Laurent series, the theorem of residues, evaluation of contour integrals

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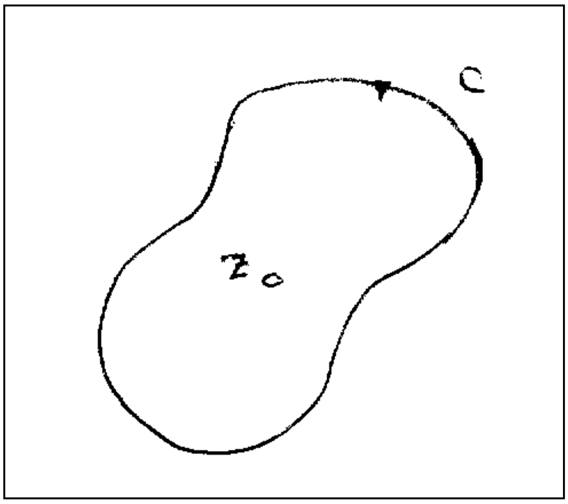
Contents

1. Cauchy's Integral theorems

We shall begin by giving a survey of some properties of analytic functions of a complex variable. These properties are treated in some detail in olewitthansen.dk/Mathematics/Holomorphic_functions.pdf



First integral theorem: Within a compact set, where $f(z)$ is analytic, the integral of $f(z)$ along a closed path C is equal to zero.

(1.1) $$\oint_C f(z)dz = 0$$

Second integral theorem: The functional value $f(z_0)$ can be determined by the integral of $f(z)$ along a closed curve in which z_0 is an inner point.

(1.2) $$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z - z_0}$$

If we make a change of variables in the expression (1.2): $z_0 \to z$ *and* $z \to \varsigma$ the formula becomes:

(1.3) $$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\varsigma)d\varsigma}{\varsigma - z}$$

Differentiating this expression with respect to z, we get the formula:

(1.4) $$f'(z) = \frac{1}{2\pi i}\oint_C \frac{f(\varsigma)d\varsigma}{(\varsigma - z)^2}$$

And by differentiating n times:

(1.5) $$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(\varsigma)d\varsigma}{(\varsigma - z)^{n+1}}$$

2. Taylor's formula for analytic functions

Any analytic function $f(z)$ has a series expansion, from a point z_0, where $f(z)$ is regular. According to Taylor's formula, we have:

(2.1) $$f(z) = a_0 + a_1(z - z_0) + a_1(z - z_0)^2 + \ldots + a_n(z - z_0)^n + \ldots\ldots$$

It is straightforward to verify that:

(2.2) $$a_n = \frac{1}{n!}f^{(n)}(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z - z_0)^{n+1}}$$

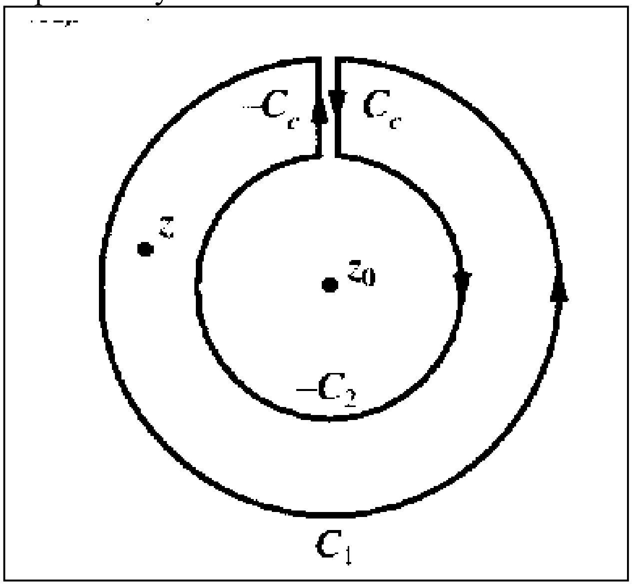
So the Taylor expansion can also be written:

