Examples of Markov chains

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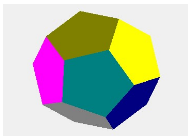

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Introduction

The theory of Markov chains is founded on the elementary part of probability. If you are not familiar with probability theory, you may for example be referred to the first four chapters of: **http://www.olewitthansen.dk/Mathematics/Probability_theory.pdf**.
An introduction to linear algebra is found in:
http://www.olewitthansen.dk/Mathematics/Eigenvalue_problems_in_**linear_algebra.pdf**

1. Definition and properties of a Markov chain

We consider a system, which may be in one of (1), (2),(3)…(n) states
The state of the system is a stochastic variable X, such that the probability the system is in the state j is: $P(X = j) = p_j$. The system changes its state stochastically, with a constant finite time interval.

The system is at every instant given by a state vector $(p_1, p_2, p_3, \ldots, p_n)$, where p_j is the probability, that the system is in the state j. Since the system must be in one of the states $1..n$ we have :

$$p_1 + p_2 + p_3 + \ldots + p_n = 1$$

The probability that the state i transits into the state j is $P(X = i \rightarrow X = j) = p_{i\,j}$.
A Markov chain is a system, which has no history. The probability of a transition of the state i to the state j depends only on the state i and the state j, but not on the states presiding the state i. Written more formally:

$$P(X_{k+1} = j_{k+1} | P(X_k = j_k, X_{k-1} = j_{k-1}, X_{k-2} = j_{k-2}, \ldots, X_0 = j_0) = P(X_{k+1} = j_{k+1} | X_k = j_k)$$

The conditional probability that the system reaches the state j_{k+1} depends only on the immediate state before, and not how the system reached that state.

An ordinary walk is not a Markov chain, since your position when you take the next step (in most circumstances) depends on some, if not all, the preceding steps you have taken.
A counterexample is the so called random walk, which we shall deal with later.

The probabilities $P(X = i \rightarrow X = j) = p_{i\,j}$ form a transition matrix. If the system has n possible states, then the transition must result to one of these states, therefore the sum of transition possibilities from a given state must add to one.

$$\sum_{j=1}^{j=n} p_{i\,j} = 1 \qquad for \;\; i = 1..n$$

The transition matrix, may be written

$$\begin{pmatrix} p_{11} & p_{12} & .. & p_{1,n-1} & p_{1,n} \\ p_{21} & p_{22} & .. & p_{2,n-1} & p_{2,n} \\ .. & ... & ... & ... & ... \\ p_{n-1,1} & p_{n-1,2} & \cdots & p_{n-1,n-1} & p_{n-1,n} \\ p_{n,1} & p_{n,2} & \cdots & p_{n,n-1} & p_{n,n} \end{pmatrix}$$

The probability distribution of being in a state *s* is written as a row vector. $(q_1, q_2,.., q_n)$.
The transition to a new state is done by matrix multiplication with the transition matrix.
If $(r_1, r_2,.., r_n)$ is the probability distribution for the new state then

$$(q_1, q_2,..., q_n)\begin{pmatrix} p_{11} & p_{12} & .. & p_{1,n-1} & p_{1,n} \\ p_{21} & p_{22} & .. & p_{2,n-1} & p_{2,n} \\ .. & ... & ... & ... & ... \\ p_{n-1,1} & p_{n-1,2} & \cdots & p_{n-1,n-1} & p_{n-1,n} \\ p_{n,1} & p_{n,2} & \cdots & p_{n,n-1} & p_{n,n} \end{pmatrix} = (r_1, r_2, ... r_n)$$

Where we have: $r_j = q_1 p_{1j} + q_2 p_{2j} + ... + q_n p_{nj}$.
The probability $q_i p_{ij}$ is the probability that the system is in state *i* times the probability of a transition to a state *j*. Written in a more compact form:

$$r_j = \sum_{i=1}^{n} q_i p_{ij} \quad j = 1..n$$

Before we go on with the theory we shall present some simple examples.

2. Conditions for a Markov-chain to be regular

In a regular Markov-chain, the sum of the probabilities of transition states should remain equal to one after each iteration. This was also the case in the examples displayed below.

First we shall look into this fact mathematically.

