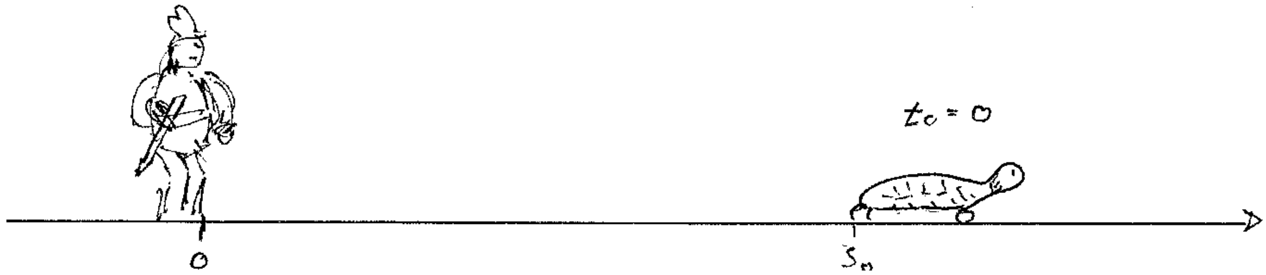
Achilles and the turtle
A mathematical analysis of a non paradox

Achilles and the turtle is one of the most famous paradoxes from ancient Greece on inconsistencies of daily life experiences
The argument is the following. Achilles should race against a turtle. However, the turtle get (fair enough) a lead. They start running, but before Achilles reaches the turtle, he must first reach the point where the turtle began its motion. Having reached that point, then before he reaches the turtle, he must first reach the point, where the turtle was after the first leap. As this line of argument can be continued infinitely, we should add up infinitely many times, and consequently Achilles shall never catch up the turtle. The weak spot is of course is of course the assumption that an infinite sum of numbers must be infinity large. Today we know that is not necessarily the case.



We assume that Achilles moves with the speed c, while the turtle moves with the lower speed u.
The turtle has a lead s_0 from the start at time $t_0 = 0$.
t_k is the time the turtle uses to move from the position where it was, corresponding to the time t_{k-1}.

Thus, we have: $t_1 = \frac{s_0}{c}$

During this time the turtle has moved the distance $s_1 = u \cdot t_1$, and we therefore have

$$t_2 = \frac{s_1}{c} = \frac{u \cdot t_1}{c} = \frac{s_0}{c}\frac{u}{c} \quad \text{And by the same token: } t_3 = \frac{s_2}{c} = \frac{u \cdot t_2}{c} = \frac{s_0}{c}\left(\frac{u}{c}\right)^2$$

And quite generally:

$$t_{k+1} = \frac{s_k}{c} = \frac{u \cdot t_k}{c} = \frac{s_0}{c}\left(\frac{u}{c}\right)^k$$

To calculate the total time that passes before Achilles reaches the turtle in infinite many steps, we must evaluate the infinite sum.

$$t = \sum_{k=1}^{\infty} t_k = \sum_{k=0}^{\infty} t_{k+1} = \frac{s_0}{c}\sum_{k=0}^{\infty}\left(\frac{u}{c}\right)^k$$

The series is, however, a geometric one, having the quotient: $k = \frac{u}{c}$ *and* $a_0 = \frac{s_0}{c}$

The formula for the sum of a geometric series is: $s = a_0 \frac{1}{1-k}$

And we therefore get:

$$t = \sum_{k=0}^{\infty} t_{k+1} = \frac{s_0}{c}\sum_{k=0}^{\infty}\left(\frac{u}{c}\right)^k = \frac{s_0}{c}\frac{1}{1-\frac{u}{c}} = \frac{s_0}{c-u}$$

The last result is easily interpreted, since it expresses that the time it takes Achilles to reach the turtle is the initial distance s_0 divided by their relative speed $c - u$.