Examples of the Differential equations of Physics
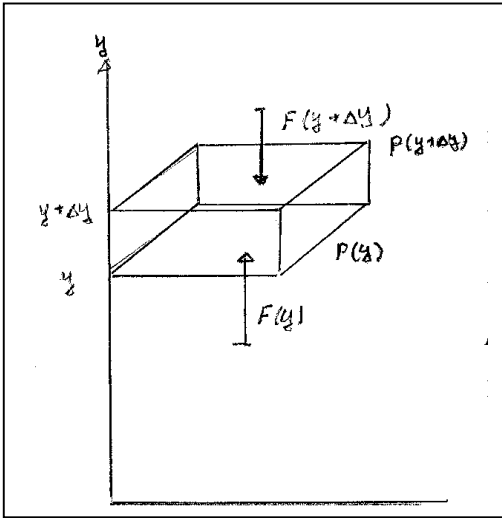
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1. The dependence of pressure with altitude

We consider a rectangular horizontal section of the atmosphere. The area of the two end faces are A. The box is situated in the height y over the ground. The height of the box is Δy. The pressure on the upper and the lower side are $p(y + \Delta y)$ and $p(y)$ respectively.

The density of the air in the height y is $\rho(y)$.

The force on a flat piece with area A is $F = pA$, where p is the pressure on the flat.

We then express that the difference in the force on the upper and the lower side is equal to the gravitational force on the air in the box, assuming that the air in the box is at rest. (g is the gravity acceleration)

$$p(y)A - p(y + \Delta y)A = m_{air}g = \rho(y)V_{air}g = \rho(y)A\Delta yg$$

So
$$p(y)A - p(y + \Delta y)A = \rho(y)A\Delta yg$$

Dividing by $A\Delta y$: $\dfrac{p(y + \Delta y) - p(y)}{\Delta y} = -\rho(y)g$, and replacing $\dfrac{p(y + \Delta y) - p(y)}{\Delta y}$ by $\dfrac{dp}{dy}$ we find:

(1.1)
$$\frac{dp}{dy} = -\rho(y)g$$

To solve this differential equation we need to know another relation between $\rho(y)$ *and* $p(y)$. This can however be obtained by:

1. The equation of state for ideal gasses: $PV = n_M RT$ (n_M is the number of moles)

2. Definition of the mole mass M: $m = n_M M \Leftrightarrow n_M = \dfrac{m}{M}$

3. The definition of density; $\rho = \dfrac{m}{V} \Leftrightarrow m = \rho V$

Insertion of the last two equations in (1) the equation of state gives:

(1.2)
$$PV = n_M RT = \frac{m}{M}RT = \frac{\rho V}{M}RT \quad \Rightarrow \quad \rho = \frac{M}{RT}P$$

This expression for the density is then used in (1.1).

(1.3)
$$\frac{dp}{dy} = -\frac{Mg}{RT}p$$

It is well known that the temperature decreases roughly by one centigrade for every 200 meters increase i altitude over the ground but Initially, we shall assume that the temperature is constant up through the atmosphere.

The differential equation (1.3) has the well known solution:

(1.4) $$p(y) = p_0 e^{-\frac{Mg}{RT}y}$$

Using the known values: $M_{air} = 29$ g/mol, $g = 9.82$ m/s^2, $R = 8.31$ J/(mol K) and $T = 273$ K, we find:

(1.5) $$p(y) = p_0 e^{-1.26 10^{-4} y}$$

Where y should be measured in meters
This results in a pressure drop of 1.3% per 100 m and a drop of 12% per 1000 m.

Next we shall look at the solution to the differential equation, where we take into account that the temperature drops linearly 1 ^0C per 200 m increase in altitude.
We put the temperature on the ground at 20 ^0C = 293 K. The temperature in the altitude y then becomes $T = T(y) = 293 - y/200$. Then the differential equation becomes:

(1.6) $$\frac{dp}{dy} = -\frac{Mg}{R(293 - \frac{y}{200})} p$$

The equation is solved in the usual way by separating the variables and integrating

$$\int_{p_0}^{p} \frac{dp}{p} = -\frac{Mg}{R} \int_0^y \frac{1}{293 - \frac{y}{200}} dy$$

$$\int_{p_0}^{p} \frac{dp}{p} = -\frac{Mg}{293R} \int_0^y \frac{1}{1 - \beta y} dy \qquad where \qquad \beta = \frac{1}{293 \cdot 200}$$

$$\ln(\frac{p}{p_0}) = \frac{Mg}{293R\beta} \ln(1 - \beta y) \qquad \Rightarrow$$

(1.6)

$$p = p_0 (1 - \beta y)^{\frac{Mg}{293R\beta}}$$

Although it looks rather different from (1.5), it turns out that it only causes a deviation from (1.5) of about 0.1 – 0.2 %.