A Markov chain is called regular if the sum probability distribution in each subsequent state equals to one.
In the example with the random walk this is actually easy to see. The transition matrix is:

$$\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

And a probability state state: (s_1, s_2) is transformed into:

$$(s_1, s_2)\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} = (s_1 p + s_2(1-p), s_1(1-p) + s_2 p)$$

Taking the sum of the state probabilities:

$$(s_1 p + s_2(1-p) + s_1(1-p) + s_2 p) = s_1(p + (1-p)) + s_2((1-p)p + p) = s_1 + s_2 = 1.$$

Doing the same calculation for a *n* x *n* matrix is a bit more circumstantial, but we shall show the same result for a 3x3 Markov matrix.

$$(s_1, s_2, s_3)\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} = (s_1 p_{11} + s_2 p_{21} + s_3 p_{31}\ ,\ s_1 p_{12} + s_2 p_{22} + s_3 p_{32},\ \ s_1 p_{13} + s_2 p_{23} + s_3 p_{33})$$

And taking the sum:

$$(s_1 p_{11} + s_2 p_{21} + s_3 p_{31} + s_1 p_{12} + s_2 p_{22} + s_3 p_{32} + \ s_1 p_{13} + s_2 p_{23} + s_3 p_{33})$$

Taking $s_{1,}\ s_{2,,} s_3$ as a common factor for the three terms, we find;

$$s_1(p_{11} + p_{12} + p_{13}) + s_2(p_{21} + p_{22} + p_{23}) + s_3(p_{31} + p_{32} + p_{33}) = s_1 + s_2 + s_3$$

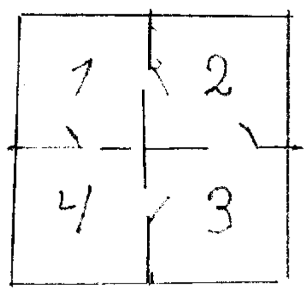
We can then see that the sum of probabilities in the transformed state vector is equal to the sum of probabilities in the original state vector, if and only if the sum of probabilities in all three rows of the transition matrix equals one.
We have secured that when the sum of probabilities in each row equals to one, then the sum of transitions probabilities also equals one.

All transitions matrices have the eigenvalue 1 and the eigenvector (1,1,1..,1) from the right
It is a simple fact, although it seldom applies in the analyze of Markov chains that all Markov matrices have the eigenvector (1,1,1,..,1)) with the eigenvalue 1 from the right.
We shall satisfy ourselves by showing that it holds for 3 x3 matrices.

$$\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} p_{11} + p_{12} + p_{13} \\ p_{21} + p_{22} + p_{33} \\ p_{31} + p_{32} + p_{33} \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Reducible and irreducible Markov chains.
A Markov chain is denoted <u>irreducible</u>, if any state can be reached from any other state. Otherwise it is called reducible. In most cases it is important that the Markov chain in question is irreducible. If not it may be decomposed into two or more irreducible Markov chains.
The irreducibility can often be decided from the transition matrix or rather from the physical phenomenon it represents.

3. Examples: The cat and the mouse in the maze



The figure to the left shows a "maze" having 4 compartments. In each compartments, there are two exits to the neighbouring compartments.
Let us assume that the cat initially is in (1). We also assume that the cat exit to any the two neighbour compartments both with probability $\frac{1}{2}$. We also assume, that at every step the compartment is shifted. Thus, we have for example:

$$P(1 \to 2) = \tfrac{1}{2} \quad and \quad P(1 \to 3) = 0$$

The 4x4 transition matrix T becomes.

$$T = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Let us assume that the cat initially is in compartment (1).
The state probabilities are then: $s_0 = (1,0,0,0)$
To find the state probabilities after one step, we multiply with the transition matrix T.

1. step. $$(1,0,0,0)\begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} = (0,\tfrac{1}{2},0,\tfrac{1}{2})$$

2. step. $$(0,\tfrac{1}{2},0,\tfrac{1}{2}) * T = (\tfrac{1}{4}+\tfrac{1}{4},0,\tfrac{1}{4}+\tfrac{1}{4},0) = (\tfrac{1}{2},0,\tfrac{1}{2},0)$$

3.step. $$(\tfrac{1}{2},0,\tfrac{1}{2},0) \bullet T = (0,\tfrac{1}{2},0,\tfrac{1}{2}).$$

We notice that the transition probabilities between the states are cyclic with period 2.
Now we shall do the same from the point of view of the mouse.
The Initial state for the mouse is (0,0,1,0).

