Numerical methods

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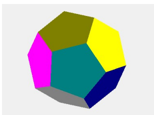

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7. Numerical solution to differential equations

It is actually the fewer differential equations in physics, which have an analytical solution.
Analytical means that the solution is given by mathematical expressions that describes the position and velocity at any time t.
The mathematical discipline which deals with numerical, that is, step by step solutions to problems is called numerical analysis.
Theoretically numerically analysis covers a large area of methods, and in contras to what one may think it is developed long before the invention of computers.

However, one should not diminish the importance of analytical solutions, even if the small mathematics computers, which has already been in use in high schools for almost ten years now (2016) are able to do differentiation, evaluate integrals and solve differential equations analytically, that my generation had to do by hand.
But of course the computers can do no more, than (some of us) could do by hand.

The alternative to analytical solutions are numerical solutions, where one roughly speaking replaces infinitesimal entities dx and dy by small changes Δx, Δt, differential quotients $\frac{dx}{dt}$ by $\frac{\Delta x}{\Delta t}$ and integrals $\int f(t)dt$, by sums $\Sigma f(t_i)\cdot\Delta t_i$
But the theory of numerical analysis relies heavily on analytic methods of course!

7.1 Taylor's formula

We shall begin by looking at numerical solution to first order differential equations.
When doing things numerically, it is imperative to be able to estimate the accuracy.
To do so, it is strictly necessary to be familiar with Taylor's formula. The formula can be written in numerous ways, but we shall apply the one where a real function $y = f(x)$ is developed around a point x_0, and h is (a small) increment to x_0. Under fairly general circumstances the formula can be written:

(7.1) $$f(x_0+h) = f(x_0) + \frac{f'(x_0)}{1!}h + \frac{f''(x_0)}{2!}h^2 + \frac{f^{(3)}(x_0)}{3!}h^3 + \ldots + \frac{f^{(n)}(x_0)}{n!}h^n + \int_0^h \frac{f^{(n+1)}(x_0+t)}{n!}t^n dt$$

The last term (the rest term) is seen to be proportional to h^{n+1}, which we write as $O(h^n)h$, where the symbol $O(h^n)$ is to be read as "order of h^n ". If we omit the rest term, we get an approximation to $f(x_0+h)$. Depending how many terms we include, we get an 0'th, 1., 2., order approximation.

$$f(x_0+h) = f(x_0) + O(h^0)h$$
$$f(x_0+h) \approx f(x_0)$$

(7.2) $$f(x_0+h) = f(x_0) + f'(x_0)h + O(h)h$$
$$f(x_0+h) \approx f(x_0) + f'(x_0)h$$

(7.3) $$f(x_0+h) = f(x_0) + f'(x_0)h + \tfrac{1}{2}f''(x_0)h^2 + O(h^2)h$$
$$f(x_0+h) \approx f(x_0) + f'(x_0)h + \tfrac{1}{2}f''(x_0)h^2$$

(7.4) $$f(x_0+h) = f(x_0) + f'(x_0)h + \tfrac{1}{2}f''(x_0)h^2 + \tfrac{1}{6}f^{(3)}(x_0)h^3 + O(h^3)h$$
$$f(x_0+h) \approx f(x_0) + f'(x_0)h + \tfrac{1}{2}f''(x_0)h^2 + \tfrac{1}{6}f^{(3)}(x_0)h^3$$

7.2 Numerical solution to first order differential equations

If we wish to solve the firs order equation

(7.5) $$\frac{dy}{dx} = f'(x) = g(x,y)$$

Where we know the initial value (x_0, y_0), then it can be done by applying (7.1), since:

(7.6) $$f(x_0+h) \approx f(x_0) + f'(x_0)h = y_0 + g(x_0,y_0)h$$

Where $f(x_0+h)$ is calculated with an accuracy of order h^2.

$$(x_1, y_1) = (x_0+h, f(x_0+h)) = (x_0+h, f(x_0)+f'(x_0)h) = (x_1, y_0+g(x_0, y_0))$$

The new value can thereafter be used to calculate:

$$(x_2, y_2) = (x_1+h, f(x_1+h)) = (x_1+h, f(x_1)+f'(x_1)h) = (x_1+h, y_1+g(x_1, y_1)\,h)$$

and so on. The method is called Euler integration.
Euler's formula is however hardly ever used, because the errors accumulate, especially when $f''(x)$ is constant. To obtain a better approximation, one may use Aitken's formula:

(7.5) $$f'(x_0) \approx \frac{f(x_0+\frac{h}{2}) - f(x_0-\frac{h}{2})}{h} \Leftrightarrow f(x_0+\tfrac{h}{2}) = f(x_0-\tfrac{h}{2}) + f'(x_0)h$$

Expanding $f(x_0+\frac{h}{2}) - f(x_0-\frac{h}{2})$ by Taylor's formula, we find this time:

$$f(x_0+\tfrac{h}{2}) - f(x_0-\tfrac{h}{2}) = f(x_0) + f'(x_0)\tfrac{h}{2} + \tfrac{1}{2}f''(x_0)\tfrac{h^2}{4} - (f(x_0) + f'(x_0)(-\tfrac{h}{2}) + \tfrac{1}{2}f''(x_0)\tfrac{h^2}{4}) + O(h^2)h$$

(7.6) $$f(x_0+\tfrac{h}{2}) - f(x_0-\tfrac{h}{2}) = f'(x_0)h + O(h^2)h$$

As it is seen this formula is correct until order h^3, in contrast to Euler's h^2 approximation, but using the same first order derivatives.
If $h = \frac{1}{10}$, then is the correction (the error) of order $h^3 = \frac{1}{1000}$ i stead of the Euler integration, where the correction is of order $h^2 = \frac{1}{100}$. Otherwise the procedure is the same as the Euler integration, apart from the initial step, where one must calculate $f(x_0+\dfrac{h}{2})$, by Euler's method.

(7.7) $$f(x_0+\frac{h}{2}) = f(x_0-\frac{h}{2}) + f'(x_0)h = f(x_0-\frac{h}{2}) + g(x_0,y_0)h$$

To scientific and practical purposes, one applies almost always the method of Runge-Kutta, which is far more complicated to account for, than the methods described above. But it has the definite advantage that the corrections (the errors) is of order h^4.