2}}=$

$\dfrac{\sqrt{a}+\dfrac{1}{\sqrt{a}}}{\sqrt{a-1}}-\sqrt{3-2\sqrt{2}}=\dfrac{(\sqrt{a})^2+1}{\sqrt{a}\sqrt{a-1}}-\sqrt{3-2\sqrt{2}}=\dfrac{\sqrt{5}+3}{\sqrt{a}\sqrt{a-1}}-\sqrt{3-2\sqrt{2}}=$

$\dfrac{\sqrt{5}+3}{\sqrt{7+3\sqrt{5}}}-\sqrt{3-2\sqrt{2}}$

$a(a-1)=(\sqrt{5}+2)(\sqrt{5}+1)=7+3\sqrt{5}$

$\sqrt{3-2\sqrt{2}}:\ 3-2\sqrt{2}=(a-b\sqrt{2})^2=a^2+2b^2-2ab\sqrt{2}\quad\Rightarrow\quad ab=1\quad\Rightarrow\quad a=b=1$

$(1-\sqrt{2})^2=3-2\sqrt{2}.$

$\dfrac{\sqrt{\sqrt{5}+2}+\sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}}=\sqrt{3-2\sqrt{2}}=\dfrac{\sqrt{5}+3}{\sqrt{7+3\sqrt{5}}}-\sqrt{3-2\sqrt{2}}=\dfrac{\sqrt{5}+3}{\sqrt{7+3\sqrt{5}}}-\sqrt{(1-\sqrt{2})^2}=$

$\dfrac{\sqrt{5}+3}{\sqrt{7+3\sqrt{5}}}-(\sqrt{2}-1)=\dfrac{\sqrt{5}+3-\sqrt{2}(\sqrt{7+3\sqrt{5}}}{\sqrt{7+3\sqrt{5}}}+1=\dfrac{\sqrt{5}+3-\sqrt{14+6\sqrt{5}}}{\sqrt{7+3\sqrt{5}}}+1$

$14+6\sqrt{5}=(a+b\sqrt{5})^2=a^2+5b^2+2ab\sqrt{5}\quad\Rightarrow\quad a=3\ \ and\ \ b=1:$

$(3+\sqrt{5})^2=9+5+6\sqrt{5}=14+6\sqrt{5}$

$\dfrac{\sqrt{5}+3-\sqrt{14+6\sqrt{5}}}{\sqrt{7+3\sqrt{5}}}+1=\dfrac{\sqrt{5}+3-(\sqrt{5}+3)}{\sqrt{7+3\sqrt{5}}}+1=1$

293. Evaluate $\left(\dfrac{1+i}{3+i}\right)^2$

$\left(\dfrac{1+i}{3-i}\right)^2=\left(\dfrac{(1+i)(3+i)}{(3+i)(3-i)}\right)^2=\left(\dfrac{3+i+3i-1}{3^2-i^2}\right)^2=\left(\dfrac{2+4i}{10}\right)^2=\left(\dfrac{4+16i-16}{100}\right)=\dfrac{-12+16i}{100}$

$\mathrm{Re}\dfrac{-12+16i}{100}=-\dfrac{3}{25}$

294. $8^x+\dfrac{1}{8^x}=9$ **Determine** $2^{9x}+\dfrac{1}{2^{9x}}$

$2^{9x}+\dfrac{1}{2^{9x}}=(2^3)^{3x}+\dfrac{1}{(2^3)^{3x}}=8^{3x}+\dfrac{1}{8^{3x}}$

$$(8^{2x}+\frac{1}{8^{2x}})(8^x+\frac{1}{8^x})=79\cdot 9=8^{3x}+\frac{1}{8^{3x}}+8^x+\frac{1}{8^x}$$

$$8^{3x}+\frac{1}{8^{3x}}=79\cdot 9-(8^x+\frac{1}{8^x})=79\cdot 9-9=78\cdot 9$$

295. Determine *n*, such that: $(1+i)^n=(1-i)^n$ **, where *i* is the imaginary unit**

$$(1+i)^n=(1-i)^n \quad \Leftrightarrow \quad \left(\frac{(1+i)}{(1-i)}\right)^n=1$$

$\frac{(1+i)(1+i)}{(1-i)(1+i)}=\frac{1+i+i-1)}{(1^2-i^2)}=\frac{2i}{2}=i$, so: $\left(\frac{(1+i)}{(1-i)}\right)^n=1$ for *i* = 4, since: $i^4=i^2\cdot i^2=(-1)(-1)=1$

