Contents

154. solve for *x*: $2^{\sin^2 x} + 2^{\cos^2 x} = 3$

$2^{\sin^2 x} + 2^{\cos^2 x} = 3$ we put: $\cos^2 x = 1 - \sin^2 x$ and we find:

$2^{\sin^2 x} + 2^{1-\sin^2 x} = 3$ we then substitute $y = \sin^2 x$ and we get:

$2^y + 2^{1-y} = 3 \quad \Leftrightarrow \quad (2^y)^2 + 2 = 3 \cdot 2^y$

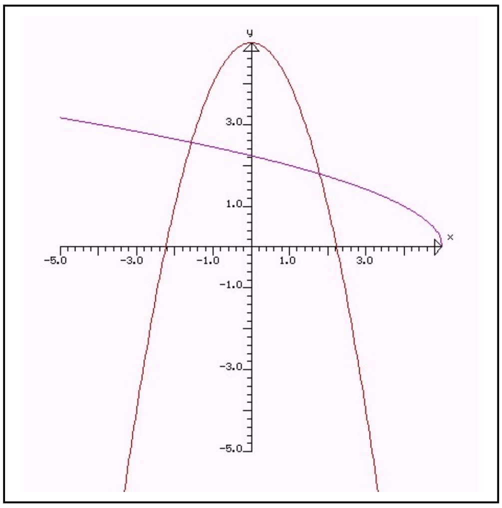
This is a quadratic equation in 2^y, and for convince we put $z = 2^y$. We then have:

$$z^2 + 2 - 3z = 0 \;\; ; \;\; d = 4 + 12 = 16 \quad z = \frac{-2 \pm 4}{2} \quad z = -3 \;\; or \;\; z = 1 \quad \Leftrightarrow \quad 2^y = 1 \quad \Leftrightarrow \quad y = 0$$

$$y = \sin^2 x = 0 \quad \Leftrightarrow \quad x = p\frac{\pi}{2}$$

153. Solve: $\sqrt{5 - x} = 5 - x^2$

Before we start, we plot the graphs of: $f(x) = \sqrt{5 - x} \quad and \quad g(x) = 5 - x^2$



We can see that there are two solutions, but none of them looks like rational numbers.

If we square both sides of $\sqrt{5 - x} = 5 - x^2$ we en up with a 4th degree polynomial. But there are no general methods to solve, besides an adapted Cardano formula. We get:

$$5 - x = 25 + x^4 - 10x^2 \quad \Leftrightarrow \quad x^4 - 10x^2 + x + 20 = 0$$

The idea is the to try to factorize this expression into two 2. degree polynomial – if possible.

$x^4 - 10x^2 + x + 20 = (x^2 + ax + b)(x^2 - ax + c)$

Since there are no term with x^3, we have put $ax \;\; and \;\; -ax$ since it will insure that that the terms with x^3 will cancel, By multiplying the two polynomials, we find:

$x^4 - 10x^2 + x + 20 = x^4 . - ax^3 + cx^2 + ax^3 - a^2x^2 + acx + bx^2 - bax + bc =$

$x^4 + cx^2 - a^2x^2 + acx + bx^2 - bax + bc$

So we identify the coefficients to the power of *x*:

$$c + b - a^2 = -10 \;\; ; \;\; ac - ba = 1 ; \;\; bc = 20$$

From these equations, we may get an expression for *b* and *c* expressed by *a*.

$$c + b = a^2 - 10 = ; \quad (c - b) = \frac{1}{a} ; \quad bc = 20$$

$$2c = a^2 - 10 + \frac{1}{a} \qquad 2b = a^2 - 10 - \frac{1}{a} \quad \Leftrightarrow \quad c = \tfrac{1}{2}(a^2 - 10 + \frac{1}{a}) \qquad b = \tfrac{1}{2}(a^2 - 10 - \frac{1}{a})$$

And we thus find an equation to determine a. $c = \tfrac{1}{2}(a^2 - 10 + \frac{1}{a}) \qquad b = \tfrac{1}{2}(a^2 - 10 - \frac{1}{a})$

$$bc = 20 \quad \Leftrightarrow \quad \tfrac{1}{4}(a^2 - 10 + \frac{1}{a})(a^2 - 10 - \frac{1}{a}) = \tfrac{1}{4}((a^2 - 10)^2 - \frac{1}{a^2}) = 20 \quad \Leftrightarrow$$

$$(a^2 - 10)^2 - \frac{1}{a^2} = 80 \quad \Leftrightarrow \quad (u - 10)^2 - \frac{1}{u} = 80 \quad \Leftrightarrow \quad u^2 + 100 - 20u - \frac{1}{u} = 80 \quad \Leftrightarrow$$

$$u^3 + 20u - 20u^2 - 1 = 0 \quad \Leftrightarrow \quad \Leftrightarrow \quad u^3 - 20u^2 + 20u - 1 = 0$$

If we put $u = a^2$ we find a third order equation in u. $a = 1$ or $a = -1$.
We can immediately that $u = 1$ is a root. Polynomial division with $u - 1$ gives:

$$u^3 - 20u^2 + 20u - 1 = (x - 1)(u^2 - 19u + 1)$$

$$u^2 - 19u + 1 = 0 \ ; \quad d = 361 - 4 = 357 \ ; \quad u = \frac{19 \pm \sqrt{357}}{2}$$

We shall first concentrate on the root $u = 1$, and we calculate b and c.

$$u = a^2 \quad \Rightarrow \quad a = 1 \quad or \quad a = -1.$$

$$a = 1: \quad c = \tfrac{1}{2}(a^2 - 10 + \frac{1}{a}) = -4 \qquad b = \tfrac{1}{2}(a^2 - 10 - \frac{1}{a}) = -5$$

$$a = -1: \quad c = \tfrac{1}{2}(a^2 - 10 + \frac{1}{a}) = -5 \qquad b = \tfrac{1}{2}(a^2 - 10 - \frac{1}{a}) = -4$$

These values are inserted in:

$$x^4 - 10x^2 + x + 20 = (x^2 + ax + b)(x^2 - ax + c) = (x^2 + x - 5)(x^2 - x - 4)$$

Using $a = -1$ gives the same product, but with the factors in inverse order.

$$x^2 + x - 5 = 0; \quad d = 1 + 20 \ ; \quad x = \frac{-1 \pm \sqrt{21}}{2} \quad x = 1.79 \ or \ x = -2.79$$

$$x^2 - x - 4 = 0 \quad d = 1 + 16 \ ; \quad x = \frac{1 \pm \sqrt{17}}{2} \quad x = 2.56 \ or \ x = -1.56$$

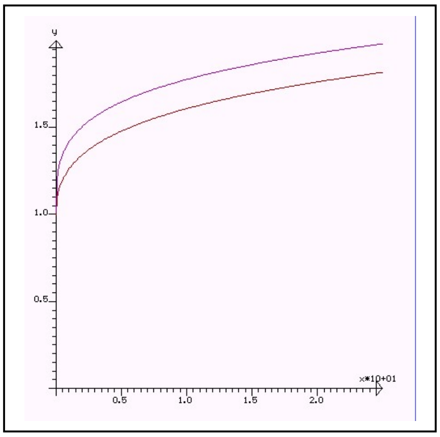
From the graph, we can see that the solutions are: $x = -1.56$ or $x = 1.79$

154. A simple exercise: $n! = n^3 - n$

For $n = 4$, we have; $n! = n^3 - n$ gives: 24 < 64 – 4
For $n = 6$, we have; $n! = n^3 - n$ gives: 720 > 216 – 6
For $n = 5$, we have; $n! = n^3 - n$ gives: 120= 125 -5.
The solution is therefore $n = 5$

155. No analytic solution: $\sqrt[3]{1+\sqrt{x}} = \sqrt{1+\sqrt[3]{x}}$

A direct approach would require to times lifting to the 6th power. Hardly the way to find the solution in a lifetime.
So we guess! Obvious $x = 0$. is a solution, but? To guess other solutions, we try to choose x, such that $1+\sqrt{x}$ is a cubic number.



$1+\sqrt{x} = 1, 8, 27, 64....$ it gives for x. 1, 4, 9, 16. and $1+\sqrt{x}$ = 2, 3, 4, 5, 6, but none of them are cubic numbers. Looking at the right side $\sqrt{1+\sqrt[3]{x}}$, then $x = 27$ gives $\sqrt{1+\sqrt[3]{x}} = 2$, but this does not comply with the left hand side.
So it seems that $x = 0$ is the only solution. This is also confined by looking at the graph to the left, plotting the left side and the right side in the same coordinate system. The only intersection point is (0,1)

156. Solve for *x*: $2^x - 3^x = \sqrt{6^x - 9^x}$

$2^x - 3^x = \sqrt{6^x - 9^x}$. We put $a = 2^x \ \ and \ \ b = 3^x$. Then the equation reads.

$$a-b = \sqrt{ab-b^2} \quad \Leftrightarrow \quad a-b = \sqrt{b}\sqrt{a-b} \quad \Leftrightarrow \quad (a-b)^2 = (\sqrt{b}\sqrt{a-b})^2 \quad \Leftrightarrow$$

$$(a-b)^2 = b(a-b) \quad \Leftrightarrow \quad a-b = b \quad \Leftrightarrow \quad a = 2b$$

$$2^x = 2\cdot 3^x \quad \Leftrightarrow \quad \left(\frac{2}{3}\right)^x = 2 \quad \Leftrightarrow \quad x = \frac{\ln 2}{\ln(\frac{2}{3})}$$