(2.3) $$\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(z_0)(z-z_0)^n$$

The Taylor series converges in any circular disc $|z-z_0|<R$, in which $f(z)$ is regular.
The criteria for convergence of the Taylor series can be copied directly from the Taylor expansion of real functions.
See e.g. olewitthansen.dk/Matematik/TaylorsFormel.pdf

3. Laurent series

Let us assume that $f(z)$ is regular in an annular between the two concentric circles C_1 and C_2 having centre at z_0.
We shall then show that if z lies between the two concentric circles, then $f(z)$ may in a uniquely be expressed by the so called *Laurent series.*



(3.1) $$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$$

Where

$$a_n=\frac{1}{2\pi i}\oint_C\frac{f(z)dz}{(z-z_0)^{n+1}}$$

To avoid negative indices, we shall separate the series in two parts.

(3.2) $$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}b_n(z-z_0)^{-n}$$

Where we now have:

(3.3) $$a_n=\frac{1}{2\pi i}\oint_C\frac{f(z)dz}{(z-z_0)^{n+1}} \quad \text{and} \quad b_n=\frac{1}{2\pi i}\oint_C(z-z_0)^{n-1}f(z)dz$$

We subsequently make a cut in the ring, so that the two circles C_1 and C_2 becomes one contour C, as shown in the figure.
Since z is an inner point in this contour, where $f(z)$ is analytic, we have according to Cauchy's integral theorem:

(3.4) $$f(z)=\frac{1}{2\pi i}\oint_C\frac{f(\varsigma)d\varsigma}{\varsigma-z}$$

We then separate the closed contour into four parts: $C = C_1 + C_c + (-C_2) + (-C_c)$ and evaluate the integral in each of these parts.

(3.5) $$f(z)=\frac{1}{2\pi i}\int_{C_1}\frac{f(\varsigma)d\varsigma}{\varsigma-z}+\frac{1}{2\pi i}\int_{C_c}\frac{f(\varsigma)d\varsigma}{\varsigma-z}-\frac{1}{2\pi i}\int_{C_2}\frac{f(\varsigma)d\varsigma}{\varsigma-z}-\frac{1}{2\pi i}\int_{C_c}\frac{f(\varsigma)d\varsigma}{\varsigma-z}$$

The contributions from C_c and $(-C_c)$ will cancel each other, and we find thus:

(3.6) $$f(z)=\frac{1}{2\pi i}\int_{C_1}\frac{f(\varsigma)d\varsigma}{\varsigma-z}-\frac{1}{2\pi i}\int_{C_2}\frac{f(\varsigma)d\varsigma}{\varsigma-z}$$

We shall then rewrite the denominator in both integrals.

(3.7) $$f(z)=\frac{1}{2\pi i}\int_{C_1}\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)-(z-z_0)}-\frac{1}{2\pi i}\int_{C_2}\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)-(z-z_0)}$$

Which we further transform into:

$$f(z)=\frac{1}{2\pi i}\int_{C_1}\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)\left(1-\dfrac{z-z_0}{\varsigma-z_0}\right)}-\frac{1}{2\pi i}\int_{C_2}\frac{f(\varsigma)d\varsigma}{(z-z_0)\left(\dfrac{\varsigma-z_0}{z-z_0}-1\right)}$$

$$f(z)=\frac{1}{2\pi i}\int_{C_1}\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)\left(1-\dfrac{z-z_0}{\varsigma-z_0}\right)}+\frac{1}{2\pi i}\int_{C_2}\frac{f(\varsigma)d\varsigma}{(z-z_0)\left(1-\dfrac{\varsigma-z_0}{z-z_0}\right)}$$

Now both fractions $\dfrac{z-z_0}{\varsigma-z_0}$ and $\dfrac{\varsigma-z_0}{z-z_0}$ are less than 1, and we may further rewrite the integrals, separating the integrands into two fractions.

$$f(z)=\frac{1}{2\pi i}\int_{C_1}\frac{1}{1-\dfrac{z-z_0}{\varsigma-z_0}}\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)}+\frac{1}{2\pi i}\int_{C_2}\frac{1}{1-\dfrac{\varsigma-z_0}{z-z_0}}\frac{f(\varsigma)d\varsigma}{(z-z_0)}$$

