About chasing and escaping

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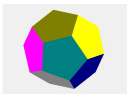

Ole Witt-Hansen 2019

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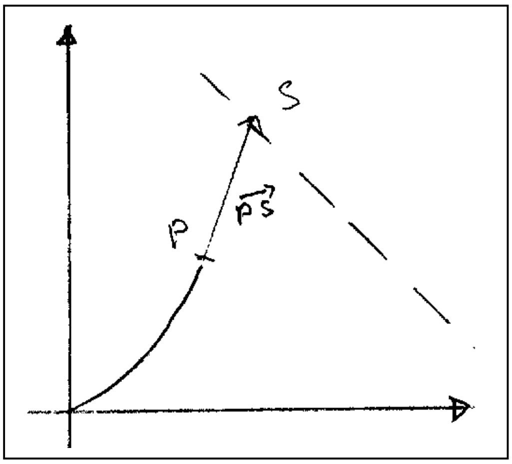
1. Stating of the problem

The problem of pursuit and escape is a classical one, and it has several different formulations. One is that a pirate ship is chasing a merchant ship, assuming that the pirate ship can go faster than the merchant ship. The merchant ship follows a predetermined curve. The pirate ship, however, follows, at each moment, the immediate direction from the pirate ship to the merchant ship. This is the mathematical definition of a pursuit curve.



Assuming that the pirate ship is at point P and the merchant ship is at point S, both at time t.
The two ships have the positions:

$$\vec{OP} = \vec{f}(t) = (x(t), y(t)) \quad and \quad \vec{OS} = \vec{g}(t) = (u(t), v(t))$$

The vector from the pirate to the merchant is then:

$$\vec{PS} = \vec{OS} - \vec{OP} = \vec{g}(t) - \vec{f}(t)$$

A unit vector having this direction is then:

$$\vec{e}_{PS} = \frac{\vec{g}(t) - \vec{f}(t)}{|\vec{g}(t) - \vec{f}(t)|}$$

If the pirate ship sails with speed v_P ,and the merchant ship sails with velocity v_S, then we can make the following first order differential equation for the pursuit curve the pirate ship must follow.

$$\vec{f}'(t) = v_P \frac{\vec{g}(t) - \vec{f}(t)}{|\vec{g}(t) - \vec{f}(t)|}$$

Or written out in coordinates:

$$(x'(t), y'(t)) = v_P \frac{(u(t) - x(t)\, ,\, v(t) - y(t))}{\sqrt{(u(t) - x(t))^2 + (v(t) - y(t))^2}}$$

It is not really possible to think of a general analytic solution to this first order coupled two dimensional differential equation. However, in some lecture notes from the University of Aarhus, (the notes are from 1974, in the period of minicomputers without graphics capabilities), however, in these notes there is actually delivered an analytic solution, where the merchant ship sails on a vertical line, and the pirate ship is initially on a position on the x-axis. The solution is obtained by putting $x(t) = x$, treating x as an independent variable, and assuming that $x = x(t)$ is monotone, then solving the resulting differential equation with respect to x, using some mathematical tricks.
The pirate ship sails at any instant with the direction of:

$$\vec{PS} = \vec{OS} - \vec{OP} = (u(t) - x(t)\, ,\, v(t) - y(t))$$

If the pirate ship sails with speed v_P, and the merchant ship sails with speed v_S, then we may reason that the pirate will always catch the merchant. This can be seen as follows: In the time dt the relative displacement of the two ships is: $(\vec{v}_S - \vec{v}_P)dt$.

The projection on this vector on $\vec{PS}$, will, however, always be opposite to $\vec{PS}$, so that the distance between the two ships will decrease. The merchant ship will eventually be caught.