157. Solve: $\log x = \sqrt{\log x^{\frac{13}{6}} - 1}$

$\log x = \sqrt{\log x^{\frac{13}{6}} - 1}$. We square both sides and find:

$$\log^2 x = \log x^{\frac{13}{6}} - 1 \quad \Leftrightarrow \quad \log^2 x = \frac{13}{6}\log x - 1 \quad \Leftrightarrow \quad 6\log^2 x - 13\log x + 6 = 0$$

We put $y = \log x$ and find:

$$6y^2 - 13y + 6 = 0 \qquad d = 169 - 144 = 25 \qquad y = \frac{13 \pm 5}{2} \quad \Leftrightarrow \quad y = 9 \quad or \quad y = 4 \quad \Leftrightarrow$$

$$\log x = 9 \quad or \quad \log x = 4 \quad \Leftrightarrow \quad x = 10^9 \quad or \quad x = 10^4$$

157. Solve for *x*: $x^x = 2^{\frac{1}{x}}$

We take $\log_2$ on both sides: $x\log_2 x = \frac{1}{x} \Leftrightarrow x^2\log_2 x = 1 \Leftrightarrow \log_2 x^{x^2} = 1$

$$x^{x^2} = 2$$

It is easy to see that the solution is $x = \sqrt{2}$, since: $\sqrt{2}^{\sqrt{2}^2} = 2$

158. Determine *f* from the equation: $f\left(\frac{\sqrt{x^2+1}-x}{x}\right) = x^2$

We put:

$$y = \frac{\sqrt{x^2+1}-x}{x} \Leftrightarrow yx = \sqrt{x^2+1}-x \Leftrightarrow yx + x = \sqrt{x^2+1} \Leftrightarrow$$

$$(yx+x)^2 = (\sqrt{x^2+1})^2 \Leftrightarrow y^2x^2 + x^2 + 2yx^2 = x^2+1 \Leftrightarrow$$

$$x^2(y^2+2y) = 1 \Leftrightarrow x^2 = \frac{1}{(y^2+2y)}$$

$$f(y) = \frac{1}{(y^2+2y)} \quad f(x) = \frac{1}{(x^2+2x)}$$

159. Determine *a* and, such that: $2^a - 2^b = 2016$

Since $2^{10} = 1024$, we make a try with $a = 11$. $2^{11} = 2048$ and $2048 - 2016 = 32 = 2^5$

So: $a = 11$ *and* $b = 5$.

160. Simplify $\sqrt{247^2 - 153^2}$

$$\sqrt{247^2-153^2} = \sqrt{(247-153)(247+153)} = \sqrt{94\cdot 400} = 20\sqrt{94}$$

161. Solve: $2^{x^2} + 4^{x^2} = 8^{x^2}$

We put $y = 2^{x^2}$ and then we get:

$$y + y^2 = y^3 \Leftrightarrow y^3 - y^2 - y = 0 \Leftrightarrow y(y^2-y-1) = 0 \Leftrightarrow$$

$$y = 0 \quad or \quad y^2 - y - 1 = 0 \quad d = 1+4 = 5; \quad y = \frac{1\pm\sqrt{5}}{2} \Leftrightarrow x^2 = \frac{1+\sqrt{5}}{2} \Rightarrow$$

$$x = \pm\sqrt{\frac{1+\sqrt{5}}{2}}$$

162. Simplify $\frac{3334\cdot 6663\cdot 3331 + 3527}{3333^2}$

We put $x = 3333$ and then we have:

$3334 = x+1$; $6663 = 2x-3$; $3331 = x-2$ $\quad 3327 = x-6$

We then replace the numbers with their expression with x.

$$\frac{(x+1)\cdot(2x-3)\cdot(x-2)+x-6}{x^2}$$

$(x+1)\cdot(2x-3)=2x^2-3x+2x-3=2x^2-x-3$

$(x-2)(2x^2-x-3)=2x^3-x^2-3x-4x^2+2x+6=$

$2x^3-x^2-x-4x^2+6=\ 2x^3-5x^2-x+6$

Now we add $x-6$: $2x^3-5x^2-x+6+x-6=\ 2x^3-5x^2$

$$\frac{(x+1)\cdot(2x-3)\cdot(x-2)+x-6}{x^2}=\frac{2x^3-5x^2}{x^2}=2x-5=6666-5=6661$$

163. Solve for *x*: $x^2-18x-17\sqrt{x}=0$

$x^2-18x-17\sqrt{x}=0$ We put $\sqrt{x}=y$, then the equation reads:

$y^4-18y^2-17y=0 \quad \Leftrightarrow \quad y=0 \quad \vee \quad y^3-18y-17=0$

It is obvious to guess the solution: $y=-1$

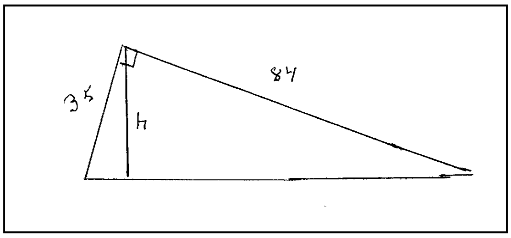
We then make polynomial division with $y+1$.

$$\begin{array}{l} y+1\,|\,y^3-18y-17\,|\,y^2-y-17 \\ \quad y^3+y^2 \\ \quad\quad -y^2-18y \\ \quad\quad -y^2-y \\ \quad\quad\quad -17y-17 \\ \quad\quad\quad -17y-17 \end{array}$$

$y^2-y-17=0$; $d=1+68=69$; $y=\dfrac{1\pm\sqrt{69}}{2}=\dfrac{1\pm\sqrt{69}}{2}$

$$x=y^2=\left(\frac{1\pm\sqrt{69}}{2}\right)^2$$

164. In a right angled triangle the *a* = 35, *b* = 84, Find the height *h*



This is a easy one. The area of the triangle may be written in two ways: $\frac{1}{2}ab=\frac{1}{2}hc \quad\Rightarrow\quad ab=hc$. At the same time: $c=\sqrt{a^2+b^2}$, so $h=\dfrac{ab}{c}=\dfrac{2240}{91}=24.62$

165. $a^2 - b^2 = 9$ *and* $ab = 3$. **Determine** $a + b$

This can be solved but the numbers are not very friendly.

$a^2 - b^2 = 9 \quad \Leftrightarrow \quad (a-b)(a+b) = 9$

$(a+b)^2 - (a-b)^2 = 4ab \quad \wedge \quad (a-b)^2 = \dfrac{81}{(a+b)^2} \quad \Rightarrow$

$(a+b)^2 - \dfrac{81}{(a+b)^2} - 4ab = 0 \quad \Leftrightarrow$

$(a+b)^4 - 4ab(a+b)^2 - 81 = 0 \quad we \;\; put \quad y = (a+b)^2 \quad \Leftrightarrow$

$y^2 - 12y - 81 = 0; \qquad d = 144 + 4 \cdot 81 = 468$

$y = \dfrac{12 \pm \sqrt{468}}{2} \quad \Rightarrow \quad y = 3 \pm \sqrt{117} \quad \Rightarrow \quad (a+b)^2 = 3 + \sqrt{117} \quad \Rightarrow$

$a + b = \sqrt{3 + \sqrt{117}}$

166. Solve for *x*: $3^x + 4^x - 6^x = 1$

$3^x + 4^x - 6^x = 1 \quad \Leftrightarrow \quad 3^x + (2^x)^2 - 2^x 3^x = 1$

We put $3^x = a$; $2^x = b$ and then we get:

$a + b^2 - ab = 1 \quad \Leftrightarrow \quad b^2 - 1 - a(b-1) = 0 \quad \Leftrightarrow \quad (b-1)(b+1) - a(b-1) \quad \Leftrightarrow \; (b-1)(b+1-a) = 0 \quad \Leftrightarrow$

$b = 1 \quad \vee \quad b - a = -1$

$2^x = 1 \quad \vee \quad 2^x - 3^x = -1 \quad \Leftrightarrow \quad x = 0 \quad \vee \quad x = 1$

Since $2^x - 3^x$ is an decreasing function the only solution is $x = 1$

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$2^{\sin^2 x} + 2^{\cos^2 x} = 3$ we put: $\cos^2 x = 1 - \sin^2 x$ and we find:

$2^{\sin^2 x} + 2^{1-\sin^2 x} = 3$ we then substitute $y = \sin^2 x$ and we get:

$2^y + 2^{1-y} = 3 \quad \Leftrightarrow \quad (2^y)^2 + 2 = 3 \cdot 2^y$

This is a quadratic equation in 2^y, and for convince we put $z = 2^y$. We then have:

$z^2 + 2 - 3z = 0 \;\; ; \; d = 4 + 12 = 16 \quad z = \dfrac{-2 \pm 4}{2} \quad z = -3 \;\; or \;\; z = 1 \quad \Leftrightarrow \quad 2^y = 1 \quad \Leftrightarrow \quad y = 0$

$y = \sin^2 x = 0 \quad \Leftrightarrow \quad x = p\dfrac{\pi}{2}$

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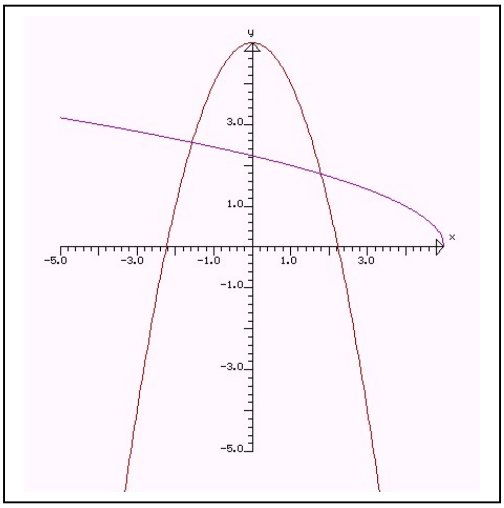