For an infinite geometric series: $\sum_{n=0}^{\infty}q^n$, where the quotient $|q| < 1$, we have: $\sum_{n=0}^{\infty}q^n=\dfrac{1}{1-q}$

We apply then this formula, read from the right to the left, to the two denominators in the integrals.

$$\frac{1}{1-\dfrac{z-z_0}{\varsigma-z_0}}=\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\varsigma-z_0}\right)^n \quad\text{and}\quad \frac{1}{1-\dfrac{\varsigma-z_0}{z-z_0}}=\sum_{n=0}^{\infty}\left(\frac{\varsigma-z_0}{z-z_0}\right)^n$$

Then we move the terms that do not depend on ς outside the integral.

$$f(z)=\frac{1}{2\pi i}\int_{C_1}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\varsigma-z_0}\right)^n\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)}+\frac{1}{2\pi i}\int_{C_2}\sum_{n=0}^{\infty}\left(\frac{\varsigma-z_0}{z-z_0}\right)^n\frac{f(\varsigma)d\varsigma}{(z-z_0)}$$

$$f(z)=\sum_{n=0}^{\infty}(z-z_0)^n\frac{1}{2\pi i}\int_{C_1}\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)^{n+1}}+\sum_{n=0}^{\infty}(z-z_0)^{-n-1}\frac{1}{2\pi i}\int_{C_2}\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)^{-n}}$$

Or by changing the order of the factors:

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{C_1}\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)^{n+1}}(z-z_0)^n+\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{C_2}\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)^{-n}}(z-z_0)^{-n-1}$$

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{C_1}\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)^{n+1}}(z-z_0)^n+\sum_{n=-1}^{\infty}\frac{1}{2\pi i}\int_{C_2}\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)^{-n+1}}(z-z_0)^{-n}$$

If we in the second term replace *n* by –*n* we get:

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{C_1}\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)^{n+1}}(z-z_0)^n+\sum_{-\infty}^{n=-1}\frac{1}{2\pi i}\int_{C_2}\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)^{n+1}}(z-z_0)^n$$

Then the two expressions can be synthesized in one expression:

$$f(z)=\sum_{n=-\infty}^{\infty}\frac{1}{2\pi i}\int_{C}\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)^{n+1}}(z-z_0)^n$$

And we arrive at the usual expression for the Laurent series:

(3.8) $$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n \qquad \text{where} \qquad a_n=\frac{1}{2\pi i}\int_C\frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)^{n+1}}$$

Note in particular that: $a_{-1}=\frac{1}{2\pi i}\int_C f(\varsigma)d\varsigma$

If $f(z)$ is analytic in the area confined by the closed contour *C*, then $a_{-1} = 0$ according to Cauchy's integral theorem, in the other case a_{-1} is denoted the *residue* $f(z)$.

We recall that in a circular disc $F=\{z\,||\,z-z_0\,|\le r\}$, which has the border $C=\{z\,||\,z-z_0\,|=r\}$ having positive orientation, and if we evaluate the integral of $(z - z_0)^n$ for integral *n*, we have:

(3.3) $$\int_C (z-z_0)^n dz=\begin{cases}0 & for \quad n\neq -1\\ 2\pi i & for \quad n=-1\end{cases}$$

This may easily be proven if we apply the parametric: $z-z_0=re^{it} \quad 0\le t\le 2\pi$.

(3.4) $$\int_C (z-z_0)^n dz=\int_0^{2\pi} r^n e^{\text{int}} rie^{it}dt=r^{n+1}i\int_0^{2\pi}e^{i(n+1)t}dt=\begin{cases}0 & for \;\; n\neq -1\\ 2\pi i & for \;\; n=-1\end{cases}$$

So if we evaluate the contour integral $f(z)$ along a closed curve *C*, which has z_0 as an inner point, we obtain:

(3.5) $$\frac{1}{2\pi i}\int_C f(z) = \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty} a_n \int_C (z-z_0)^n = \frac{a_{-1}}{2\pi i}2\pi i = a_{-1}$$

This result appears to have a great impact on the evaluation of definite contour integrals..