1. step. $(0,0,1,0) \bullet T = (0,\frac{1}{2},0,\frac{1}{2})$
2. step: $((0,\frac{1}{2},0,\frac{1}{2}) \bullet T = (\frac{1}{2},0,\frac{1}{2},0)$
3. step: $(\frac{1}{2},0,\frac{1}{2},0) \bullet T = (0,\frac{1}{2},0,\frac{1}{2})$

Not surprisingly the sequence of states for the mouse is also cyclic with period 2. We notice that in the states (1) and (3), the cat and the mouse have the same probability distribution.
Therefore the probability that the cat catches the mouse is equal to: $(0,\frac{1}{2},0,\frac{1}{2}) \cdot (0,\frac{1}{2},0,\frac{1}{2}) = \frac{1}{2}$.

We may alter the transitions probabilities, such that the cat (and the mouse), have the probability $\frac{1}{2}$ of staying in the compartment and the probability $\frac{1}{4}$ of leaving through one of the two exits.
The transition table then becomes:

$$(1,0,0,0)\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = (\tfrac{1}{2},\tfrac{1}{4},0,\tfrac{1}{4})$$

$$(\tfrac{1}{2},\tfrac{1}{4},0,\tfrac{1}{4})\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = (\tfrac{3}{8},\tfrac{2}{8},\tfrac{1}{8},\tfrac{2}{8})$$

$$(\tfrac{3}{8},\tfrac{2}{8},\tfrac{1}{8},\tfrac{2}{8})\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = (\tfrac{10}{32},\tfrac{8}{32},\tfrac{6}{32},\tfrac{8}{32})$$

And similarly for the mouse:

$$(0,0,1,0)\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = (0,\tfrac{1}{4},\tfrac{1}{2},\tfrac{1}{4})$$

$$(0,\tfrac{1}{4},\tfrac{1}{2},\tfrac{1}{4})\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = (\tfrac{1}{8},\tfrac{3}{8},\tfrac{2}{8},\tfrac{2}{8})$$

$$(\tfrac{1}{8},\tfrac{3}{8},\tfrac{2}{8},\tfrac{2}{8})\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = (\tfrac{7}{32},\tfrac{9}{32},\tfrac{9}{32},\tfrac{7}{32})$$

If we should calculate the possibility that the cat and the mouse are in the same compartment, say after 3 steps we must calculate:

$$(\tfrac{7}{32},\tfrac{9}{32},\tfrac{9}{32},\tfrac{7}{32}) \bullet (\tfrac{10}{32},\tfrac{8}{32},\tfrac{6}{32},\tfrac{8}{32}) = 63/256\,.$$

4. Examples: Choice of a brand of goods

This is rather a mathematical example which hardly may be found in reality.
Let us assume that customers buy the same commodity, which is delivered in three different brands *A*, *B* and *C*. We also assume that this is given by the transition table below.
If the consumers one week have used the brand *A, B* or *C* then probabilities of changing to another brand are the following:

$$P(A \to A) = 0.85, P(A \to B) = 0.10 \;\; and \;\; P(A \to C) = 0.05\,.$$
$$P(B \to A) = 0.15, P(B \to B) = 0.80 \;\; and \;\; P(C \to C) = 0.05$$
$$P(C \to A) = 0.20, P(C \to B) = 0.10 \;\; and \;\; P(C \to C) = 0.65$$

Let us assume that three customers buy the commodity with an interval of one week, then we may describe the "system" as a Markov chain, having the transition matrix.

$$\begin{matrix} A & B & C \end{matrix}$$
$$\begin{pmatrix} 0.85 & 0.10 & 0.05 \\ 0.15 & 0.80 & 0.05 \\ 0.20 & 0.15 & 0.65 \end{pmatrix}$$

We can see that the customers (1) (2) and (3) favour the brand *A, B, C*, respectively, and we are interested in whether, this line of habit change from the Markov probability point of view.
For example will the product *C* eventually be deselected? Will the state probabilities eventually become a stationary state?
The latter is one of the main results of certain classes in the theory of Markov chains.
Below is shown a computer simulation of the situation.
We assume that the demand for the three brands initially have the following probabilities:

A	*B*	*C*
0.200	0.300	0.500

```
Probability table

 0.850   0.100   0.050
 0.150   0.800   0.050
 0.150   0.800   0.050

initial state:

 0.200   0.300   0.500
Sum of probabilities in initial state=  1.000

Probabilities for new state 1
 0.290   0.660   0.050
sum of probabilities for state  1   1.000

Probabilities for new state:   2
 0.353   0.597   0.050
sum of probabilities for state  2   1.000

Probabilities for new state:   3
```

```
 0.397   0.553   0.050
sum of probabilities for state  3   1.000

Probabilities for new state:  15
 0.499   0.451   0.050
sum of probabilities for state  15   1.000

Probabilities for new state:  16
 0.499   0.451   0.050
sum of probabilities for state  16   1.000
```

From the tenth iteration it is plausible that: (0.499, 0.451, 0.053) is in fact a stationary state.