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So we identify the coefficients to the power of *x*:

$$c + b - a^2 = -10 \quad ; \quad ac - ba = 1; \quad bc = 20$$

From these equations, we may get an expression for *b* and *c* expressed by *a*.

$$c + b = a^2 - 10 = ; \quad (c - b) = \frac{1}{a}; \quad bc = 20$$

$2c = a^2 - 10 + \frac{1}{a} \qquad 2b = a^2 - 10 - \frac{1}{a} \quad \Leftrightarrow \quad c = \frac{1}{2}(a^2 - 10 + \frac{1}{a}) \qquad b = \frac{1}{2}(a^2 - 10 - \frac{1}{a})$

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$bc = 20 \quad \Leftrightarrow \quad \frac{1}{4}(a^2 - 10 + \frac{1}{a})(a^2 - 10 - \frac{1}{a}) = \frac{1}{4}((a^2 - 10)^2 - \frac{1}{a^2}) = 20 \quad \Leftrightarrow$

$(a^2 - 10)^2 - \frac{1}{a^2} = 80 \quad \Leftrightarrow \quad (u - 10)^2 - \frac{1}{u} = 80 \quad \Leftrightarrow \quad u^2 + 100 - 20u - \frac{1}{u} = 80 \quad \Leftrightarrow$

$u^3 + 20u - 20u^2 - 1 = 0 \quad \Leftrightarrow \quad \Leftrightarrow \quad u^3 - 20u^2 + 20u - 1 = 0$

If we put $u = a^2$ we find a third order equation in *u*. $a = 1$ or $a = -1$.

We can immediately that $u = 1$ is a root. Polynomial division with $u - 1$ gives:

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We shall first concentrate on the root $u = 1$, and we calculate b and c.

$$u = a^2 \quad \Rightarrow \quad a = 1 \quad or \quad a = -1.$$

$$a = 1: \quad c = \tfrac{1}{2}(a^2 - 10 + \frac{1}{a}) = -4 \qquad b = \tfrac{1}{2}(a^2 - 10 - \frac{1}{a}) = -5$$
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Using a = -1 gives the same product, but with the factors in inverse order.

$$x^2 + x - 5 = 0; \quad d = 1 + 20 \quad ; \quad x = \frac{-1 \pm \sqrt{21}}{2} \quad x = 1.79 \;\; or \;\; x = -2.79$$
$$x^2 - x - 4 = 0 \quad d = 1 + 16 \quad ; \quad x = \frac{1 \pm \sqrt{17}}{2} \quad x = 2.56 \;\; or \;\; x = -1.56$$

From the graph, we can see that the solutions are: x = -1.56 or x = 1.79

154. A simple exercise: $n! = n^3 - n$

For n = 4, we have; $n! = n^3 - n$ gives: 24 < 64 – 4
For n = 6, we have; $n! = n^3 - n$ gives: 720 > 216 – 6
For n = 5, we have; $n! = n^3 - n$ gives: 120= 125 -5.
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A direct approach would require to times lifting to the 6th power. Hardly the way to find the solution in a lifetime.
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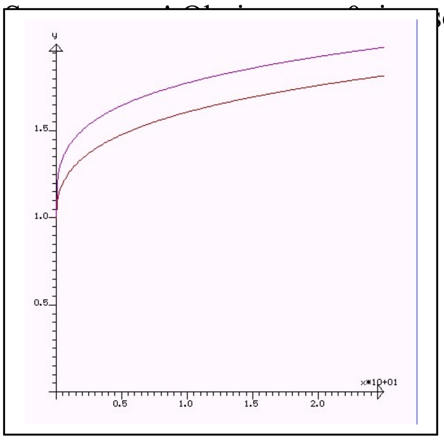


$1+\sqrt{x} = 1, 8\ 27\ 64....$ it gives for x. 1, 4, 9, 16. and $1+\sqrt{x}$ = 2, 3, 4, 5, 6, but none of them are cubic numbers. Looking at the right side $\sqrt{1+\sqrt[3]{x}}$, then x = 27 gives $\sqrt{1+\sqrt[3]{x}} = 2$, but this does not comply with the left hand side.
So it seems that x = 0 is the only solution. This is also confined by looking at the graph to the left, plotting the left side and the right side in the same coordinate system.

The only intersection point is (0,1)

156. Solve for *x*: $2^x - 3^x = \sqrt{6^x - 9^x}$

$2^x - 3^x = \sqrt{6^x - 9^x}$. We put $a = 2^x$ *and* $b = 3^x$. Then the equation reads.

$a - b = \sqrt{ab - b^2} \quad \Leftrightarrow \quad a - b = \sqrt{b}\sqrt{a-b} \quad \Leftrightarrow \quad (a-b)^2 = (\sqrt{b}\sqrt{a-b})^2 \quad \Leftrightarrow$

$(a-b)^2 = b(a-b) \quad \Leftrightarrow \quad a - b = b \quad \Leftrightarrow \quad a = 2b$

$2^x = 2 \cdot 3^x \quad \Leftrightarrow \quad \left(\frac{2}{3}\right)^x = 2 \quad \Leftrightarrow \quad x = \frac{\ln 2}{\ln(\frac{2}{3})}$

157. Solve: $\log x = \sqrt{\log x^{\frac{13}{6}} - 1}$

$\log x = \sqrt{\log x^{\frac{13}{6}} - 1}$. We square both sides and find:

$\log^2 x = \log x^{\frac{13}{6}} - 1 \quad \Leftrightarrow \quad \log^2 x = \frac{13}{6}\log x - 1 \quad \Leftrightarrow \quad 6\log^2 x - 13\log x + 6 = 0$

We put $y = \log x$ and find:

$6y^2 - 13y + 6 = 0 \quad d = 169 - 144 = 25 \quad y = \frac{13 \pm 5}{2} \quad \Leftrightarrow \quad y = 9 \quad or \quad y = 4 \quad \Leftrightarrow$

$\log x = 9 \quad or \quad \log x = 4 \quad \Leftrightarrow \quad x = 10^9 \quad or \quad x = 10^4$

157. Solve for *x*: $x^x = 2^{\frac{1}{x}}$

We take $\log_2$ on both sides: $x\log_2 x = \frac{1}{x} \quad \Leftrightarrow \quad x^2\log_2 x = 1 \quad \Leftrightarrow \quad \log_2 x^{x^2} = 1$

$x^{x^2} = 2$

It is easy to see that the solution is $x = \sqrt{2}$, since: $\sqrt{2}^{\sqrt{2}^2} = 2$

158. Determine *f* from the equation: $f\left(\frac{\sqrt{x^2+1} - x}{x}\right) = x^2$

We put:

$y = \frac{\sqrt{x^2+1} - x}{x} \quad \Leftrightarrow \quad yx = \sqrt{x^2+1} - x \quad \Leftrightarrow \quad yx + x = \sqrt{x^2+1} \quad \Leftrightarrow$

$(yx + x)^2 = (\sqrt{x^2+1})^2 \quad \Leftrightarrow \quad y^2x^2 + x^2 + 2yx^2 = x^2 + 1 \quad \Leftrightarrow$

$x^2(y^2 + 2y) = 1 \quad \Leftrightarrow \quad x^2 = \frac{1}{(y^2 + 2y)}$

$f(y) = \frac{1}{(y^2 + 2y)} \quad f(x) = \frac{1}{(x^2 + 2x)}$

159. Determine *a* and, such that: $2^a - 2^b = 2016$

Since $2^{10} = 1024$, we make a try with a = 11. $2^{11} = 2048$ and 2048 - 2016 = 32= 2^5
So: $a = 11 \quad and \quad b = 5$.

160. Simplify $\sqrt{247^2 - 153^2}$

$$\sqrt{247^2 - 153^2} = \sqrt{(247-153)(247+153)} = \sqrt{94 \cdot 400} = 20\sqrt{94}$$

161. Solve: $2^{x^2} + 4^{x^2} = 8^{x^2}$

We put $y = 2^{x^2}$ and then we get:

$$y + y^2 = y^3 \quad \Leftrightarrow \quad y^3 - y^2 - y = 0 \quad \Leftrightarrow \quad y(y^2 - y - 1) = 0 \quad \Leftrightarrow$$

$$y = 0 \quad or \quad y^2 - y - 1 = 0 \quad d = 1 + 4 = 5; \quad y = \frac{1 \pm \sqrt{5}}{2} \quad \Leftrightarrow \quad x^2 = \frac{1+\sqrt{5}}{2} \quad \Rightarrow$$

$$x = \pm\sqrt{\frac{1+\sqrt{5}}{2}}$$

162. Simplify $\dfrac{3334 \cdot 6663 \cdot 3331 + 3527}{3333^2}$

We put $x = 3333$ and then we have:

$3334 = x + 1$; $6663 = 2x - 3$; $3331 = x - 2$ $\quad 3327 = x - 6$
We then replace the numbers with their expression with x.

$$\frac{(x+1)\cdot(2x-3)\cdot(x-2) + x - 6}{x^2}$$

$$(x+1)\cdot(2x-3) = 2x^2 - 3x + 2x - 3 = 2x^2 - x - 3$$

$$(x-2)(2x^2 - x - 3) = 2x^3 - x^2 - 3x - 4x^2 + 2x + 6 =$$

$$2x^3 - x^2 - x - 4x^2 + 6 = 2x^3 - 5x^2 - x + 6$$

Now we add $x - 6$: $2x^3 - 5x^2 - x + 6 + x - 6 = 2x^3 - 5x^2$

$$\frac{(x+1)\cdot(2x-3)\cdot(x-2) + x - 6}{x^2} = \frac{2x^3 - 5x^2}{x^2} = 2x - 5 = 6666 - 5 = 6661$$