2. Finding an analytic expression for the pursuit curve

Assuming that the merchant ship sails with the constant speed v_S, along the y axis, such that $\vec{v}_S = (0, v_S)$, and that the pirate ship follows the pursuit curve: $y = f(x)$, we may write:

$$\frac{dy}{dx} = \frac{v - y}{u - x} \quad \text{or} \quad f'(x) = \frac{v(x) - f(x)}{-x}$$

Since the two ships sail with speeds v_S and v_P, we must have:

$$v_S dt = \sqrt{du^2 + dv^2} \quad and \quad v_P dt = \sqrt{dx^2 + dy^2}$$

From which we find: $dt = \dfrac{\sqrt{du^2 + dv^2}}{v_S}$ *and* $dt = \dfrac{\sqrt{dx^2 + dy^2}}{v_P}$ so that

$$\sqrt{du^2 + dv^2} = \frac{v_S}{v_P}\sqrt{dx^2 + dy^2}$$. Dividing by dx

$$\sqrt{(\frac{du}{dx})^2 + (\frac{dv}{dx})^2} = \frac{v_S}{v_P}\sqrt{1 + (\frac{dy}{dx})^2}$$

Since $(\dfrac{du}{dx}) = 0$, we get the equation:

$$v'(x) = -\frac{v_S}{v_P}\sqrt{1 + f'(x)^2}$$, (minus because v is a decreasing function of x.)

By differentiating the equation: $f'(x) = \dfrac{v(x) - f(x)}{-x} \Leftrightarrow -xf'(x) = v(x) - f(x)$ we find:

$$-f'(x) - xf''(x) = v'(x) - f'(x) \quad \Leftrightarrow \quad -xf''(x) = v'(x)$$

And by inserting the expression for $v'(x)$, we finally have:

$$xf''(x) = \frac{v_S}{v_P}\sqrt{1 + f'(x)^2} \quad \Leftrightarrow \quad xy'' = \alpha\sqrt{1 + y'^2}$$

Where we have put: $\dfrac{v_S}{v_P} = \alpha$. Replacing y' by y, we solve the equation by separating the variables.

$$xy' = \alpha\sqrt{1 + y^2} \quad \Leftrightarrow \quad x\frac{dy}{dx} = \alpha\sqrt{1 + y^2} \quad \Leftrightarrow \quad \frac{dy}{\sqrt{1 + y^2}} = \frac{\alpha}{x}dx$$

And integrating on both sides: $$\int \frac{dy}{\sqrt{1 + y^2}} = \int \frac{\alpha}{x}dx$$

The integral $\int \frac{dy}{\sqrt{1+y^2}}$ can be evaluated by the substitution: $y = \sinh x \quad \Rightarrow \quad dy = \cosh x dx$

Since $\cosh^2 x = 1 + \sinh^2 x$, the integral can be rewritten as:

$$\int \frac{dy}{\sqrt{1+y^2}} = \int \frac{\cosh x dx}{\sqrt{1+\sinh^2 x}} = \int \frac{\cosh x dx}{\cosh x} = \int dx = x + c$$

We only need to express *y* by *x*.

$y = \sinh x \quad \Leftrightarrow \quad x = \sinh^{-1} y$, so we solve $y = \sinh x$ with respect to *x*. $\sinh x = \frac{1}{2}(e^x - e^{-x})$

$$y = \tfrac{1}{2}(e^x - e^{-x}) \quad \Leftrightarrow \quad e^{2x} - 2ye^x - 1 = 0$$

$$d = 4y^2 + 4 \quad \Rightarrow \quad e^x = \frac{2y \pm 2\sqrt{1+y^2}}{2} = y + \sqrt{1+y^2} \quad \Rightarrow \quad x = \ln(y + \sqrt{1+y^2})$$

So we have that: $\int \frac{dy}{\sqrt{1+y^2}} = \ln(y + \sqrt{1+y^2})$

This can of course also be verified directly by differentiation:

$$(\ln(y + \sqrt{1+y^2}))' = \frac{1}{y+\sqrt{1+y^2}}(y' + \frac{yy'}{\sqrt{1+y^2}}) = \frac{y'}{y+\sqrt{1+y^2}}(1 + \frac{y}{\sqrt{1+y^2}}) =$$

$$\frac{y'}{y+\sqrt{1+y^2}} \frac{\sqrt{1+y^2}+y}{\sqrt{1+y^2}} = \frac{y'}{\sqrt{1+y^2}}$$