5. Example. The random walk.

The random walk appears frequently in statistical mechanics, but there, it is often a random walk in two or three dimensions. See for example: www.olewitthansen.dk/Physics/Brownian_movement ,

In the context of Markov chains, we shall only consider random walks in one dimension,
There are only two states: Forward and backward, with probabilities p and 1 - p (The next step is forward or the next state is backward)..

The transition matrix is thus very simple. $\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$

The initial state we put to (1,0) assuming that this state belongs to the position 0.
The next state goes to the right or the left.

$$(1,0)\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} = (p, 1-p)$$

And after two steps:

$$(p, 1-p)\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} = (p^2 + (1-p)^2,\ p(1-p) + p(1-p) = (p^2 + (1-p)^2, 2p(1-p))$$

The sum of probabilities should add to one: First step: $(p + 1 - p) = 1$, and for the second step:

$$(p^2 + (1-p)^2 + \ 2p(1-p)) = (p + (1-p)^2 = 1$$

Indicating that the sum of probabilities has a binomial distribution.

We confine ourselves to prove it for the third step.

$$(p^2 + (1-p)^2, 2p(1-p))\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} =$$

$$(p^3 + p(1-p)^2 + 2p(1-p)^2\ ,\ p^2(1-p) + (1-p)^3 + 2p^2(1-p)$$

If we take the sum of the probabilities and rearrange the terms we can see;

$$p^3 + 3p^2(1-p) + 3p(1-p)^2 + (1-p)^3 = (p + (1-p))^3 = 1$$

So when *n* steps has been taken the probabilities of having taking *j* steps to the right and *n* - *j* steps to the left has a binomial distribution:

$$\begin{pmatrix} j \\ n-j \end{pmatrix} p^j (1-p)^{n-j}$$

Or if $p = \frac{1}{2}$ we have

$$\begin{pmatrix} j \\ n-j \end{pmatrix} \left(\frac{1}{2}\right)^n$$

If we want the mean distance taken in one step, we may consider a stochastic variable *X*: where $X(0) = 1$. Then we have if the position is *j*:

$$P(X(j \to j+1) = p \quad \text{and} \quad P(X(j \to j-1) = 1-p$$

The mean value after one step is thus: $E(X) = (j+1)p + (j-1)(1-p) = j + 2p - 1$
If $p = \frac{1}{2}$ then $E(X) = j$, as it should be.

We shall then calculate the variance $\sigma^2(1)$ on one step. Since the steps are independent of each other we have $\sigma^2(n) = n\sigma^2(1)$
$\sigma^2(1) = E(X^2) - E(X)^2 = (+1)^2 p + (-1)^2(1-p) - (2p-1)^2 = -4p(p-1)$
Not so transparent, however, but if we put $p = \frac{1}{2}$ then $\sigma^2(1)$=1.
But $\sigma^2(1) = 1$, implies that $\sigma^2(n) = n\sigma^2(1)$= *n*, So the standard variation after *n* steps is $\sigma(n) = \sqrt{n}$.
The last result is a general conclusion, also for a random walk also in two or three dimensions.

6. Example: The length of a queue

Let us assume that we have a queue to the counter in a bank. The length of the queue changes in a fixed interval, as the queue increases with one, when one person enters the queue with probability *p*. Or the length of the queue decreases with one, because a customer leaves the counter.
The problem is essentially the same as the random walk.

We are dealing with two states of the queue. It increases with one customer with probability $P(j \to j+1) = p$ or the queue decrease by one customer with probability: $P(j \to j-1) = 1-p$.

But this corresponds to the transition of the random walk. We assume initially the queue has a length *j*, corresponding to the state (1, 0). Then the transition matrix becomes.

$$\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

After 3 steps the sum of state vector probabilities becomes.

$$p^3 + 3p^2(1-p) + 3p(1-p)^2 + (1-p)^3$$

We recognize it as a binomial probability distribution. Below we have made a table showing the probability distribution of the length of the queue after 3 steps.