163. Solve for *x*: $x^2 - 18x - 17\sqrt{x} = 0$

$x^2 - 18x - 17\sqrt{x} = 0$ We put $\sqrt{x} = y$, then the equation reads:

$$y^4 - 18y^2 - 17y = 0 \quad \Leftrightarrow \quad y = 0 \quad \vee \quad y^3 - 18y - 17 = 0$$

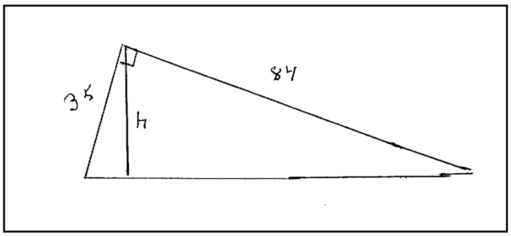
It is obvious to guess the solution: y = -1
We then make polynomial division with y +1.

$$y+1 \mid y^3-18y-17 \mid y^2-y-17$$
$$y^3+y^2$$
$$-y^2-18y$$
$$-y^2-y$$
$$-17y-17$$
$$-17y-17$$

$y^2-y-17=0$; $d=1+68=69$; $y=\dfrac{1\pm\sqrt{69}}{2}=\dfrac{1\pm\sqrt{69}}{2}$

$x=y^2=\left(\dfrac{1\pm\sqrt{69}}{2}\right)^2$

164. In a right angled triangle the *a* = 35, *b* = 84, Find the height *h*



This is a easy one. The area of the triangle may be written in two ways: $\frac{1}{2}ab=\frac{1}{2}hc \Rightarrow ab=hc$. At the same time: $c=\sqrt{a^2+b^2}$, so $h=\dfrac{ab}{c}=\dfrac{2240}{91}=24.62$

165. $a^2-b^2=9$ *and* $ab=3$. **Determine** $a+b$

This can be solved but the numbers are not very friendly.

$a^2-b^2=9 \Leftrightarrow (a-b)(a+b)=9$

$(a+b)^2-(a-b)^2=4ab \wedge (a-b)^2=\dfrac{81}{(a+b)^2} \Rightarrow$

$(a+b)^2-\dfrac{81}{(a+b)^2}-4ab=0 \Leftrightarrow$

$(a+b)^4-4ab(a+b)^2-81=0$ *we put* $y=(a+b)^2 \Leftrightarrow$

$y^2-12y-81=0$; $d=144+4\cdot 81=468$

$y=\dfrac{12\pm\sqrt{468}}{2} \Rightarrow y=3\pm\sqrt{117} \Rightarrow (a+b)^2=3+\sqrt{117} \Rightarrow$

$a+b=\sqrt{3+\sqrt{117}}$

166. Solve for *x*: $3^x+4^x-6^x=1$

$3^x+4^x-6^x=1 \Leftrightarrow 3^x+(2^x)^2-2^x3^x=1$

We put $3^x=a$; $2^x=b$ and then we get:

$a+b^2-ab=1 \quad \Leftrightarrow \quad b^2-1-a(b-1)=0 \quad \Leftrightarrow \quad (b-1)(b+1)-a(b-1) \quad \Leftrightarrow \quad (b-1)(b+1-a)=0 \quad \Leftrightarrow$
$b=1 \quad \vee \quad b-a=-1$

$2^x=1 \quad \vee \quad 2^x-3^x=-1 \quad \Leftrightarrow \quad x=0 \quad \vee \quad x=1$

Since 2^x-3^x is an decreasing function the only solution is $x = 1$

167. Integer solution to

$$\frac{x-6}{2020}+\frac{x-5}{2021}+\frac{x-4}{2022}=3$$

Operating with large integer numbers is somewhat troublesome, so we put $a=2021$ *and* $y=x-5$, then we get:

$$\frac{y-1}{a-1}+\frac{y}{a}+\frac{y+1}{a+1}=3$$

To get rid of the denominators, we multiply the equation with $(a-1)a(a+1)$. We then get:

$(y-1)a(a+1)+y(a-1)(a+1)+(y+1)(a-1)a=3(a-1)a(a+1)$

$ya(a+1)-a(a+1)+y(a-1)(a+1)+(y+1)(a-1)a=3(a-1)a(a+1)$

$y\left(a^2+a+a^2-1+a^2-a\right)-a^2-a+a^2-a=3(a-1)a(a+1)$

$y\left(3a^2-1\right)=a\left(3a^2-1\right)$
$y=a$
$x=y+5=2026$

168. $x^2-2y^2=1$, where *x* and *y* are primes

The solution is based on guesswork. The solution is $x=3$ *and* $y=2$, since: $3^2-2\cdot 2^2=1$

169. Ridiculous easy. Determine *x*: $3^{88}+3^{88}+3^{88}=3^x$

$3^{88}+3^{88}+3^{88}=3^x \quad \Leftrightarrow \quad 3\cdot 3^{88}=3^x \quad \Leftrightarrow \quad 3^{88+1}=3^x \quad \Leftrightarrow \quad x=89$

170. A simple 2. order differential equation: $y''=y'+y$

In one of my articles: *The differential equations of physic*: I have shown the well known theorem: The solution of any differential equation of *n*th order with constant coefficients can be reduced to a solving (using complex numbers) to an algebraic equation of order *n* by putting $y=e^{kx}$, where *k* is a complex number. This is demonstrated below:
$y''=y'+y$. We put $y=e^{kx}$, and we get: $k^2e^{kx}-ke^{kx}-e^{kx}=0 \quad \Leftrightarrow \quad k^2-k-1=0$

$k^2-k-1=0; \quad d=1+4=5, \quad k=\dfrac{1\pm\sqrt{5}}{2}$

$y=ce^{\frac{1+\sqrt{5}}{2}x} \quad \vee \quad y=ce^{\frac{1-\sqrt{5}}{2}x}$

171. Solve for *x*: $4^x + 16^x = 272$ **(Easy one)**

$4^x + 16^x = 272 \quad \Leftrightarrow \quad 4^x + (4^x)^2 = 272$. We put $y = 4^x$.

$y + y^2 - 272 = 0; \quad d = 1 + 4 \cdot 272 = 1189 = 33^2 \; ; \quad y = \dfrac{-1 \pm 33}{2} \quad y = 16 \;\vee\; y = -17$

$4^x = 16 \quad \Leftrightarrow \quad x = 2$

172. Simplify: $\dfrac{a^3 + 3ab^2}{b^3 + 3a^2b} = 1$

$\dfrac{a^3 + 3ab^2}{b^3 + 3a^2b} = 1 \quad \Leftrightarrow \quad a^3 + 3ab^2 = b^3 + 3a^2b \quad \Leftrightarrow \quad a^3 + 3ab^2 - b^3 - 3a^2b \;=\; (a-b)^3$

173. Solve: $2^x + (\frac{2}{3})^x + (\frac{3}{4})^x = 3$**. (Cheat problem)**

$2^x + (\frac{2}{3})^x + (\frac{3}{4})^x = 3$.

This equation cannot be solved by traditional means, but you notice that $2 \cdot \frac{2}{3} \cdot \frac{3}{4} = 1$ and thus:

$2^x \cdot (\frac{2}{3})^x \cdot (\frac{3}{4})^x = 1^x = 1$

If we put $2^x = a$,, $(\frac{2}{3})^x = b \;\; and \;\; (\frac{3}{4})^x = c$

We have two equations: $a + b + c = 3 \quad abc = 1$, but these have the only solution: $a = 1, b = 1; \; c = 1$

But this requires that : $x = 0$. Which also sees is the only solution.

174. Find x such that $x^{x^5} = 100$

I can't see any analytic solution to this equation, but some qualified guesses led to

$x = \sqrt[5]{10}$, as we can see: $x^5 = 10 \;\; and \;\; x^{10} = (\sqrt[5]{10})^{10} = 10^2 = 100$.

175. Solve for *x* $\log_2 x + \log_4 x = 3$

$\log_2 x + \log_4 x = 3$

$y = \log_4 x \quad \Leftrightarrow \quad x = 4^y \quad \Rightarrow \quad \log_2 x = y \log_2 4 \;\; = \log_4 x \log_2 4 \quad \Leftrightarrow \quad \log_4 x = \dfrac{\log_2 x}{\log_2 4}$

$\log_2 x + \log_4 x = 3 \quad \Leftrightarrow \quad \log_2 x + \dfrac{\log_2 x}{\log_2 4} = 3 \quad \Leftrightarrow \quad \log_2 x + \dfrac{\log_2 x}{2} = 3 \quad \Leftrightarrow$

$\frac{3}{2} \log_2 x = 3 \quad \Leftrightarrow \quad \log_2 x = 2 \quad \Leftrightarrow \quad x = 4$

176. Determine *f*(*x*) from $f(f(x)) = x^2 + x + 1$ **(Not a friendly exercise)**

$f(f(x)) = x^2 + x + 1 \quad \Leftrightarrow \quad f^{-1}(f(f(x)) = f^{-1}(x^2 + x + 1) \quad \Leftrightarrow \quad f(x) = f^{-1}(x^2 + x + 1)$

$y = x^2 + x + 1 \quad \Leftrightarrow \quad f^{-1}(y) = x$

$y = x^2 + x + 1 \quad \Leftrightarrow \quad x^2 + x + 1 - y = 0 \quad , \quad d = 1 - 4(1 - y) = y - 3;$