So we have:

$$\int \frac{dy}{\sqrt{1+y^2}} = \int \frac{\alpha}{x} dx \quad \Leftrightarrow \quad \ln(y + \sqrt{1+y^2}) = \alpha \ln x + \ln c = \ln cx^\alpha \quad \Leftrightarrow$$

$$y + \sqrt{1+y^2} = cx^\alpha \quad \Leftrightarrow \quad \sqrt{1+y^2} = cx^\alpha - y \quad \Leftrightarrow$$

$$1 + y^2 = c^2x^{2\alpha} + y^2 - 2cx^\alpha y \quad \Leftrightarrow$$

$$y = f'(x) = \frac{1}{2cx^\alpha}(c^2x^{2\alpha} - 1) = \frac{1}{2}(cx^\alpha - \frac{1}{c}x^{-\alpha})$$

$$f(x) = \frac{c}{\alpha+1}x^{\alpha+1} - \frac{1}{c(-\alpha+1)}x^{-\alpha+1} + b$$

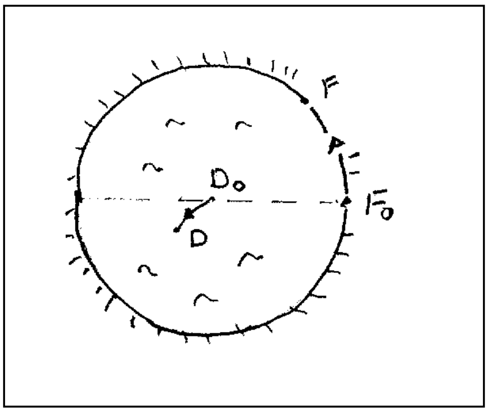
However, the solution is mostly of academic interest, since it is much easier to look at a numeric graphic solution.
But we can find out, when the pirate will catch up the merchant ship. The merchant ship will always being obtained, if $v_S < v_P$, and the pirate follow the pursuit curve defined above.
We have the equation: $v(x) = f(x) - xf'(x)$.

Since $v(x) = v_S t(x)$, we may obtain an expression for $t(x) = (f(x) - xf'(x))/v_S$.

When if $v_S < v_P$ then the pursuit curve hits the y-axis in (0,b), at time: $t_{catch\ up} = \dfrac{b}{v_S}$

This value is also mostly of academic interest, since it must comply with the initial conditions.

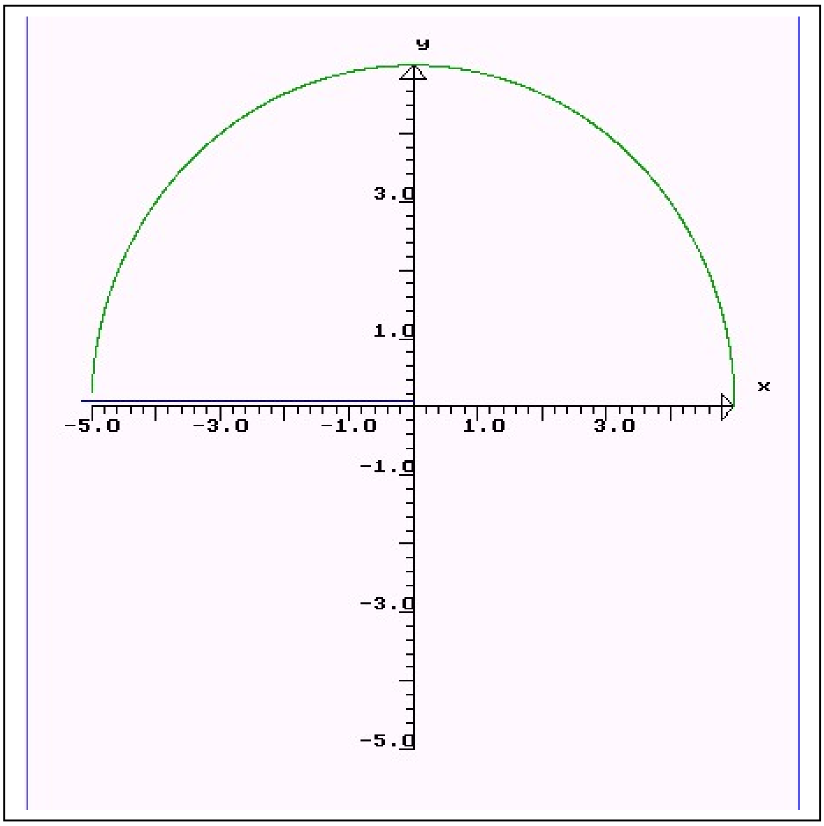
3. The duck and the fox. Numeric graphic representations