+3	+2	+1	0	-1	-2	-3
p^3	0	$3p^2(1-p)$	0	$3p(1-p)^2$	0	$(1-p)^3$

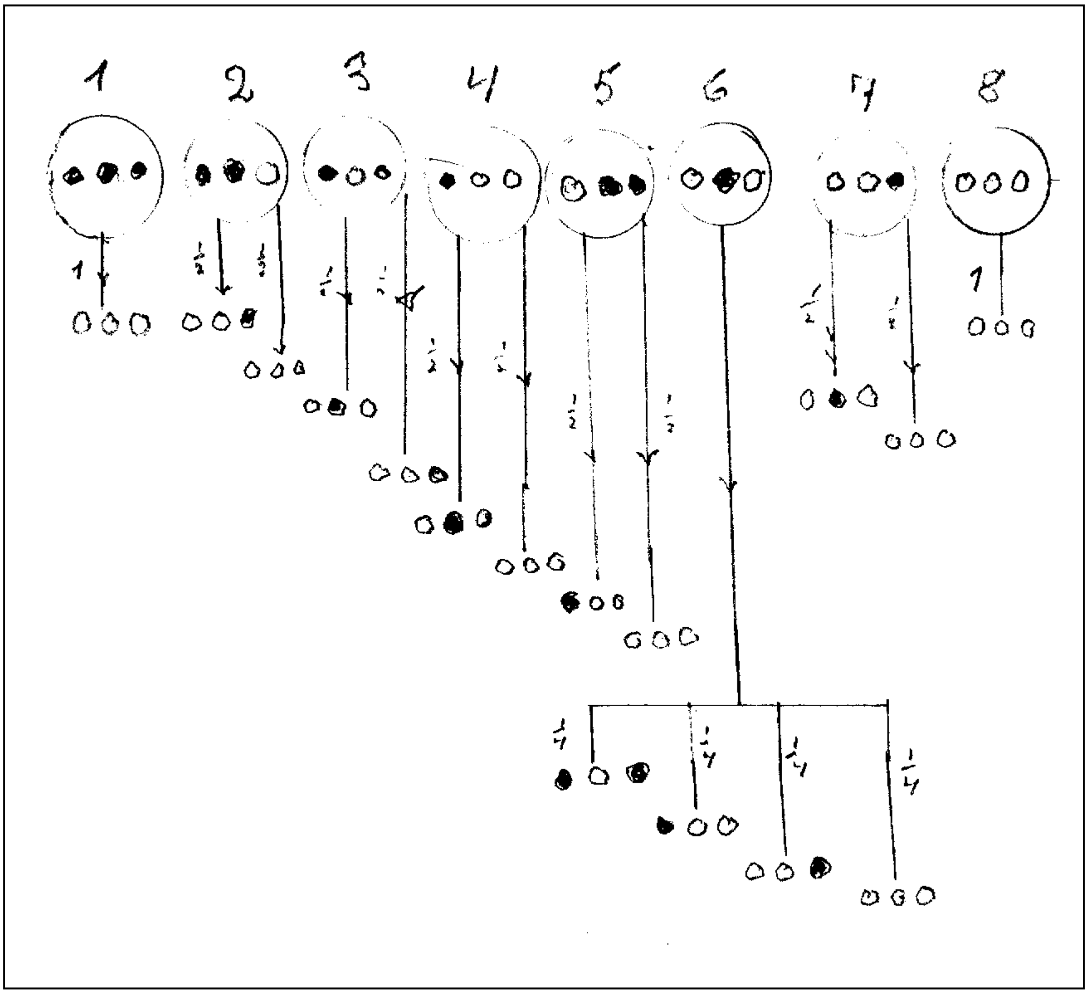
As for the mean and variance, it is the same as for the random walk.

7. Example: Prevalence of a contagious disease

We shall then look at a Markov chain as a "desk top model" for the prevalence of a contagious disease.
We consider three persons which can be either sick or healthy. A healthy person is marked with a small open circle, whereas a sick person is marked with a small filled circle.
The order of the three persons is respected. Since any person may be sick or healthy there are $2^3 = 8$ possible states denoted 1..8. The eight states are shown graphically below.
If the system has a given state at $t = n$ there is a transition to a new state at $t = n+1$, obeying the following rules.

1. If a person is sick at $t = n$, then he has recovered (is healthy) at $t = n+1$.
2. If a healthy person has a sick neighbour at $t = n$, then the person becomes sick at $t = n+1$ with probability p, and remains healthy with probability $1 - p$.
3. A healthy person with no sick neighbours remains healthy.