$x = \dfrac{-1 \pm \sqrt{y-3}}{2} \qquad f^{-1}(y) = \dfrac{-1 \pm \sqrt{y-3}}{2} \quad \Rightarrow \quad f^{-1}(x) = \dfrac{-1 \pm \sqrt{x-3}}{2}$

$f(x) = f^{-1}(x) = \dfrac{-1 \pm \sqrt{x-3}}{2}$

177. Solve for *x*: $3^x + 9^x = 27^x$

$3^x + 9^x = 27^x \quad \Leftrightarrow \quad 3^x + (3^x)^2 = (3^x)^3$

We put: $y = 3^x$, and we get:

$y + y^2 = y^3 \quad \Leftrightarrow \quad y^3 - y^2 - y = 0 \quad \Leftrightarrow \quad y(y^2 - y - 1) = 0 \quad \Leftrightarrow$

$y = 0 \quad \vee \quad y^2 - y - 1 = 0 \ , \quad d = 1 + 4 = 5 \quad y = \dfrac{1 \pm \sqrt{5}}{2}$

$3^x = 0 \quad \vee \quad 3^x = \dfrac{1 \pm \sqrt{5}}{2} \quad \Leftrightarrow \quad 3^x = \dfrac{1 + \sqrt{5}}{2} \quad \Leftrightarrow \quad x = \dfrac{\ln(\frac{1+\sqrt{5}}{2})}{\ln 3}$

278. Determine *f*(*x*) from the equation: $f(x + \frac{1}{x}) = x^5 + \frac{1}{x^5}$

We put $y = x + \dfrac{1}{x}$

$y^2 = (x + \frac{1}{x})^2 = x^2 + \frac{1}{x^2} + 2$

$y^2 - 2 = x^2 + \frac{1}{x^2}$

$(y^2 - 2)^2 = (x^2 + \frac{1}{x^2})^2 = x^4 + \frac{1}{x^4} + 2$

$(y^2 - 2)^2 - 2 = x^4 + \frac{1}{x^4}$

$((y^2 - 2)^2 - 2)y = (x^4 + \frac{1}{x^4})(x + \frac{1}{x}) = x^5 + x^3 + \frac{1}{x^3} + \frac{1}{x^5} = x^5 + \frac{1}{x^5} + x^3 + \frac{1}{x^3}$

$(y^2 - 2)y = (x^2 + \frac{1}{x^2})(x + \frac{1}{x}) = x^3 + \frac{1}{x^3} + x + \frac{1}{x}$

$(y^2 - 2)y = (x^2 + \frac{1}{x^2})(x + \frac{1}{x}) = x^3 + \frac{1}{x^3} + x + \frac{1}{x}$

$(y^2 - 2)y - y = x^3 + \frac{1}{x^3}$

$((y^2 - 2)^2 - 2)y - (y^2 - 2)y + y = x^5 + \frac{1}{x^5}$

$f(y) = ((y^2 - 2)^2 - 2)y - (y^2 - 2)y + y$

$$f(y) = ((y^2-2)^2-2)y-(y^2-2)y+y$$
$$((y^2-2)^2-2)y-(y^2-2)y+y = x^5+\frac{1}{x^5}$$
$$f(y) = ((y^2-2)^2-2)y-(y^2-2)y+y$$
$$((y^2-2)^2-2)y-(y^2-2)y+y = x^5+\frac{1}{x^5}$$
$$f(y) = ((y^2-2)^2-2)y-(y^2-2)y+y$$
$$f(x) = ((x^2-2)^2-2)x-(x^2-2)x+x$$

279. Solve for *x*: $x^2+x+6\sqrt{x+2}=18$

To try to find an analytic solution will lead nowhere, but at a sight we can see that $x = 2$.
Is a solution since; $2^2+2+6\sqrt{2+2}=18$
As the function is increasing, there can be only one solution.

180. $x = 45678^3 - 45676^3$ **. Determine** $\sqrt{\frac{x-2}{6}}$ **.**

We put $a = 45676$, and then we have;

$$x = (a+2)^3 - a^3 = a^3+8+6a^2+12a-a^3 \quad \Leftrightarrow$$
$$x-2 = 6+6a^2+12a = 6(a^2+2a+1) = 6(a+1)^2 \Rightarrow$$
$$\sqrt{\frac{x-2}{6}} = a+1 = 45677$$

181. Solve: $x^2+x+6\sqrt{x+2}=18$

It seems a waste of good intellect go try to solve this analytically. However, it is obvious hat $x = 2$ is a solution, since: $2^2+2+6\sqrt{2+2}=18$.

182. Solve for x*:* $16^x+20^x=25^x$

$$16^x+20^x=25^x \quad \Leftrightarrow \quad \frac{16^x}{16^x}+\frac{20^x}{16^x}=\frac{25^x}{16^x} \quad \Leftrightarrow$$
$$1+\left(\frac{5}{4}\right)^x = \left(\left(\frac{5}{4}\right)^x\right)^2$$

We put: $y = \left(\frac{5}{4}\right)^x$ and then we have.

$$1+y=y^2 \quad \Leftrightarrow \quad y^2-y-1=0; \quad d=1+4=5 \quad y=\frac{1\pm\sqrt{5}}{2} \quad \Leftrightarrow$$

$$\left(\frac{5}{4}\right)^x = \frac{1+\sqrt{5}}{2} \quad \Leftrightarrow \quad x = \frac{\ln(\frac{1+\sqrt{5}}{2})}{\ln\frac{5}{4}}$$

182. Solve for *x*. $\sqrt{x^{\sqrt{x}}} = x^{\frac{1}{\sqrt{x}}}$

$\sqrt{x^{\sqrt{x}}} = x^{\frac{1}{\sqrt{x}}}$. We put $y = \sqrt{x}$, and we then have:

$$\sqrt{x^y} = x^{\frac{1}{y}} \quad \Leftrightarrow \quad x^y = x^{\frac{2}{y}} \quad \Leftrightarrow \quad y = \frac{2}{y} \quad \Leftrightarrow \quad y^2 = 2 \quad \Leftrightarrow \quad x = 2$$

183. Solve: $7x^{-5} = 98 \quad \Leftrightarrow \quad x^5 = \frac{7}{98} \quad x^5 = \frac{1}{14} \quad \Leftrightarrow \quad x = \sqrt[5]{\frac{1}{14}}$

184. Solve: $2^x 3^{x^2} = 6$

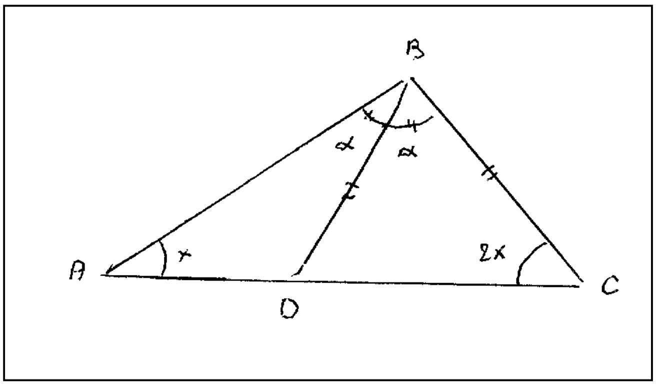
$$2^x 3^{x^2} = 6 \quad \Leftrightarrow \quad x\ln 2 + x^2 \ln 3 = \ln 6 \quad \Leftrightarrow \quad x^2 \ln 3 + x \ln 2 - \ln 6 = 0$$

$$d = \ln^2 2 + 4\ln 3 \ln 6. \quad ; \quad x = \frac{-\ln 2 \pm \sqrt{\ln^2 2 + 4\ln 3 \ln 6}}{2\ln 3}$$

185. Solve: $e^{e^x} = 2$

$$e^{e^x} = 2 \quad \Leftrightarrow \quad e^x = \ln 2 \quad \Leftrightarrow \quad x = \ln(\ln 2)$$

186. Determine the angle *x* in the triangle shown below.



Since the triangle *DBC* is isosceles, we must have $\angle CDB = 2x$. The angle. $\angle ADB = 180 - 2x$.
In the triangle *ABD* we thus have:

$$x + 180 - 4x + 180 - 2x = 180 \quad \Leftrightarrow$$
$$.-5x = -180 \quad \Leftrightarrow \quad x = 36$$

187. Simplify: $\frac{x^5 + x + 1}{x^2 + x + 1}$

It is obvious to make polynomial division. Since the division succeeds, it is actually a very simple exercise.

$x^2+x+1 \,|\, x^5+x+1 \,|\, x^3-x^2+1$

$x^5+x^4+x^3$

$-x^4-x^3$

$-x^4-x^3-x^2$

x^2+x+1

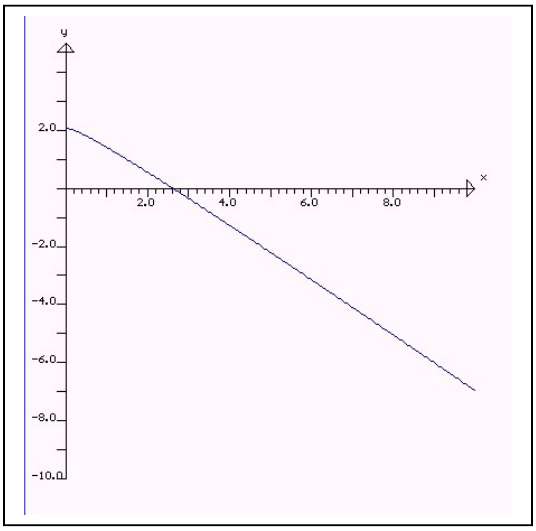
x^2+x+1

And therefore: $\frac{x^5+x+1}{x^2+x+1}=x^3-x^2+1$

188. Simplify: $(99\tfrac{1}{2})^2=(100-\tfrac{1}{2})^2=10^4+\tfrac{1}{4}-100$

189. Solve for *x*: $\sqrt{1+\sqrt{x}}=x-1$

$\sqrt{1+\sqrt{x}}=x-1 \quad \Leftrightarrow \quad 1+\sqrt{x}=x^2+1-2x \quad \Leftrightarrow \quad \sqrt{x}=x^2-2x$



We put $y=\sqrt{x} \quad \Leftrightarrow \quad x=y^2$, and then we have:

$\sqrt{x}=x^2-2x \quad \Leftrightarrow \quad y=y^4-2y^2 \quad \Leftrightarrow$

$y^4-2y^2-y=0 \quad \Leftrightarrow$

$y^3-2y-1=0$

The last equation has no simple solution and must be solved with Cardano's formula.
Although I have presented a derivation on Cardano's formula, in the mathematics section of my homepage, it is not worth the effort to repeat it here. Instead a graph of the function is shown below.