2. Radioactive chains of decay

The differential equation for the number N of radioactive Nuclei, which have not yet decayed is well known from elementary high school.

$$(2.1) \qquad \frac{dN}{dt} = -kN$$

It has the equally well known solution

$$(2.2) \qquad N(t) = N_0 e^{-kt}$$

The *activity* is the rate of decay

$$(2.3) \qquad A(t) = -\frac{dN}{dt} = kN(t)$$

Th constant k is the decay constant, and is equal to ln2 divided by the half life $T_{\frac{1}{2}}$, since

$$(2.4) \qquad \tfrac{1}{2} N_0 = N_0 e^{-kT_{\frac{1}{2}}} \text{ gives:} \qquad k = \frac{\ln 2}{T_{\frac{1}{2}}}$$

We shall then look at a chain, where the original nucleus decays into another radioactive nucleon. This is well known from the common chains of decay: The Uranium-, the Thorium-, and the Actinium series.

If we denote the two nuclei by (1) and (2), we may establish two differential equations.
The first one is identical to (2.1), with decay constant k_1, whereas the second expresses that nucleus (2) is produced with a speed that is equal the activity of nucleus (1), subsequently decays with the decay constant k_2.

$$\frac{dN_1}{dt} = -k_1 N_1 \quad and \quad \frac{dN_2}{dt} = -\frac{dN_1}{dt} - k_2 N_2 \quad \Rightarrow$$

$$(2.2) \qquad \frac{dN_2}{dt} = k_1 N_1 - k_2 N_2$$

The last differential equation has the form

$$(2.3) \qquad \frac{dy}{dx} = -ky + h(x)$$

It is solved by moving the term $-k \cdot y$ to the left hand side, multiplying the equation by e^{kx} , and rewrite it as a single differential quotient.

$$(2.4) \qquad \frac{dy}{dx} = -ky + h(x) \quad \Leftrightarrow \quad \frac{dy}{dx}e^{k \cdot x} + ke^{k \cdot x}y = h(x)e^{k \cdot x} \quad \Leftrightarrow \quad \frac{d(ye^{k \cdot x})}{dx} = h(x)e^{k \cdot x}$$

If $H(x) = \int h(x)e^{k \cdot x} dx$, then the differential equation has the solution:

(2.5) $\qquad ye^{k \cdot x} = H(x) + c \qquad \Leftrightarrow \qquad y = H(x)e^{-k \cdot x} + ce^{-k \cdot x}$

The constant c is the usual constant of integration, which is to be determined by the initial conditions.

Replacing x with t, y with N_2, and $h(x)$ with $N_1(t)$ in (2.3) and following the same manipulations with the new variables, we find:

$$\frac{dN_2}{dt} = k_1 N_1 - k_2 N_2 \qquad \wedge \qquad N_1 = N_0 e^{-k_1 \cdot t} \qquad \Rightarrow$$

$$e^{k_2 \cdot t} \cdot \frac{dN_2}{dt} + k_2 e^{k_2 \cdot t} \cdot N_2 = k_1 N_0 e^{-k_1 \cdot t} e^{k_2 \cdot t} \qquad \Leftrightarrow$$

$$\frac{d(e^{k_2 \cdot t} \cdot N_2)}{dt} = k_1 N_0 e^{(k_2 - k_1) \cdot t} \qquad \Leftrightarrow$$

$$e^{k_2 \cdot t} \cdot N_2 = N_0 \frac{k_1}{k_2 - k_1} e^{(k_2 - k_1) \cdot t} + c \quad \Leftrightarrow$$

$$N_2 = N_2(t) = N_0 \frac{k_1}{k_2 - k_1} e^{-k_1 \cdot t} + ce^{-k_2 \cdot t} \qquad \Leftrightarrow$$

The constant c is determined by $N_2(0) = 0 \Rightarrow c = -N_0 \frac{k_1}{k_2 - k_1}$, and the solution is hereafter:

$$N_2(t) = N_0 \frac{k_1}{k_2 - k_1} e^{-k_1 \cdot t} - N_0 \frac{k_1}{k_2 - k_1} e^{-k_2 \cdot t} \qquad \Leftrightarrow$$

(2.6)

$$N_2(t) = N_0 \frac{k_1}{k_2 - k_1} (e^{(k_2 - k_1) \cdot t} - 1)e^{-k_2 \cdot t}$$

Notice that $N_2 > 0$ for $t > 0$, whether $k_2 > k_1$ or not. (The case $k_2 = k_1$, has only academic interest, but the solution is: $N_2 = k_1 N_0 t e^{-k_2 \cdot t}$).