In the graphically representation below is indicated which states, along with the probability that any state can go into in a transition following the rules 1 – 3.
For example the state 4 (black, white, white), can either go into (white, black, white) or (white, white, white) with probabilities p and $1 - p$. In the figure below is chosen $p = ½$.



The transition probabilities are obtained by looking at the schema above.
For example the transition probability $P(4 \to 6) = 0.5$ and the transition probability $P(4 \to 8) = 0.5$ .All other transition probabilities from (4) are zero.

Below is shown the transition matrix for a computer simulation of the Markov chain for the contagious disease example. Since the sum of the transition probabilities in each row equals to one, it fulfils the conditions that the sum of the probabilities of the states remain the same (equal to one).

Below is shown the Markov transition table from the code of the computer simulation. In this computer run we have chosen $p = p12 = 0.5,\ q12 = 1 - p12,\ and\ \ p14 = p12/2$.
The choice of transition probabilities does, however, (within limits) not change the general picture.

```
((0.0,0.0,0.0,0.0,0.0,0.0,0.0,1.0),
(0.0,0.0,0.0,0.0,0.0,0.0,p12,p12),
(0.0,0.0,0.0,0.0,0.0,p12,0.0,q12),
(0.0,0.0,0.0,0.0,0.0,p12,0.0,q12),
(0.0,0.0,0.0,p12,0.0,0.0,0.0,q12),
(0.0,0.0,p14,p14,0.0,0.0,p14,p14),
(0.0,0.0,0.0,0.0,0.0,p12,0.0,q12),
(0.0,0.0,0.0,0.0,0.0,0.0,0.0,1.0));
```

Below is shown some of the output of the results the computer simulation of the first 5 transitions.

probability table

0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000
0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.500
0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500
0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500
0.000	0.000	0.000	0.500	0.000	0.000	0.000	0.500
0.000	0.000	0.250	0.250	0.000	0.000	0.250	0.250
0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500
0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000

initial state
0.400 0.200 0.100 0.200 0.100 0.000 0.000 0.000
sum of probabilities= 1.000

probabilities for state no 1
0.000 0.000 0.000 0.050 0.000 0.150 0.100 0.700
sum of probabilities= 1.000

probabilities for state no 2
0.000 0.000 0.038 0.038 0.000 0.075 0.038 0.813
sum of probabilities= 1.000

probabilities for state no 3
0.000 0.000 0.019 0.019 0.000 0.056 0.019 0.888
sum of probabilities= 1.000

probabilities for state no 4
0.000 0.000 0.014 0.014 0.000 0.028 0.014 0.930
sum of probabilities= 1.000

probabilities for state no 5
0.000 0.000 0.007 0.007 0.000 0.021 0.007 0.958
sum of probabilities= 1.000

probabilities for state no 6
0.000 0.000 0.005 0.005 0.000 0.011 0.005 0.974
sum of probabilities= 1.000

probabilities for state no 11
0.000 0.000 0.000 0.000 0.000 0.001 0.000 0.998
sum of probabilities= 1.000

We can see that already after three iterations, almost everyone has recovered, and we have reached a stationary state, where state (7) is an absorbing state – indeed.

8. Example: Balls in two bowls.

First we shall consider the (next) simplest case. Let us assume we have two bowls *A* and *B*, each having 3 balls. There are in all 3 red balls and 3 white balls.

We move from one state to another by taking a ball at random from each, and after switching the two balls they are replaced in the bowl. We shall consider the number of red balls in the bowl *A* as a stochastic variable with the values (0,1,2,3).
We shall then find the probabilities $P(X = j \mid X = i) = P(i \to j)$ for $i, j = 0,1,2,3$ in a systematic way. Generally we shall only write probabilities, which are different from zero.

$P(0 \to 0) = 0$. (Since all the red balls are in bowl *B*).
$P(0 \to 1) = 1$. (Since when you take a ball from *B*, it is red with certainty).
$P(1 \to 0) = \frac{1}{3}\frac{1}{3} = \frac{1}{9}$ (There is 1 red in *A*, and 1 white in *B*)
$P(1 \to 1) = \frac{1}{3}\frac{2}{3} + \frac{2}{3}\frac{1}{3} = \frac{4}{9}$ (Either you take one red from *A* and receive one red from *B* or you take one white and receive one white).
$P(1 \to 2) = \frac{2}{3}\frac{2}{3} = \frac{4}{9}$ (Take one white from *A* ($p = \frac{2}{3}$) and get one red from *B* ($p = \frac{2}{3}$).
$P(2 \to 1) = \frac{2}{3}\frac{2}{3} = \frac{4}{9}$ (You take one red ($p = \frac{2}{3}$) and receive one white ($p = \frac{2}{3}$)
$P(2 \to 2) = \frac{2}{3}\frac{1}{3} + \frac{2}{3}\frac{1}{3} = \frac{4}{9}$ (As above $P(1 \to 1)$)
$P(2 \to 3) = \frac{1}{9}$

Notice that in each case, the sum of probabilities $P(i \to j)$, $j = 0..3$ add up to one.