If the *Laurent* series does not have any negative exponents of *n* in $(z-z_0)^n$, then $f(z)$ is regular in the area of the outer circular disc.
If there are a finite number *m* of negative exponents, there are two possibilities.

(a) If $f(z)$ is regular, no matter how small we chose the radius of the inner circle, we say that $f(z)$ has an isolated pole at z_0 of *m*'th order, and $(z-z_0)^m f(z)$ will therefore be regular in the vicinity of z_0. For example $f(z) = 1/\sin^2 z$ has poles of order 2 at $z = 0, \pm\pi, \pm 2\pi, ...$

(b) If $f(z)$ has infinitely many negative exponents of $(z-z_0)$, then we say that $f(z)$ has an essential pole (singularity) in z_0.
For example $e^{\frac{1}{z}}$ has an essential singularity at 0.

If $f(z)$ has an isolated pole at z_0, it applies, as described above, that the *residue* at z_0 is:

(3.6) $$a_{-1} = \frac{1}{2\pi i}\int_C f(z)dz$$

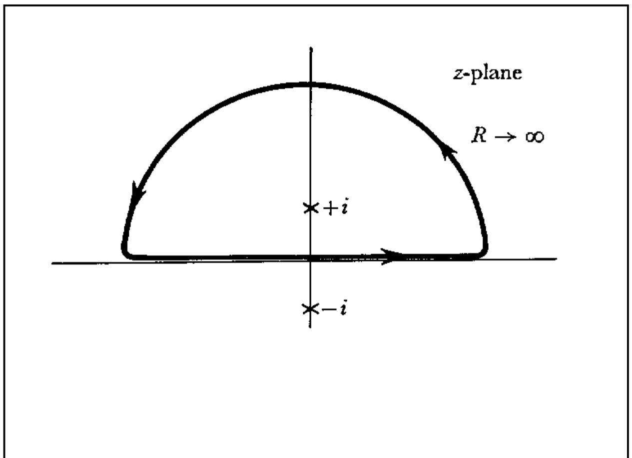
If $f(z)$ has several isolated poles within the closed contour *C*, then the residue theorem applies.

(3.7) $$\int_C f(z)dz = 2\pi i \sum residues$$

The residue theorem can be a very important tool, when evaluating definite integrals of real functions.

4.Contour integrals

We shall demonstrate the method by 3 examples:



Example

(4.1) $$I = \int_0^\infty \frac{dx}{1+x^2}$$

This integral may, however, be evaluated by traditional methods, using the substitution: $x = \tan t \Rightarrow dx = 1 + \tan^2 t$

(4.2) $$I = \int_0^\infty \frac{dx}{1+x^2} = \int_0^{\frac{\pi}{2}} \frac{1+\tan^2 t}{1+\tan^2 t}dt = \int_0^{\frac{\pi}{2}} dt = \frac{\pi}{2}$$

Since the integrand $\frac{1}{1+x^2}$ is an even function, we have: $\int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Next we do the integral $\oint_C \frac{dz}{1+z^2}$ using the contour shown in the figure above.

The contribution from the semi circle will go to zero as the radius R goes to infinity, as can be seen if we use the parametric: $z = R\,e^{i\theta}$, $0 \le \theta \le \pi$, and then evaluate the integral over the semi circle.

$$\int_0^\pi \frac{R i e^{i\theta} d\theta}{1+R^2 e^{i2\theta}} \approx \int_0^\pi \frac{R i e^{i\theta} d\theta}{R^2 e^{i2\theta}} = \frac{i}{R}\int_0^\pi e^{-i\theta} d\theta \to 0 \quad for \quad R \to \infty$$

We then evaluate the integral, using the theorem of residues: $\int_C f(z)dz = 2\pi i \sum residues$.