296. Solve for (*x,y*) $3^{4x+5y}-200=43 \quad and \quad 7^{8x-4y}-300=43$

$3^{4x+5y}-200=43 \quad and \quad 7^{8x-4y}-300=43 \quad \Leftrightarrow \quad 3^{4x+5y}=243 \quad and \quad 7^{8x-4y}=343$

Now; $243=3^5$ *and* $343=7^3$, so we have:

$3^{4x+5y}=3^5 \quad and \quad 7^{8x-4y}=7^3 \quad \Leftrightarrow \quad 4x+5y=5 \quad and \quad 8x-4y=3$

We multiply the first equation by (-4), and add the two equations to get:

$$-8x-10y=-10 \quad and \quad 8x-4y=3 \quad \Rightarrow$$

$$-14y=-7 \quad \Leftrightarrow \quad y=\tfrac{1}{2} \quad \Rightarrow \quad 4x+\frac{5}{2}=5 \quad \Leftrightarrow \quad x=\frac{5}{8}$$

$$(x,y)=(\frac{5}{8},\frac{1}{2})$$

297. Determine *a*, and *b*, such that: $2^a-2^b=2016$

Well $2^{10}=1024, so\ 2^{11}=2048, \quad 2048-2016=32=2^5 \quad so \quad a=11 \quad and \quad b=5$

298. Determine (*x* +*y*) from: $x^2+y^2=7 \quad and \quad x^3+y^3=10$

$$(x+y)^3=x^3+y^3+2xy(x+y) \quad \Leftrightarrow (x+y)((x+y)^2-2xy)-10=0 \quad \Leftrightarrow$$

$$(x+y)(x^2+y^2+2xy-2xy)-10=0 \quad \Leftrightarrow \quad (x+y)\cdot 7-10=0 \quad \Leftrightarrow (x+y)=\frac{10}{7}$$

$$(x+y)^2=\frac{100}{49}= \quad x^2+y^2+2xy=7+2xy \quad \Rightarrow \quad 2xy=\frac{100}{49}-\frac{343}{49}=-\frac{243}{49} \quad \Leftrightarrow \quad xy=-\frac{243}{98}$$

$$x+y=a \quad xy=b \quad \Rightarrow \quad x+y=a \quad and \quad y=\frac{b}{x} \quad \Rightarrow \quad x+\frac{b}{x}=a \quad \Leftrightarrow \quad x^2-ax+b=0$$

$$d=a^2-4b=\frac{100}{49}+\frac{486}{49}=\frac{586}{49} \ ; \quad x=\frac{a\pm\sqrt{d}}{2} \quad y=\frac{10}{7}-x$$

299. Solve for *x*: $x^4 - 4x - 1 = 0$

$x^4 - 4x - 1 = 0 \quad \Leftrightarrow$

$(x^2+1)^2 - 2x^2 - 1 - 4x - 1 = 0 \quad \Leftrightarrow$

$(x^2+1)^2 - 2(x^2 + 2x + 1) = 0 \quad \Leftrightarrow$

$(x^2+1)^2 - 2(x+1)^2 = 0$

$(x^2+1)^2 = 2(x+1)^2 \quad \Leftrightarrow$

$\sqrt{(x^2+1)^2} = \sqrt{2}\sqrt{(x+1)^2} \quad \Leftrightarrow$

$(x^2+1) = \sqrt{2}(x+1) \quad \Leftrightarrow$

$x^2 - \sqrt{2}x + 1 - \sqrt{2} = 0; \quad d = 2 - 4(1-\sqrt{2}) = 4\sqrt{2} - 2$

$x^2 = \dfrac{\sqrt{2} \pm \sqrt{4\sqrt{2}-2}}{2} \quad \Leftrightarrow \quad x^2 = \dfrac{\sqrt{2} + \sqrt{4\sqrt{2}-2}}{2} \quad \Leftrightarrow$

$x = \pm\sqrt{\dfrac{\sqrt{2} + \sqrt{4\sqrt{2}-2}}{2}}$

300. Solve for x: $\sqrt{x+6} - \sqrt{11-x} = 3$

$\sqrt{x+6} - \sqrt{11-x} = 3$.

Well, a traditional approach will result in a 4th degree equation, so we shall resort to a qualified guess. $x = 10$ seems to be a possible guess, and; $\sqrt{10+6} - \sqrt{11-10} = 4 - 1 = 3$