8.1 Results from a computer simulation

The initial Markov chain is started with three red balls in *A*,
From the printout below, we can see that after only three iterations the probability distribution is almost constant to

0 red	1 red	2 reds	3 reds
0.050	0.450	0.450	0.050

This may be considered, (but strictly speaking not proved), as a stationary distribution (or equivalently) an eigenvector to the transition matrix.

probability table

0.000	1.000	0.000	0.000
0.111	0.444	0.444	0.000
0.000	0.444	0.444	0.111
0.000	0.000	1.000	0.000

initial state
0.000 0.000 0.000 1.000

State no 1
0.000 0.000 1.000 0.000

State no 2
0.000 0.444 0.444 0.111

State no 3
0.049 0.395 0.506 0.049

State no 4
0.044 0.450 0.450 0.056

State no 5
0.050 0.444 0.456 0.050

State no 6
0.049 0.450 0.450 0.051

State no 7
0.050 0.449 0.451 0.050

State no 8
0.050 0.450 0.450 0.050

The last probability distribution may be considered as a stationary state.

9. Example. Balls in two bowls

We shall now look into a slightly different version of the diffusion problem above, as we now increase the number of white balls to four.
The probabilities differ only slightly from the previous example, although it has a different transition table. The elements in the transition table are as before $p_{ij} = P(X = j \mid X = i) = P(i \to j)$, where X is a stochastic variable, which denotes the number of red balls in bowl A.

$P(0 \to 0) = \frac{1}{4}$. (You must take a white ball from A, and since there are 4 balls in B three red and one white the probability of taking a white ball is ¼).
$P(0 \to 1) = \frac{3}{4}$ (Since there are 3 red balls in B).
$P(1 \to 0) = \frac{1}{3}\frac{1}{2} = \frac{1}{6}$ (There is one red ball in A, and two white balls in B).
$P(1 \to 1) = \frac{1}{3}\frac{1}{2} + \frac{2}{3}\frac{1}{2} = \frac{1}{2}$.(This can be obtained either from taking one red ball and getting one red ball or from taking one white ball and getting one white ball).
$P(1 \to 2) = \frac{2}{3}\frac{1}{2} = \frac{1}{3}$
$P(2 \to 0) = 0$
$P(2 \to 1) = \frac{2}{3}\frac{3}{4} = \frac{1}{2}$
$P(2 \to 2) = \frac{2}{3}\frac{1}{4} + \frac{1}{3}\frac{3}{4} = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$
$P(2 \to 3) = \frac{1}{3}\frac{1}{4} = \frac{1}{12}$

The circumstances in this example are almost the same, as in the previous example, so we restrict ourselves to show the results of a computer simulation.
Adding another white ball, seem to have little influence, since the probability distribution ends up with (becomes static) with the distribution (0,1,2,3) red balls. (0.05, 0.45,0 .45, 0.05)

<u>Transition Probability table for the number of red balls</u>

```
 0.000   1.000   0.000   0.000
 0.111   0.444   0.444   0.000
 0.000   0.444   0.444   0.111
 0.000   0.000   1.000   0.000

initial state

 0.000   0.000   0.000   1.000
sum of probabilities=  1.000

probabilities for state no  2
 0.000   0.444   0.444   0.111
sum of probabilities=  1.000

probabilities for state no  3
 0.049   0.395   0.506   0.049
sum of probabilities=  1.000

probabilities for state no  4
 0.044   0.450   0.450   0.056
sum of probabilities=  1.000

probabilities for state no  5
 0.050   0.444   0.456   0.050
sum of probabilities=  1.000

probabilities for state no  6
 0.049   0.450   0.450   0.051
sum of probabilities=  1.000

probabilities for state no  7
 0.050   0.449   0.451   0.050
sum of probabilities=  1.000

probabilities for state no  8
 0.050   0.450   0.450   0.050
sum of probabilities=  1.000
```