189. Solve for *x* and *y*: $x+y=(x-y)^2$ **large**

The right hand side $(x-y)^2$ can be: 1, 4, 9, 16,…
Intuitively the right hand side, should not be to large
If we make a try with: $(x-y)^2=9$, it leads quickly to the solution
$(x,y)=(6,3) \quad 6+3=9 \quad \textit{and} \quad (6-3)^2=9$

190. $2^x=7^y=196$. **Determine:** $\frac{x+y}{xy}$

First we notice that: $196=2^2\cdot 7^2$.

$2^x = 196 \quad \Leftrightarrow \quad x = \frac{\ln 196}{\ln 2} \quad and \quad 7^y = 196 \quad \Leftrightarrow \quad y = \frac{\ln 196}{\ln 7} \quad \Rightarrow$

$x + y = \frac{\ln 196}{\ln 2} + \frac{\ln 196}{\ln 7} = \ln 196(\frac{1}{\ln 2} + \frac{1}{\ln 7}) = \ln 196 \frac{\ln 2 + \ln 7}{\ln 2 \ln 7}$

$xy = \frac{\ln^2 196}{\ln 2 \ln 7}$

$\frac{x+y}{xy} = \frac{\ln 196 \frac{\ln 2 + \ln 7}{\ln 2 \ln 7}}{\frac{\ln^2 196}{\ln 2 \ln 7}} = \frac{\ln 2 + \ln 7}{\ln 196} = \frac{\ln 14}{\ln 196}$

From $196 = 2^2 \cdot 7^2 \quad \Rightarrow \quad \ln 196 = 2 \ln 14$, so $\frac{\ln 14}{\ln 196} = \frac{1}{2}$

191. $x = 3 + 2\sqrt{2}.$ **Determine:** $\sqrt{x} - \frac{1}{\sqrt{x}}$

$x = 3 + 2\sqrt{2} \quad \Rightarrow \quad \frac{1}{x} = \frac{1}{3 + 2\sqrt{2}} = \frac{1}{(3 + 2\sqrt{2})} \frac{(3 - 2\sqrt{2})}{(3 - 2\sqrt{2})} = \frac{(3 - 2\sqrt{2})}{3^2 - 8} = 3 - 2\sqrt{2}$

$\sqrt{x} - \frac{1}{\sqrt{x}} = \sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}$

$\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 = 2\sqrt{2} + 3 + 3 - 2\sqrt{2} - x + 2\sqrt{3 + 2\sqrt{2}}\sqrt{3 - 2\sqrt{2}} = 6 - 2\sqrt{9 - 8} = 4$, so

$\sqrt{x} - \frac{1}{\sqrt{x}} = 2$

192. Simplify: $\sqrt{46 - 12\sqrt{14}}$

We shall try to write: $46 - 12\sqrt{14}$ as $(a\sqrt{2} - b\sqrt{7})^2 = 2a^2 + 7b^2 - 2ab\sqrt{14}$

We then have $ab = 6$ *and* $2a^2 + 7b^2 = 46$

The first equation could be: $a = 3$ *and* $b = 2$, and indeed $2a^2 + 7b^2 = 18 + 28 = 46$

So $\sqrt{46 - 12\sqrt{14}} = 3\sqrt{2} + 2\sqrt{7}$

193. Solve: $\ln(e^x + 1) = 2x$ **(undergraduate level)**

$\ln(e^x + 1) = 2x \quad \Leftrightarrow \quad (e^x + 1) = e^{2x} \quad \Leftrightarrow \quad e^{2x} - e^x - 1 = 0$

Put: $y = e^x$ then we have:

$y^2 - y - 1 = 0 \quad ; \quad d = 1 + 4 = 5 \quad y = \frac{1 \pm \sqrt{5}}{2} \quad \Rightarrow \quad e^x = \frac{1 + \sqrt{5}}{2} \quad \Rightarrow \quad x = \ln(\frac{1 + \sqrt{5}}{2})$

194. Find the integer solutions to: $x\sqrt{y} + y\sqrt{x} = 182 \quad and \quad x\sqrt{x} + y\sqrt{y} = 183$

Since the solutions (*x, y*) must be integers, they must be chosen among the numbers: 1, 4, 9, 16,.25.. But this narrows the solutions a lot. The only one, which are close to the solution is (25,16), but this gives: $x\sqrt{y} + y\sqrt{x} = 25\cdot 4 + 16\cdot 5 = 180 \quad and \quad x\sqrt{x} + y\sqrt{y} = 5\cdot 25 + 16\cdot 4 = 189$.
This is a bit strange, since other candidates are far from the stated values.

So alternatively, we shall try an analytic solution. We put: $a = \sqrt{x} \quad and \quad b = \sqrt{y}$, and we get:

$x\sqrt{y} + y\sqrt{x} = 182 \quad \Leftrightarrow \quad a^2b + b^2a = 182 \quad and \quad x\sqrt{x} + y\sqrt{y} = 183 \quad \Leftrightarrow \quad a^3 + b^3 = 183$

$a^2b + b^2a = 182 \qquad ab(a+b) = 182$

$a^3 + b^3 = 183 \qquad a^3 + b^3 = 183$

The last equation invites to use the formula: $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

From which we find: $(a+b)^3 = 183 + 3\cdot 182 = 729 = 9^3 \quad \Rightarrow \quad a + b = 9$. So far so good!

To avoid a third degree equation we use: $ab(a+b) = 182 \quad \Rightarrow \quad ab = \frac{182}{9}$

But here arise a problem, since 9 is not a divisor in 182. However, if we continue the calculation and insert $b = 9 - a$. we find:

$-a^2 + 9a = \frac{182}{9} \quad \Leftrightarrow \quad 9a^2 - 81a + 182 = 0 \quad d = 81^2 - 36\cdot 182 = 9 - Okay!$

$a = \frac{81 \pm 3}{18} \quad \Leftrightarrow \quad a = 4.89 \quad or \quad a = 4.67$!!!!!

That was we could have expected from the preliminary analysis. There are no integer, not to speak of a quadratic number, solution.
It is a long time ago that I found this problem on the site. Since could not find an solution using the numbers 182 and 183 , I have visited the site several times to clear it up, but this problem has been taken away for a long time. But today 14.07.2022, it was there again but with *x* and *y* replaced by *a a* and *b* and 182 and 183 replaced by 180 and 189. So I was right from the beginning, although the calculations above were promising.

195. Solve for (x ,y): $\frac{1}{2a} + \frac{1}{3b} = \frac{1}{4}$

$\frac{1}{2a} + \frac{1}{3b} = \frac{1}{4} \quad \Leftrightarrow \quad \frac{1}{3b} = \frac{1}{4} - \frac{1}{2a} \quad \Leftrightarrow \quad \frac{1}{3b} = \frac{a-2}{4a}$

If the tem on the right hand side should be a genuine fraction: *a* must be equal to 3, and then we have: $\frac{1}{3b} = \frac{a-2}{4a} = \frac{1}{12} \quad so \quad a = 3 \quad and \quad b = 4$

And indeed: $\frac{1}{2a} + \frac{1}{3b} = \frac{1}{6} + \frac{1}{12} = \frac{1+2}{12} = \frac{1}{4}$

196. Solve for x: $33x^2 + 49x^{-1} - 10 = 0$ **(trivial)**

$33x^2 + 49x^{-1} - 10 = 0 \quad \Leftrightarrow \quad 33 + 49x - 10x^2 = 0 \quad \Leftrightarrow$

$10x^2 - 49x - 33 = 0 \;\; ; \;\; d = 49^2 + 4 \cdot 10 \cdot 35 = 61^2$

$x = \dfrac{49 \pm 61}{20} \quad x = 55 \;\; or \;\; x = 6$

197. Find integer solutions to: $\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{xy} = 1$

$\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{xy} = 1$ To get rid of the denominators, we multiply by x^2y^2:

$y^2 + x^2 + xy = y^2x^2$. The problem is of course the lack of the term 2*xy*. If we add the term *xy* on both sides, we find:

$y^2 + x^2 + 2xy = y^2x^2 + xy \quad \Leftrightarrow \quad (x+y)^2 = xy(xy+1)$

This shows that $xy(xy+1)$ should be a quadratic number, that is 1, 4, 9, 16, 25,36….

We look at the number $a(a+1) = k^2 \quad \Leftrightarrow \quad a^2 + a - k^2 = 0 \;\; ; \;\; d = 1 + 4k^2 \quad a = \dfrac{-1 \pm \sqrt{1+4k^2}}{2}$

$1 + 4k^2 = 1,4,9,16,...$ has no quadratic integer solution, so the exercise has no solution.

