Implicit differentiation

Illustrated with examples

Tangent to ellipse. Prism in main position

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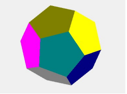

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1. Implicit differentiation

In a issue some years ago in the monthly journal for math and physics teachers in the Danish 9 -12 year high school, there was an article on orthogonal tangents (??) to the ellipse.
However the equations for the tangents were derived from rather awkward analytical geometry. When I was I high school (in 1963), the equation for the tangent to the ellipse and the hyperbola was a part of the mandatory curriculum in analytical geometry (but since 1964 it has been absent in the curriculum). However, I still remember with some awe, that the equation for the tangent were derived using implicit differentiation, and it was not until some years later at the university I learned to appreciate the technique of implicit differentiation.

Loosely formulated: If you have a function given by the equation $f(x,y)$ = c, then y is an implicit function (or functions) of x. To solve the equation for y may be difficult if not impossible, but nevertheless it is still possible to find an expression for the differential quotient dy/dx, by the implicit assumption that $y = y(x)$

Taking the differential of $f(x,y)$ = c, we find:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

And by dividing with dx:

(1.1) $$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$$

This equation can then be solved with respect to dy/dx. (And possibly solved for $dy/dx = 0$)

2. Tangent to ellipse and hyperbola

An ellipse having ”centre” at (x_0,y_0) and semi-axes a and b, has the well known equation:

(2.1) $$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

If (x_1,y_1) is a point on the ellipse, then $\frac{dy}{dx}$ calculated in x_1 is the slope for the tangent in x_1.
To determine dy/dx , we will differentiate the equation for the ellipse implicit.

$$\frac{2(x-x_0)}{a^2} + \frac{2(y-y_0)}{b^2}\frac{dy}{dx} = 0$$

And solving it with respect to dy/dx to give.

(2.2) $$\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{(x-x_0)}{(y-y_0)}$$

The equation for a line passing through (x_1,y_1), and having slope a is: $y - y_1 = a(x - x_1)$.

Inserting the expression (2.2) for the slope at x_1, we find:

$$y - y_1 = -\frac{b^2}{a^2}\frac{(x_1 - x_0)}{(y_1 - y_0)}(x - x_1)$$

After some rewriting, it gives:

(2.3) $$\frac{(x - x_1)(x_1 - x_0)}{a^2} + \frac{(y - y_1)(y_1 - y_0)}{b^2} = 0$$

What we find is a simple (easy to remember) expression for the equation of the tangent, which has a strong resemblance with the equation of the ellipse.

To actually write the equation out it requires, however, that you shall know y_1 , which must be found from the equation of the ellipse.

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1 \qquad \Leftrightarrow \qquad y_1 - y_0 = \pm b\sqrt{1 - \frac{(x_1 - x_0)^2}{a^2}}$$

For a hyperbola, having the equation:

$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1$$

The equation for the tangent is almost the same, the only difference is that the minus sign follow in the equation for the tangent.

(2.4) $$\frac{(x - x_1)(x_1 - x_0)}{a^2} - \frac{(y - y_1)(y_1 - y_0)}{b^2} = 0$$

Actually the method described can be applied as well on any equation for a geometrical object in the plane.

Given $f(x,y)$ = c, we find as before:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

And by dividing with dx:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0$$

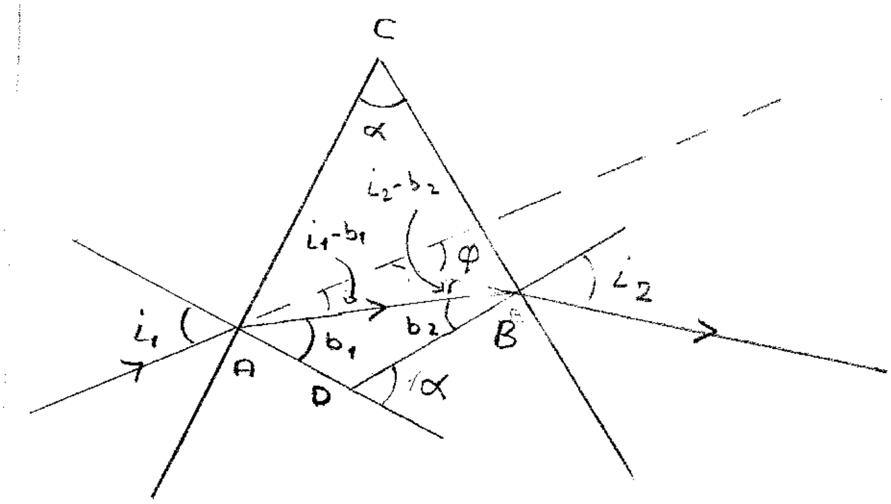
And solving for $\frac{dy}{dx}$ gives: $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$, which can then be inserted in the equation for the tangent.

$y - y_1 = a(x - x_1)$ to give $y - y_1 = \frac{dy}{dx}(x - x_1)$, but it still requires that you know y_1.