The result (2.6) is relatively easy to interpret, since the first two factors are the number of (1) nuclei, which have decayed to (2) nuclei, but have not yet decayed, and the last factor is the law of decay for the (2) nuclei.

If the chain of decay is longer than three nuclei a solution to the differential equations can in principle be found in the same manner, as one should just replace the expression for $N_1(t)$ with the expression for $N_2(t)$ in the differential equation for $N_3(t)$.

Solutions of the type (2.6) can be applied to determine the age of radioactive materials.
In praxis we know the two decay constants k_1 and k_2 together with the ratio N_2/N_1.
Then the following equation (2.7) can be applied to find the time t which has elapsed since the material N_1 was created. This has been one of the first reliable methods to determine the correct age of the earth.

(2.7) $$\frac{N_2}{N_1} = \frac{N_0 \frac{k_1}{k_2-k_1}(e^{(k_2-k_1)t}-1)e^{-k_2t}}{N_0 e^{-k_1t}} = \frac{k_1}{k_2-k_1}(1-e^{(k_1-k_2)t})$$

If $k_2 > k_1$ then: $$\frac{N_2}{N_1} \to \frac{k_1}{k_2-k_1} \quad for \ t \to \infty$$

3. Linear motion of a particle in liquids and gasses

When you analyze a mechanical system in order to determine the equation of motion, one is often referred to assume, that the system is without friction or dissipative forces.
This is usually only a realistic description to a certain degree, and sometimes completely unrealistic, but the differential equation, which describes the dynamics of the system, can only be solved in some cases, when the dissipative forces (coming from friction or viscosity) do not depend on the velocity. The latter is the case where to solid materials move relatively to each other.

The aim of this section is to draw attention some simple examples where the friction (the viscosity) is velocity dependent, either linearly or as the square of the velocity.

3.1 A ball sinking in a liquid

We shall first consider a body (characteristically a ball), which sinks in a liquid (water) under the influence of gravity.
If the speed is not too big (and for minor bodies it is not) we have a so called *laminar* streaming, and in that case, we can assume that the resistance to the movement is proportional to the speed of the sinking body.
If the speed of movement becomes larger, the resistance transforms into *turbulent* flow, where the resistance to the movement is more conspicuous, but empirically it is assumed to be proportional to the square of the speed. Turbulence is best described by the appearance of vortices in the liquid or in the air.

Incidentally turbulence is still one of partly unsolved problem hydrodynamics, since the Navier-Stokes equations (Newton's second law for hydrodynamics) do not allow a transition from laminar streaming to turbulent streaming, although both phenomena appear as solution to the equations.

A theoretical expression for the viscous force on a ball in a *laminar* flow is first given by Stoke, and is called Stokes law. If r = radius of the ball, v= the speed, η = viscosity of the fluid, then:

(3.1) $$F_{visc} = 6\pi\eta rv$$

In the following examples, we shall abbreviate the constants $6\pi\eta r$ to one, and we then write the proportionality between the force and the speed as: $F_{visc} = \alpha \cdot v$. This formula is actually independent of the shape of the falling body, as long as the flow is laminar.

For a motion along the x axis, we have the well known concepts.

Velocity: $v = \dfrac{dx}{dt}$, acceleration: $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$, and Newton's 2. law: $F_{res} = ma$

A body falling in a liquid is influenced by the following forces:

1. Gravity: $F_T = mg$.

2. The buoyancy $F_{up} = \rho_v Vg$

Which is equal to the gravity of the displaced liquid, where ρ_v is the density of the fluid and $V = \dfrac{m}{\rho}$ is the volume of a body with mass m and density ρ.