If however the exercise was:

$\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{2}{xy} = 1$ the solution would be: $(x+y)^2 = (xy)^2 \quad \Leftrightarrow \quad x + y = xy$

But this has neither a integer solution, whereas *x*+*y*+1 =*xy* has the solution (*x,y*) = (2,3)

198. Determine integer solution to. $\sqrt{x} + \sqrt{y} = 13 \;\; ; \;\; x - y = 65$

$\sqrt{x} + \sqrt{y} = 13 \;\; ; \;\; x - y = 65$ The first equation confines the first equation to:

$(\sqrt{y}, \sqrt{x}) = (2,11), (3,10). (4,9), (5.8).(6,7)$, But it is easy to see that only (4,9) complies with the second equation, since; $\sqrt{x} + \sqrt{y} = \sqrt{81} + \sqrt{16} = 9 + 4 = 13 \;\; and \;\; x - y = 81 - 16 = 65$.

This can also easily be confirmed by calculations:

$x - y = (\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) = (\sqrt{x} - \sqrt{y})13 = 65 \quad \Rightarrow \quad (\sqrt{x} - \sqrt{y}) = 5$

The two equations:

$(\sqrt{x} - \sqrt{y}) = 5$ and $(\sqrt{x} + \sqrt{y}) = 13$ may easily be solved to give: $(x, y) = (9,4)$

199. Solve the equation: $z + \dfrac{1}{z} = 4$

$z + \dfrac{1}{z} = 4 \quad \Leftrightarrow \quad z^2 - 4z + 1 = 0 \;\; ; \;\; d = 16 - 4 = 12 \quad z = \dfrac{4 \pm \sqrt{12}}{2} \quad z = 2 \pm \sqrt{3}$

200. Determine $\frac{x}{y}$ from the equation: $\frac{x^2}{y^2}+\frac{y^2}{z^2}=4$

$\frac{x^2}{y^2}+\frac{y^2}{z^2}=4$. If we put $z=\frac{x^2}{y^2}$, we have the equation $z+\frac{1}{z}=4$, which is the same as in 199.

So $\frac{x^2}{y^2}=2\pm\sqrt{3}$, and therefore: $\frac{x}{y}=\sqrt{2\pm\sqrt{3}}$

201. Solve: $\sqrt[3]{2-x}+\sqrt{x-1}=1$.

$\sqrt[3]{2-x}+\sqrt{x-1}=1$. If x is a solution, 2- x should be a cubic number, and x-1 should be a quadratic number. $2-x=0,1,8,27,64...$ and $x-1=1,4,9,16,...$ $x-1\geq 0 \quad\Rightarrow\quad x\geq 1 \quad\Rightarrow\quad 2-x\leq 1$
The only solution is therefore: x = 1.

202. Solve: $5^x 16^{\frac{x-1}{x}}=100$

$5^x 16^{\frac{x-1}{x}}=100$, well, 100 = 4·25, so it will be obvious to guess at: x =2, and indeed: $5^2\cdot 16^{\frac{1}{2}}=100$.

203. Determine *x* from the series: $1+4+7+...+x=287$

The series is algebraic , where we have. $a_{k+1}-a_k=d$, and for the sum of n terms we have the formula. $S_n=\frac{n}{2}(a_1+a_n)$.

It follows that $a_k=a_1+(k-1)d$.

We thus have: $x=1+3(n-1) \quad\Leftrightarrow\quad x=3n-2 \quad\Leftrightarrow\quad n=(x+2)/3$ And thus:

$287=\frac{n}{2}(3+x) \quad\Leftrightarrow\quad 287=\frac{n}{2}(3+3n-2) \quad\Leftrightarrow\quad 574=3n^2+n \quad\Leftrightarrow\quad 3n^2+n-574='0$

$d=1+12\cdot 574=83^2; \quad n=\frac{-1\pm 83}{6}=\frac{84}{6}=14$

And therefore $x=3n-2=40$

204. Determine *n* from the equation: $n!=n^2+11n+40$

$n!=n^2+11n+40$. To my knowledge there are no analytic method to solve this equation, but $4!=24$, $5!=120; \quad 6!=720$. So we try with n = 5. $25+55+40=120=5!$

205. Determine *x* from: $x^{x^{11}}=11$

$x^{x^{11}}=11$. Well it must be qualified guesswork, but the most obvious candidate is $x=\sqrt[11]{11}$. Since: $((\sqrt[11]{11})^{\sqrt[11]{11}})^{11}=(\sqrt[11]{11})^{11}=11$

206. Solve for *x*. $343^{3x-4}=\sqrt{7}$

We notice that $343=7^3$

$343^{3x-4} = \sqrt{7} \quad \Leftrightarrow \quad (3x-4)\ln 343 = \frac{1}{2}\ln 7 \quad \Leftrightarrow \quad 3x-4 = \frac{\frac{1}{2}\ln 7}{3\ln 7} \quad 3x-4 = \frac{1}{6} \quad \Leftrightarrow \quad x = \frac{25}{18}$

207. Solve for *x*: $\left(\frac{x}{8}\right)^x = 8^{8^8}$

It is a rather trivial exercise, if we put $y = \frac{x}{8}$, we

have: $y^{8y} = 8^{8^8} \quad \Leftrightarrow \quad (y^y)^8 = 8^{8^8} \quad \Leftrightarrow \quad y = 8 \quad \Leftrightarrow \quad x = 8^2$

208. Trivial: Find an expression for $\sqrt{20} - \sqrt{5}$

$\sqrt{20} - \sqrt{5} = \sqrt{4 \cdot 5} - \sqrt{5} = 2\sqrt{5} - \sqrt{5} = \sqrt{5}$

209. Trivial: $2^3 + 2^{3x} = 10$

$2^3 + 2^{3x} = 10$, we put: $y = 2^x$, and we get: $y^3 + y = 10$

It requires only very little insight to find that $y = 2$, since $2^3 + 2 = 10$. The solution is:

$2^x = 2 \quad \Leftrightarrow \quad x = 1$

210. Solve for *x*. $x^{\log 25} + 25^{\log x} = 10$

$x^{\log 25} + 25^{\log x} = 10$. The two terms on the lhs are equal, since:

$\log(x^{\log 25}) = \log 25 \log x = \log x \log 25 = \log 25^{\log x}$, so we have :

$2x^{\log 25} = 10 \quad \Leftrightarrow \quad x^{\log 25} = 5 \quad \Leftrightarrow \quad \log 25 \log x = \log 5 \quad \log x = \frac{\log 5}{2\log 5} = \frac{1}{2} \quad \Rightarrow \quad x = \sqrt{10}$

211. $x\sqrt{x} - 11\sqrt{x} = 10$**. Determine an expression for** $x - \sqrt{x}$

$x\sqrt{x} - 11\sqrt{x} = 10$. We put: $y = \sqrt{x}$, and then we have: $y^3 - 11y - 10 = 0$.

At a glance we see that $y = -1$ is a solution and we make polynomial division with: $y + 1$

$y+1 \mid y^3 - 11y - 10 \mid y^2 - y - 10$

$y^3 + y^2$

$-y^2 - 11y$

$-y^2 - y$

$-10y - 10$

$-10y - 10$

$y^2 - y - 10 = 0 \quad \Leftrightarrow \quad y^2 - y = 10 \quad \Leftrightarrow \quad x - \sqrt{x} = 10$

212. Evaluate the integral: $\int_0^{\frac{3}{4}} \frac{\sin\sqrt{1-x^2}}{\sqrt{1-x^2}}dx$

$\int_0^{\frac{3}{4}} \frac{\sin\sqrt{1-x^2}}{\sqrt{1-x^2}}dx$, we put $u=\sqrt{1-x^2} \quad \Rightarrow \quad du = \frac{1}{2\sqrt{1-x^2}}dx \;= \frac{1}{2u}du$

$$\int_0^{\frac{3}{4}} \frac{\sin\sqrt{1-x^2}}{\sqrt{1-x^2}}dx = \tfrac{1}{2}\int_1^{x=\frac{3}{4}} \frac{\sin u}{u}du$$

This integral has no analytic solution, but (as far as I remember) $\int_1^x \frac{\sin u}{u}du$ has a name.

I have however proved in the article on Laurent-series and contour inttegrals: in www.olewitthansen.dk that

$$\int_1^{\infty} \frac{\sin u}{u}du = \frac{\pi}{2}$$

I therefore find this exercise non appropriate, in the context of Olympic mathematical problems.