3. Prism in the main position

Some years ago, I wrote some notes on geometrical optics for use as an optional subject for use in the Danish high school. The revised notes have later become an article in my home page www.olewitthansen.dk (Geometrical optics).
These notes included optical experiments, and I had to apply the well known experimental fact that the bending of a light ray through a prism is least when the prism is in its main position, that is, when the path of the light ray through the prism is symmetric.

It ought to be simple (and perhaps it is), but back then I could find no theoretical explanation in the litterature, and it was only years later that I thought of implicit differentiation as a possible way out. Elementary no, but you reach a theoretical explanation, and the aim here is more to demonstrate the power of implicit differentiation. Below is shown a prism with a non symmetric passage of a light ray.



We consider the square *ACBD*. We shall apply that the supplement angle (180^0 minus the angle) to the angle in triangle is (evidently) equal to the sum of the other two angles in the triangle.
If we apply this to ΔABD , and since A=B=90^0 it appears from the figure. (and from the geometry).

$$D = 180 - \alpha \;\; and \;\; 180 - D = b_1 + b_2 \Rightarrow \alpha = b_1 + b_2$$

The deviation φ of the beam is seen to be:

(3.1) $$\varphi = i_1 - b_1 + i_2 - b_2 = i_1 + i_2 - (b_1 + b_2) = i_1 + i_2 - \alpha$$

One may experimentally show, (and theoretically prove, as demonstrated below) that the deviation of the beam is least, when the passage of the beam through the prism is symmetric, that is, when:

$$i_1 = i_2 = i \quad \text{and} \quad b_1 = b_2 = b.$$

This is called the main position of the prism. In that case:

(3.2) $$\varphi = \varphi_{\min} = 2 \cdot i - \alpha, \quad \text{so} \quad i = \frac{\varphi_{\min} + \alpha}{2} \quad \text{and} \quad b = \frac{\alpha}{2}.$$

Applying this to the law of refraction to these angles, we get:

(3.3) $$n = \frac{\sin i}{\sin b} = \frac{\sin(\frac{\varphi_{\min} + \alpha}{2})}{\sin(\frac{\alpha}{2})}$$

3.1 A theoretical argument for the main position

According to the derivation above, we have in all cases

$$\varphi = i_1 + i_2 - (b_1 + b_2) \quad \textit{and} \quad \alpha = (b_1 + b_2) \ ,$$

And according to the law of refraction:

$$\frac{\sin i_1}{\sin b_1} = n \quad \textit{and} \quad \frac{\sin i_2}{\sin b_2} = n \quad \Rightarrow$$
$$\sin i_1 = n \sin b_1 \quad \textit{and} \quad \sin i_2 = n \sin b_2 = n \sin(\alpha - b_1)$$

From the last equation, we may however consider i_2 as an implicit function of b_1, and from the first equation we may consider b_1 as an implicit function of i_1. Written formally:

$$i_2 = i_2(b_1) \quad \text{and} \quad b_1 = b_1(i_1) \quad => \quad i_2 = i_2(b_1(i_1))$$

The functional dependence of the various variables has been stated above.
Hereafter we may implicitly express φ as a function of i_1 and by differentiating implicit, we may find the minimum φ.

$$\varphi = i_1 + i_2 - (b_1 + b_2) \quad \textit{and} \quad \alpha = (b_1 + b_2) \quad \Rightarrow$$
$$\varphi = i_1 + i_2 - \alpha = i_1 + i_2(b_1(i_1)) - \alpha$$

Doing implicit differentiation of the composite function with respect to i_1 , we get:

$$\frac{d\phi}{di_1} = 1 + \frac{di_2}{db_1}\frac{db_1}{di_1}$$

By implicit differentiation of the equations:

$$\sin i_1 = n \sin b_1 \qquad \textit{and} \qquad \sin i_2 = n \sin b_2 = n \sin(\alpha - b_1)$$

We find:

$$\cos i_2 \frac{di_2}{db_1} = -n \cos(\alpha - b_1) \qquad \textit{and} \qquad n \cos b_1 \frac{db_1}{di_1} = \cos i_1$$

Solving these equations with respect to $\frac{di_2}{db_1}$ and $\frac{db_1}{di_1}$ inserting in the expression for φ, we have

$$\begin{aligned}\frac{d\phi}{di_1} &= 1 + \frac{di_2}{db_1}\frac{db_1}{di_1} \\ &= 1 - \frac{\cos i_1}{\cos i_2}\frac{\cos(\alpha - b_1)}{\cos b_1} \\ &= 1 - \frac{\cos i_1}{\cos i_2}\frac{\cos b_2}{\cos b_1}\end{aligned}$$

We can now see from above that $\frac{d\phi}{di_1} = 0$ has the solution: $i_1 = i_2 \ \wedge \ b_1 = b_2$, since in that case both fraction becomes equal to one, and the solution correspond to path of the beam, when the prism is in its main position.