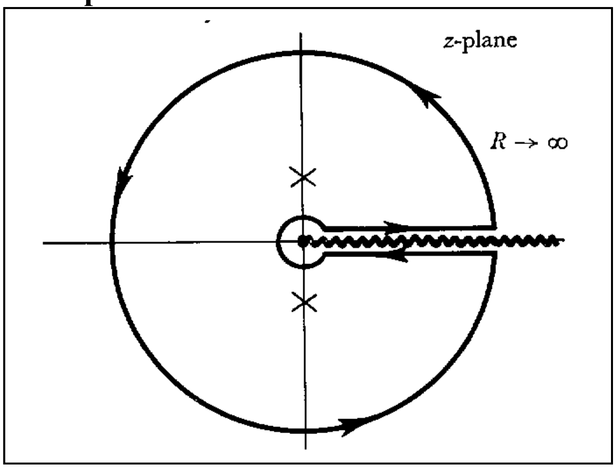
The function $$f(z) = \frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}$$

Has a pole at $z = i$, which is the only pole $f(z)$ has inside the contour.

The residue becomes $\frac{1}{2i}$, and the integral $\int_0^\infty \frac{dx}{1+x^2}$ is therefore equal to $\frac{1}{2} 2\pi i \frac{1}{2i} = \frac{\pi}{2}$.

(4.3) $$I = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

Exampel:



We shall make an attempt on the integral:

(4.4) $$I = \int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2}$$

In contrary to the first example, there exist no traditional methods to do this integral. Instead we shall apply the residue theorem, using the contour shown to the left.
The integral along the circle can numerically be estimated by the expression:

$$\frac{\sqrt{R}}{1+R^2} 2\pi R \to 0 \quad for \quad R \to \infty$$

The sum of the two integrals along the real axis is:

$$\int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2} - \int_\infty^0 \frac{\sqrt{x}\,dx}{1+x^2} = 2\int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2}.$$

The function:

$$f(z) = \frac{\sqrt{z}}{1+z^2} = \frac{\sqrt{z}}{(z-i)(z+i)}$$

has two poles within the contour at $z = i$ and $z = -i$ having the two residues: $\frac{\sqrt{i}}{2i}$ *and* $\frac{\sqrt{-i}}{-2i}$

$\sqrt{z}$, however, is not a unique number for a complex *z*, but rather represents one of the two solutions to the binome equation.

$$z^2 = i \iff e^{2ix} = e^{\frac{\pi}{2}i} \iff x = \tfrac{\pi}{4} \vee x = \tfrac{5\pi}{4},$$

so

$$z = \cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4} \vee z = \cos\tfrac{5\pi}{4} + i\sin\tfrac{5\pi}{4} \iff z = \tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{2}}{2} \vee z = -\tfrac{\sqrt{2}}{2} - i\tfrac{\sqrt{2}}{2}$$

Correspondingly, we have:

$$z^2 = -i \iff e^{2ix} = e^{-\frac{\pi}{2}i} \iff x = -\tfrac{\pi}{4} \vee x = \tfrac{3\pi}{4}$$

$$z = \cos(-\tfrac{\pi}{4}) + i\sin(-\tfrac{\pi}{4}) \vee z = \cos\tfrac{3\pi}{4} + i\sin\tfrac{3\pi}{4} \iff z = \tfrac{\sqrt{2}}{2} - i\tfrac{\sqrt{2}}{2} \vee z = -\tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{2}}{2}$$

We should apply the solution in the first quadrant for the first residue, getting $\frac{\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}{2i}$ and the solution in the second quadrant for the second residue, getting $\frac{-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}{-2i}$.

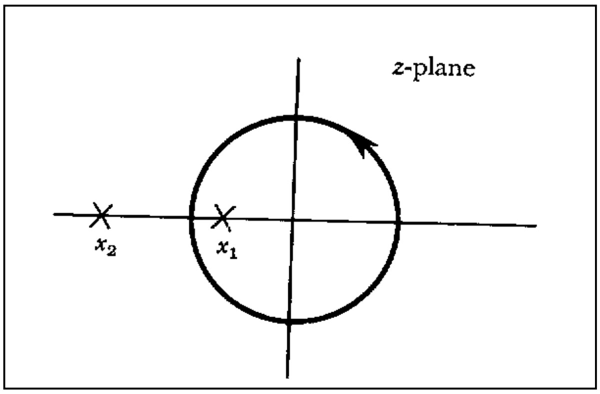
The sum of the two residues becomes: $\frac{\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}{2i} + \frac{\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}{2i} = \frac{\sqrt{2}}{2i}$

$$\int_C f(z)dz = 2\pi i \sum residues = 2\pi i \frac{\sqrt{2}}{2i} = \pi\sqrt{2}$$