213. Verify that: $\int_{-1}^{1}\sqrt{1-x^2}\,dx = \int_{-1}^{1}\frac{1}{1+x^2}dx$

$\int_{-1}^{1}\sqrt{1-x^2}\,dx$. We put

$x=\cos t \quad \Rightarrow \quad \sqrt{1-x^2}=\sin t \quad dx=-\sin t\,dt \quad x=-1 \quad \Rightarrow \quad t=\pi \quad and \quad x=1 \quad \Rightarrow \quad t=0$

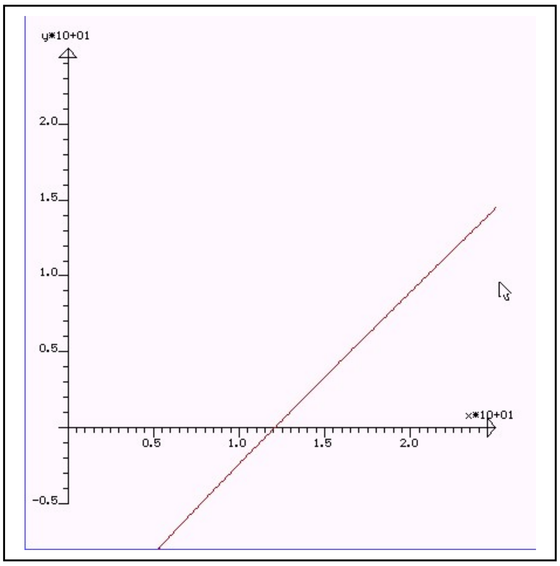
$$\int_{-1}^{1}\sqrt{1-x^2}\,dx = \int_0^{\pi}\sin t\sin t\,dt = \int_0^{\pi}\sin^2 t\,dt = \int_0^{\pi}\frac{1-\cos 2t}{2}dt = \left[\frac{t}{2}-\frac{\sin 2t}{4}\right]_0^{\pi} = \frac{\pi}{2}$$

$$\int_{-1}^{1}\frac{1}{1+x^2}dx = \left[\tan^{-1}x\right]_{-1}^{1} = \frac{\pi}{4}-(-\frac{\pi}{4}) = \frac{\pi}{2}$$

214. Solve $x+\sqrt{x+\sqrt{x}} = 16$

It is not difficult to verify that there are no integer solutions to this equation. Neither have $\sqrt{x+\sqrt{x}}$ no integer solution

If we put $y = x+\sqrt{x}$, we end up with an equation: $y^2-\sqrt{x}+y=16$, which leads nowhere:
A graph shows why. It has an intersection about 12.06



215. Solve: $2^{\frac{2x-1}{x-1}}+2^{\frac{3x-2}{x-1}}=24$

$2^{\frac{2x-1}{x-1}}+2^{\frac{3x-2}{x-1}}=24 \quad\Leftrightarrow\quad 2^{\frac{x+x-1}{x-1}}+2^{\frac{x+2x-2}{x-1}}=24 \quad\Leftrightarrow\quad 2^{\frac{x}{x-1}+\frac{x-1}{x-1}}+2^{\frac{x}{x-1}+\frac{2x-2}{x-1}}=24 \quad\Leftrightarrow$

$2^{\frac{x}{x-1}+1}+2^{\frac{x}{x-1}+2}=24 \quad\Leftrightarrow\quad 2\cdot 2^{\frac{x}{x-1}}+4\cdot 2^{\frac{x}{x-1}}=24$

We put $y=\dfrac{x}{x-1}$.

$2\cdot 2^{y}+4\cdot 2^{y}=24 \quad\Leftrightarrow\quad 6\cdot 2^{y}=24 \quad\Leftrightarrow\quad 2^{y}=4 \quad\Leftrightarrow\quad y=2 \quad \dfrac{x}{x-1}=2 \quad\Leftrightarrow\quad x=2$

216. Solve: $x^4-4x-1=0$

$x^4-4x-1=0$. We notice that: $(x^2+1)^2=x^4+1+2x^2 \quad\Leftrightarrow\quad x^4=(x^2+1)^2-1-2x^2$

Is inserted in: $x^4-4x-1=0$ to give:

$(x^2+1)^2-1-2x^2-4x-1=0 \quad\Leftrightarrow\quad (x^2+1)^2-2(x^2+2x+1)=0$

$(x^2+1)^2-2(x+1)^2=0 \quad\Leftrightarrow\quad (x^2+1)^2=2(x+1)^2 \quad\Leftrightarrow$

$(x^2+1)=\sqrt{2}(x+1) \quad\Leftrightarrow\quad x^2-\sqrt{2}x-\sqrt{2}+1=0$

$d=2+4(\sqrt{2}-1)=4\sqrt{2}-2.$

$$x=\frac{\sqrt{2}\pm\sqrt{4\sqrt{2}-2}}{2}$$

217. Solve for *x*: $x^2+\left(\dfrac{x}{1+x}\right)^2=15$

$x^2+\left(\dfrac{x}{1+x}\right)^2=15 \quad\Leftrightarrow\quad x^2\left(1+\dfrac{1}{(1+x)^2}\right)=15 \quad\Leftrightarrow\quad \left(\dfrac{x^2}{(1+x)^2}\right)\left((1+x)^2+\dfrac{1}{(1+x)^2}\right)=15$

$$x^2\left(1+\frac{1}{(1+x)^2}\right)=15 \quad\Leftrightarrow\quad x^2\left(\frac{(1+x)^2+1}{(1+x)^2}\right)=15 \quad\Leftrightarrow\quad x^2\left(\frac{x^2+2x+1+1}{(1+x)^2}\right)=15 \quad\Leftrightarrow$$

$$x^2\left(\frac{x^2}{(1+x)^2}+\frac{2(x+1)}{(1+x)^2}\right)=15 \quad\Leftrightarrow\quad x^2\left(\frac{x^2}{(1+x)^2}+\frac{2}{(1+x)}\right)=15 \quad\Leftrightarrow$$

$$\frac{x^4}{(1+x)^2}+\frac{2x^2}{(1+x)}=15 \quad\Leftrightarrow\quad \left(\frac{x^2}{(1+x)}\right)^2+2\frac{x^2}{(1+x)}-15=0$$

We put: $y=\frac{x^2}{(1+x)}$ and we thus find:

$$y^2+2y-15=0; \quad d=2+4\cdot 15=64; \quad y=\frac{-2\pm 8}{2} \quad\Leftrightarrow\quad y=-5 \quad\vee\quad y=3$$

$$\frac{x^2}{(1+x)}=-5 \quad\vee\quad \frac{x^2}{(1+x)}=3 \quad\Leftrightarrow\quad x^2+5x+5=0 \quad\vee\quad x^2-3x-3=0$$

$$d=25-20=5 \quad or \quad d=9+12=21; \quad x=\frac{-5\pm\sqrt{5}}{2} \quad\vee\quad x=\frac{3\pm\sqrt{21}}{2}$$

218. Simplify: $\frac{5}{\sqrt[3]{25}}$ (Very easy)

$$\frac{5}{\sqrt[3]{25}}=\frac{\sqrt[3]{125}}{\sqrt[3]{25}}=\sqrt[3]{5}$$

219. Solve for *x*: $\sqrt{x^1}+\sqrt{x^2}=\sqrt{x^3}+\sqrt{x^4}$

$\sqrt{x^1}+\sqrt{x^2}=\sqrt{x^3}+\sqrt{x^4}$.

We put: $y=\sqrt{x}$, and we then have:

$$y+y^2=y^{3\prime}+y^4 \quad\Leftrightarrow\quad y(y+1)=y^3(y+1) \quad\Leftrightarrow$$

$$y=y^3 \quad y^2=1 \quad\vee\quad y=0 \quad\Leftrightarrow\quad y=0 \quad\vee\quad y=1.$$

$$\sqrt{x}=0 \quad\vee\quad \sqrt{x}=1 \quad x=0 \quad\vee\quad x=1$$

220. Solve for *x*: $(2+\sqrt{3})^x+(2-\sqrt{3})^x=4$ (Simple).

$(2+\sqrt{3})^x+(2-\sqrt{3})^x=4$. We notice that:

$$(2+\sqrt{3})^x(2-\sqrt{3})^x=((2+\sqrt{3})(2-\sqrt{3}))^x=(4-3)^x=1^x=1$$

We put: $a=(2+\sqrt{3})^x$ *and* $b=(2-\sqrt{3})^x$, and we thus have:

$$a+b=4 \quad ab=1 \quad\Rightarrow\quad a+\frac{1}{a}=4 \quad\Leftrightarrow\quad a^2-4a+1=0; \quad d=16-4=12$$

$$a=\frac{4\pm 2\sqrt{3}}{2}=2\pm\sqrt{3} \quad and \quad b=\frac{1}{a}=2\mp\sqrt{3}.$$

$$a=(2\pm\sqrt{3})^x \quad\Leftrightarrow\quad 2\pm\sqrt{3}=(2\pm\sqrt{3})^x \quad\Leftrightarrow\quad x=1$$

221. Solve for *x,y*: $6^x + 6^y = 42$, *and* $x + y = 3$

$6^x + 6^y = 42$, *and* $x + y = 3 \quad \Rightarrow \quad 6^{x+y} = 6^3 \quad \Leftrightarrow \quad 6^x \cdot 6^y = 216$.

We put: $a = 6^x$ *and* $b = 6^y$ The two equations are then:

$$a + b = 42 \quad and \quad ab = 216 \quad \Rightarrow \quad a + \frac{216}{a} = 42 \quad \Leftrightarrow \quad a^2 - 42a + 216 = 0 \ ; d = 42^2 - 4 \cdot 216 = 900$$

$$a = \frac{42 \pm 30}{2} \quad \Leftrightarrow \quad a = 36 \quad \vee \quad a = 6 \quad \Leftrightarrow \quad x = 2 \quad \vee \quad x = 1$$

222. $3^{2x} = 2^{3x} = 5184$

First we notice that $5184 = 72^2$.

$$3^{2x} = (3^2)^x = (2^3)^x = 5184 \quad \Leftrightarrow \quad 9^x = 8^x = 72^2 \quad \Rightarrow$$

$$9 = (72^2)^{\frac{1}{x}} \quad 8 = (72^2)^{\frac{1}{y}} \quad \Rightarrow \quad 8 \cdot 9 = (72^2)^{\frac{1}{x}+\frac{1}{y}} \quad \Leftrightarrow \quad 72 = (72^2)^{\frac{x+y}{xy}} \quad \Leftrightarrow$$

$$72^{\frac{xy}{x+y}} = 72^2 \quad \Leftrightarrow \quad \frac{xy}{x+y} = 2$$