10. The general diffusion problem.

We shall then generalize the problem with 3 balls in each of two bowls, to n balls in each bowl. If we look at the A bowl, we have n balls. There are $2n$ in all and $2r$ red balls and $2w$ white balls. In A we have at every instant p red balls and q white balls, where $p + q = n$.
We shall the find the probabilities: $P(p \to p-1)$, $P(p \to p)$ and $P(p \to p+1)$, where p is the number of red balls in A.

$$P(p \to p-1) = \frac{p}{n}\frac{2w-q}{n}$$ (Takes one red, and receives one white)

$$P(p \to p) = \frac{p}{n}\frac{2r-p}{n} + \frac{q}{n}\frac{2w-q}{n}$$ (Takes one red and gets one red or takes one white and gets one white)

$$P(p \to p+1) = \frac{q}{n}\frac{2r-p}{n}$$

One may easily convince your self, that these formulas give the same result as we obtained earlier from the for the 3 balls example. There we had n = 3, and p = 0, 1, 2, 3 ; 2n =6, $q = n - p$. 2r = 3, 2w = 3. For example, using the general formula:

$$P(1 \to 0) = \frac{p}{n}\frac{2w-q}{n} = \frac{1}{3}\frac{1}{3} = \frac{1}{9} \quad P(1 \to 1) = \frac{p}{n}\frac{2r-p}{n} + \frac{q}{n}\frac{2w-q}{n} = \frac{1}{3}\frac{2}{3} + \frac{2}{3}\frac{1}{3} = \frac{4}{9}$$

$$P(1 \to 2) = \frac{q}{n}\frac{2r-p}{n} = \frac{2}{3}\frac{2}{3} = \frac{4}{9}$$

We may also compare the stationary distribution to a binomial distribution, where a ball is taken at random with probability ½. The probability distribution for the red ball, with 3 repetitions, is:

0	1	2	3
0.1250	0.3750	0.3750	0.1250

This stochastic probability distribution differs markedly from the (deterministic) Markov chain.

11. Example: A game about a crown. Absorbing states

We shall now consider a one crown game with two participants A and B. A coin is tossed and if it shows heads A wins one crown, otherwise B wins a crown. The game is initiated by A and B both having 3 crowns. The game ends when one of the participants has lost all of his crowns,
Let q the number of crowns that A have, so q can have the values (0, 1, 2, 3 , 4, 5, 6)

The probability of going from state q to state q +1 is p, (where p = ½), and otherwise and zero for all other transitions.

Likewise the probability of going from state q to state q - 1 is p - 1, and otherwise zero, with the exceptions, that we put $P(0 \to 0) = 1$ and $P(6 \to 6) = 1$, since then the game is over.

Since the game ends when either one of the plays have 6 crowns or when he has 0 crowns, there are no transitions from state (1,1) and (6,6) to other states. So we expect that in the long run the sum of probabilities for being in these two states must be equal to one.
State (1,1) and (6,6) are therefore called ***absorbing states***.

```
probability for the number of crowns that A has gained

probability table
 1.000   0.000   0.000   0.000   0.000   0.000   0.000
 0.500   0.000   0.500   0.000   0.000   0.000   0.000
 0.000   0.500   0.000   0.500   0.000   0.000   0.000
 0.000   0.000   0.500   0.000   0.500   0.000   0.000
 0.000   0.000   0.000   0.500   0.000   0.500   0.000
 0.000   0.000   0.000   0.000   0.500   0.000   0.500
 0.000   0.000   0.000   0.000   0.000   0.000   1.000

initial state:
 0.000   0.000   0.000   1.000   0.000   0.000   0.000
```

```
new state   1
 0.000   0.000   0.500   0.000   0.500   0.000   0.000
new state   2
 0.000   0.250   0.000   0.500   0.000   0.250   0.000
new state   3
 0.125   0.000   0.375   0.000   0.375   0.000   0.125
new state   4
 0.125   0.188   0.000   0.375   0.000   0.188   0.125
new state   5
 0.219   0.000   0.281   0.000   0.281   0.000   0.219
new state   6
 0.219   0.141   0.000   0.281   0.000   0.141   0.219
new state   7
 0.289   0.000   0.211   0.000   0.211   0.000   0.289
new state   8
 0.289   0.105   0.000   0.211   0.000   0.105   0.289
new state   9

Jump to state 43

 0.499   0.000   0.001   0.000   0.001   0.000   0.499
new state  44
 0.499   0.001   0.000   0.001   0.000   0.001   0.499
```

Notice the symmetry in the probability of winning the game, (as it should be).