A classic example, which has many different formulations of which we choose the following:
A duck sits in the middle of a circular pond, having radius r. A fox waits on the edge of the pond to grab the dock when it reaches the edge. The duck swims with velocity v_D, and the fox runs with velocity v_F , where v_F is assumed to be higher than v_D. The duck can now invent several strategies to escape, where the simplest is to swim to the opposite shore of where the fox is.
The answer is very simple: The time for the duck to reach the shore is $t_D = \dfrac{r}{v_D}$, and the time for the fox to reach the same position is $t_F = \dfrac{\pi r}{v_F}$. The duck will reach the shore earlier and escape if:

$$t_D < t_F \quad \Leftrightarrow \quad \frac{r}{v_D} < \frac{\pi r}{v_F} \quad \Leftrightarrow \quad v_D > \frac{v_F}{\pi}$$

This is (trivially) illustrated in the computer graphic solution shown below:



The duck D may, however, also choose another strategy, namely at each instant to swim away in a direction of the vector $\vec{FD}$ from the fox F to the duck:
If the coordinates of the duck is (x, y) , and the coordinates of fox is:
$(u, v) = (r\cos(\frac{v_F}{r}t)\,,\, r\sin(\frac{v_F}{r}t))$, then the vector from the fox to the duck is:

$$\vec{FD} = (x - r\cos(\tfrac{v_F}{r}t)\,,\, y - r\sin(\tfrac{v_F}{r}t))$$

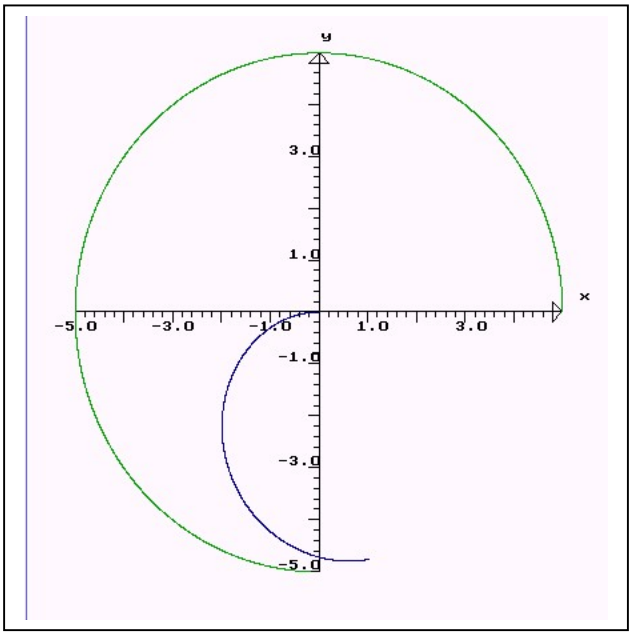
A unit vector in this direction is consequently:

$$\left(\frac{x - r\cos(\frac{v_F}{r}t)}{\sqrt{(x - r\cos(\frac{v_F}{r}t))^2 + (y - r\sin(\frac{v_F}{r}t))^2}}\,,\,\frac{y - r\sin(\frac{v_F}{r}t)}{\sqrt{(x - r\cos(\frac{v_F}{r}t))^2 + (y - r\sin(\frac{v_F}{r}t))^2}}\right)$$