So we find

$$2\int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2} = \pi\sqrt{2} \iff \int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2} = \frac{\pi\sqrt{2}}{2}$$

Example



We shall then try to evaluate the integral:

$$I = \int_0^\pi \frac{d\theta}{a + b\cos\theta}, \quad \text{where } a > b > 0$$

The integrand is an even function, so

$$2I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$$

We shall integrate along a contour, being a unit circle, where we apply the parametric.

$$z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta. \quad \cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) = \tfrac{1}{2}(z + \tfrac{1}{z})$$

Then we get:

$$2I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \oint_C \frac{dz/(iz)}{a + b/2(z + 1/z)} = \frac{2}{i}\oint_C \frac{dz}{bz^2 + 2az + b}$$

The integrand has the poles giving by the roots in the denominator:

$$bz^2 + 2az + b = 0 \quad \Leftrightarrow \quad z^2 + 2\frac{a}{b}z + 1 = 0 \quad \Leftrightarrow \quad z = -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1},$$

The roots are real, since $a > b$. The largest root lies inside the contour, whereas the least root does not. If α *and* β are roots in the quadratic polynomial $ax^2 + bx + c$, we have:

$$ax^2 + bx + c = a(x - \alpha)(x - \beta).$$

The residue is therefore: $\frac{2}{i}\frac{1}{b(z_2 - z_1)} = \frac{2}{i}\frac{1}{2b\sqrt{\frac{a^2}{b^2} - 1}} = \frac{1}{i}\frac{1}{\sqrt{a^2 - b^2}}$

So we have:

$$2I = 2\pi i \sum residues = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

The integral becomes:

$$I = \int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{a^2 - b^2}}$$

4.1 Contour integrals, where the pole lies on the contour

The integral $\int_2^4 \frac{dx}{x-3}$ has no mathematical definition, but to avoid integrating over the pole at x = 3 we may consider the two integrals:

$$\int_2^{3-\delta} \frac{dx}{x-3} = \left[\ln|x-3|\right]_2^{3-\delta} \ \ln|-\delta| - \ln|-2| = \ln\delta - \ln 2 \quad \text{and} \quad \int_{3+\delta}^{4} \frac{dx}{x-3} = \ln 4 - \ln\delta$$

However the sum of the two integrals ln4 - ln2 is independent of δ, and therefore:

$$\lim_{\delta\to 0}\left(\int_2^{3-\delta} \frac{dx}{x-3} + \int_{3+\delta}^{4} \frac{dx}{x-3}\right) = \ln 4 - \ln 2$$

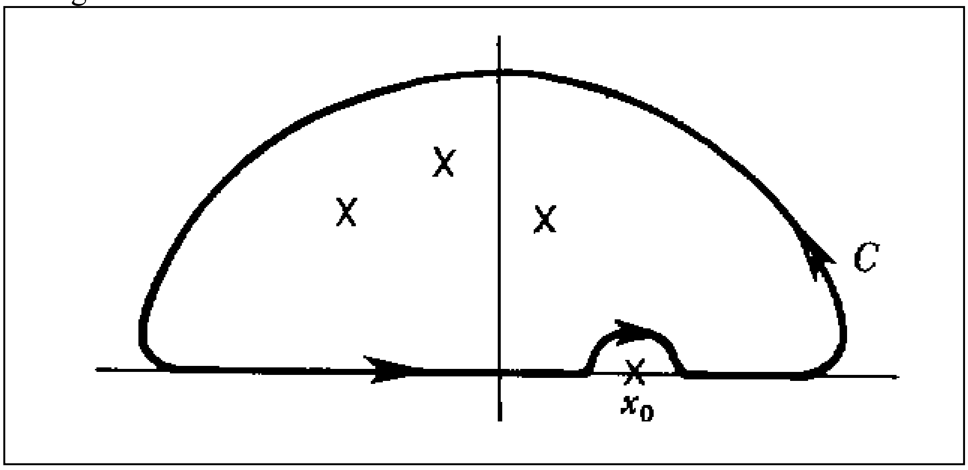
The limiting value is called Cauchy's principal value, and the integral is written with a prefixed *P*.

This is not surprising, since the two large numbers $\ln\delta$ cancel each other, and this will also be the case in the limit $\delta \to 0$.

In general, Cauchy's principal value is defined in the following manner, where $f(x)$ is defined and has an integral in the interval $[a,b]$, and x_0 is in $]a,b[$

$$P\int_a^b \frac{f(x)}{x-x_0}dx = \lim_{\delta\to 0}\left(\int_a^{x_0-\delta}\frac{f(x)}{x-x_0}dx + \int_{x_0+\delta}^{b}\frac{f(x)}{x-x_0}dx\right)$$

The integral path for Cauchy's principal value, may be a part of a contour integral, in which the end points on the x-axis, $x_0 \pm \delta$ are connected by a small semi circle with radius δ, as shown in the figure below.



Assuming that the integral of the large semi circle goes to zero when the radius goes to infinity, then the contour integral becomes: $2\pi i \sum_{contour} residues$.

However we may directly evaluate the integral of the small semi circle, having its centre at x_0 and its radius δ. For small δ we may approximate $f(z)$ in the vicinity of x_0:

$$f(z) \to \frac{a_{-1}}{z-x_0},$$

where a_{-1} is the residue at the pole $z = x_0$.

We apply the parametric: $z - x_0 = re^{i\theta}$, $dz = rie^{i\theta}d\theta$

$$\int_{semi\,circle} f(z)dz \to \int_{semi\,circlel} \frac{a_{-1}}{z-x_0}dz = -a_{-1}\int_0^{\pi}\frac{rie^{i\theta}}{re^{i\theta}}d\theta = -ia_{-1}\int_0^{\pi}d\theta = -\pi i a_{-1}$$

We find thus:

$$\oint_C f(z)dz = P\int f(z)dz - \pi i(residue\ at\ x_0) = 2\pi i\sum(residues\ in\ the\ contour\ C)$$

$$P\int f(z)dz = 2\pi i\sum(residuer\ i\ konturen\ C + \tfrac{1}{2}residue\ i\ x_0)$$

Example

We shall apply the method to evaluate the integral:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$$

The integrand has at pole at $x = 0$, but that does not lead to infinite values, since we know that:

$$\frac{\sin x}{x} \to 1 \ \ for \ \ x \to 0 .$$

To do so, we look at the integral:

$$\int_C \frac{e^{iz}}{z} dz$$

Using a contour, as in the figure above, but where the pole now is at $z = 0$.

When $z = x + iy$ and therefore: $e^{i(x+iy)} = e^{ix}e^{-y}$, the factor e^{-y} will insure that the integral along the large semi circle goes to zero for $R \to \infty$. We shall then apply the theorem above:

$$P\int f(z)dz = 2\pi i \sum (residues\ in\ \ C + \tfrac{1}{2} residue\ i\ x_0) .$$

The function $\frac{e^{iz}}{z}$ does not have any poles in the contour C, so we only get at contribution from the pole z = 0.

$$\int_C \frac{e^{iz}}{z} dz = \pi i e^0 \quad \Rightarrow \quad \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi i$$