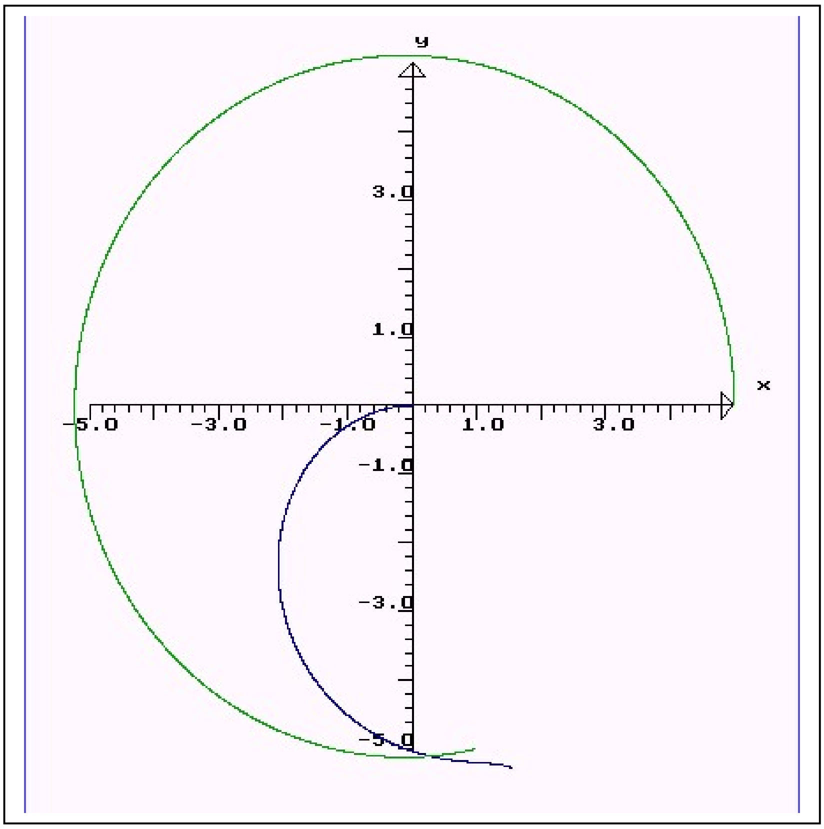
The duck swims along this unit vector with the speed v_D, and this gives the differential equation:

$$(x', y') = v_D\left(\frac{x - r\cos(\frac{v_F}{r}t)}{\sqrt{(x - r\cos(\frac{v_F}{r}t))^2 + (y - r\sin(\frac{v_F}{r}t))^2}}\,,\,\frac{y - r\sin(\frac{v_F}{r}t)}{\sqrt{(x - r\cos(\frac{v_F}{r}t))^2 + (y - r\sin(\frac{v_F}{r}t))^2}}\right)$$

There is allegedly no analytic solution to this differential equation.
We shall however present a numeric graphic solution, where we choose $r = 5$, $v_D = 0.30$ and $v_F < \pi v_D$. If we compare to the first strategy, the duck should escape if $v_F < 0.942$, but this turns out not to be the case. Even if $v_F = 0.90$ the duck will not escape, as seen below:



If we put $v_D = 0.31$ and $v_F = 0.90$, the duck just barely escapes to the shore before the fox.
It is however not easily explained, why the simple strategy is always more advantageous.