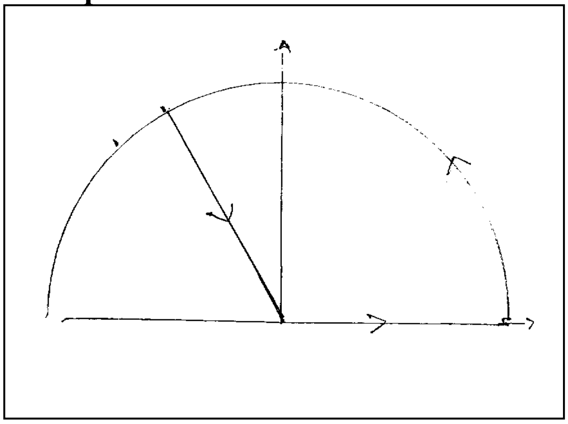
Taking the real and imaginary part on both sides, we find:

$$\int_{-\infty}^{\infty} \frac{\cos x}{x} dx = 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

The first integral is (trivially) equal to 0, since (cos x)/x is an odd function, while since (sin x)/x is an even function, so we have:

$$\int_0^{\infty} \frac{\sin x}{x} dx = \tfrac{1}{2}\pi$$

Example



Sometimes one is referred to some simple tricks to evaluate an integral. That is the case with the integral.

$$I = \int_0^\infty \frac{dx}{1+x^3}$$

We shall choose a contour *J*, as shown in the figure comprised by the positive real axis, a circle segment from 0 to $2\pi/3$ and the line $z = e^{i2\pi/3}t,\quad 0<t<R$. Letting $R \to \infty$, the integral along the circle will go to zero, and the integral along the real axis is then the integral that we seek.

We then calculate the integral along the line $z = e^{i2\pi/3}t$.

$$\int_{line} \frac{dz}{1+z^3} = -\int_0^\infty \frac{e^{i2\pi/3}}{1+e^{2\pi i}t^3}dt = -e^{i2\pi/3}\int_0^\infty \frac{1}{1+t^3}dt = -e^{i2\pi/3}I$$

We have thus:

$$J = \int_J \frac{dz}{1+z^3} = \int_{x-axis} \frac{dz}{1+z^3} + \int_{circle} \frac{dz}{1+z^3} + \int_{line} \frac{dz}{1+z^3} = I + 0 - e^{i2\pi/3}I$$

$$J = (1-e^{i2\pi/3})I = (1+\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2})I = \tfrac{1}{2}(3-i\sqrt{3})$$

On the other hand, then $\dfrac{dz}{1+z^3}$ has a single isolated pole within the contour *J*. To locate the pole, we must solve the equation.

$$z^3 = -1 \quad\Leftrightarrow\quad e^{i3x} = e^{i\pi} \quad\Leftrightarrow\quad 3x = \pi \;\vee\; 3x = \pi + 2\pi \;\vee\; 3x = \pi + 4\pi \quad\Leftrightarrow$$

The only pole, which lies in the contour *J* is $x = \frac{\pi}{3}$, so we find for the residue:

$$\frac{1}{(z_2 - e^{i\pi})(z_3 - e^{i5\pi/3})} = \frac{1}{(e^{i\pi/3} - e^{i\pi})(e^{i\pi/3} - e^{-i\pi/3})} = \frac{2}{(3\sqrt{3}+3i)}$$

since: $(e^{i\pi/3} - e^{i\pi})(e^{i\pi/3} - e^{-i\pi/3}) = (e^{i\pi/3}+1)2i\sin\frac{\pi}{3} = i(\frac{3}{2} + i\frac{\sqrt{3}}{2})\sqrt{3} = i\frac{1}{2}(3\sqrt{3}+3i)$

The value of the contour integral is:

$$J = 2\pi i \frac{2}{(3\sqrt{3}+3i)}$$

At the same time, we have as demonstrated above:

$$J = (1 - e^{i2\pi/3})I = \tfrac{1}{2}(3 - i\sqrt{3})\, I$$

So now, we may calculate *I*.

$$I = \frac{2J}{(3 - i\sqrt{3})} = \frac{8\pi i}{(3 - i\sqrt{3})i(3\sqrt{3} + 3i)} = \frac{8\pi}{9\sqrt{3} + 9i - 9i + 3\sqrt{3}} = \frac{8\pi}{12\sqrt{3}} = \frac{2\pi}{3\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}$$

And finally get the result:

$$I = \int_0^\infty \frac{dx}{1 + x^3} = \frac{2\pi}{3\sqrt{3}}$$