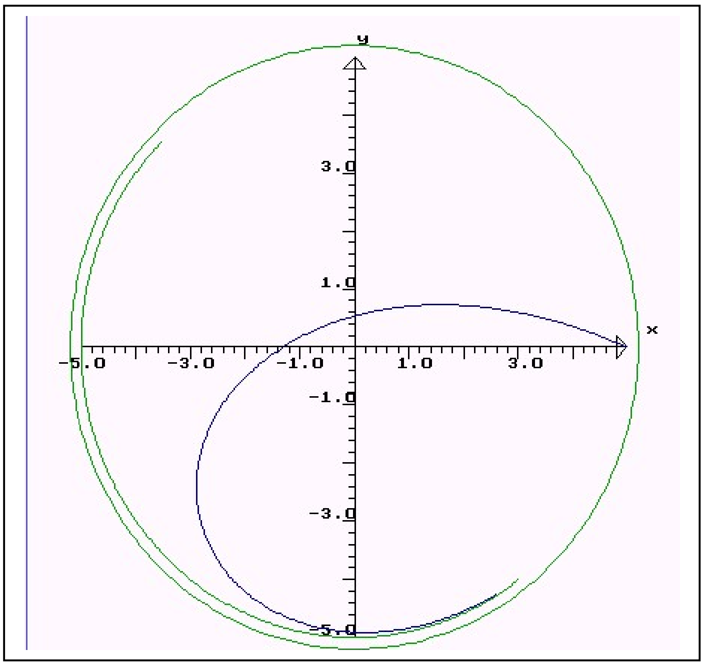
4. The lion and the prey

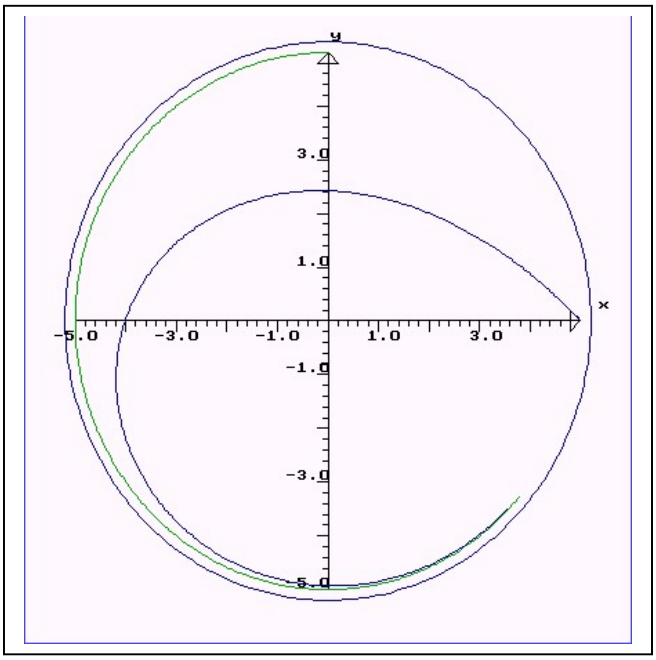
This is another classical problem of which there (as far as I know) exists no analytic solution.
A prey (possibly a human being) is placed at the border of a circular arena. A lion is let into the arena and it begins immediately to chase the prey. The prey tries to escape by running along the border. The Lion runs faster than the prey and follows a pursuit curve defined earlier, where the immediate direction of the velocity is along the line from the lion to the prey. The prey will, as it was the case in the other examples eventually be caught, but the interesting point is the shape of the pursuit curve.

The prey has the coordinates: $\vec{OP} = (u(t), v(t)) = r(\cos(\frac{v_P}{r}t + \alpha), \sin(\frac{v_P}{r}t + \alpha))$, and it moves with speed v_P, and the lion has the coordinates $\vec{OL} = (x(t), y(t))$. The lion moves along the direction of $\vec{LP} = (u - x, v - y)$. The differential equation then:

$$(x'(t), y'(t)) = v_L\left(\frac{(u-x)}{\sqrt{(u-x)^2 + (v-y)^2}}, \frac{(v-y)}{\sqrt{(u-x)^2 + (v-y)^2}}\right)$$

This is illustrated below, where the lion runs with a speed 0.55 and the prey try to escape with the velocity 0.5. The figures have two initial points for the position of the prey.
The green curve is the curve followed by the prey and the blue curve is that of the lion.





5. The pirate and the merchant ship

This is the classic formulation of finding the mathematical pursuit curve, on which there actually exists an analytic solution. The merchant ship sails with constant speed v_S along the y-axis, or any other linear direction away from the pirate ship, and the pirate begins the pursuit somewhere on the x-axis with speed v_P, where $v_P > v_S$.

As we have reasoned above, the merchant ship will always be caught, as long as the pirate follows the curve of pursuit, which was derived in the introduction. The pursuit curves are, however much simpler than in the two previous cases. The differential equation for the pursuit is as stated earlier, and we assume a linear trajectory for the merchant ship

$$(x'(t), y'(t)) = v_P \frac{(u(t) - x(t)\,, v(t) - y(t))}{\sqrt{(u(t) - x(t))^2 + (v(t) - y(t))^2}}$$