3. The viscous force: $F_{visc} = \alpha v$.

The resulting force on the body is therefore:

$$F_T - F_{up} = mg - \rho_v V g = \rho V g - \rho_v V g = (\rho - \rho_v) V g = m_v g$$

Where $m_v g$ is the gravity of the body reduced by the buoyancy:The equation of motion is hereafter:

(3.2) $\qquad F_{res} = ma \quad \Leftrightarrow \quad m\dfrac{dv}{dt} = m_v g - \alpha v \quad \Leftrightarrow \quad \dfrac{dv}{dt} + \dfrac{\alpha}{m}v = \dfrac{m_v}{m}g$

To obtain more simplicity we put $g_v = \dfrac{m_v}{m}g$

The equation is solved by the same method, as we did in (2.4), by multiplication with $e^{\frac{\alpha}{m}\cdot t}$ and rearranging.

$$e^{\frac{\alpha}{m}t}\dfrac{dv}{dt} + \dfrac{\alpha}{m}e^{\frac{\alpha}{m}t}v = e^{\frac{\alpha}{m}t}g_v \qquad \Leftrightarrow$$

(3.3) $\qquad \dfrac{d(ve^{\frac{\alpha}{m}t})}{dt} = e^{\frac{\alpha}{m}t}g_v \qquad \Leftrightarrow$

$$ve^{\frac{\alpha}{m}t} = \dfrac{m}{\alpha}e^{\frac{\alpha}{m}t}g_v + c \qquad \Leftrightarrow$$

$$v = \dfrac{mg_v}{\alpha} + ce^{-\frac{\alpha}{m}\cdot t} \qquad \Leftrightarrow$$

(3.4)

$$v = \dfrac{m_v g}{\alpha} + ce^{-\frac{\alpha}{m}\cdot t}$$

Adding the initial condition $v(0)=0$, we find $c = -\dfrac{m_v g}{\alpha}$, which inserted in the solution (3.4) gives:

(3.5) $\qquad v = \dfrac{m_v g}{\alpha}(1 - e^{-\frac{\alpha}{m}\cdot t})$

We can see that the velocity approaches asymptotically to $v_\infty = \dfrac{m_v g}{\alpha}$.
The half life of the velocity can be found in the traditional manner:

$$t_{\frac{1}{2}} = \dfrac{\ln 2}{k} \quad og \quad k = \dfrac{\alpha}{m} \quad \Rightarrow \quad t_{\frac{1}{2}} = \dfrac{m\ln 2}{\alpha}.$$

For the majority of motions in liquids, the final velocity is obtained rather quickly.

The equation (3.5) can of course be integrated to give the distance x.

(3.6)
$$x = x_0 + \frac{m_v g}{\alpha}(t + \frac{m}{\alpha}(e^{-\frac{\alpha}{m} \cdot t} - 1))$$

If the body has a initial velocity v_0 opposite to gravity, we must change sign on the mg term in (3.4) and $c = v_0 + \frac{m_v g}{\alpha}$. In this case we find the solution:

(3.7)
$$v = v_0 e^{-\frac{\alpha}{m} \cdot t} + \frac{m_v g}{\alpha}(e^{-\frac{\alpha}{m} \cdot t} - 1)$$

But again the velocity again approaches asymptotically to $v_\infty = -\frac{m_v g}{\alpha}$.

3.2 Vertical motion with turbulent resistance in liquids and gasses

In general the equations of motion cannot be solved in two dimensions, mostly because of the non linear $-v^2$ term in the drag force, but they can actually be solved analytically for a vertical motion.
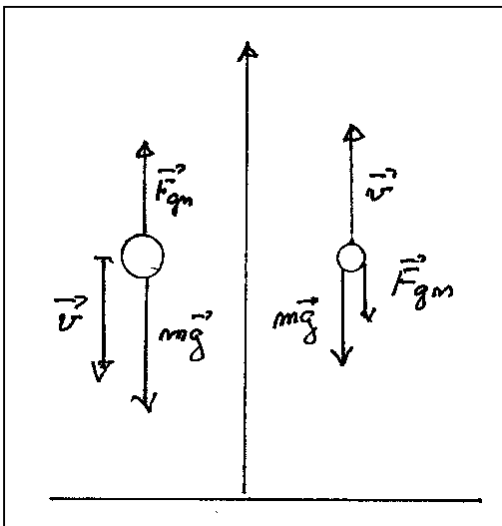
We shall look at a vertical motion of a spherical body in a liquid or a gas, either falling, or moving upwards, as a consequence of buoyancy.
The body is in all cases influenced by

1. The gravitational force. (Always directed downwards)
2. The buoyancy. (Always directed upwards)
3. The viscous resistance. (Always directed against the velocity)

Whether the ball (bullet) sinks or moves upward depends on whether the density of the ball is larger or smaller than the density of the liquid (water) or gas (air).
For the viscous drag force, we shall apply the semi empiric expression.

(3.8)
$$F_{visc} = \tfrac{1}{2} c_w \rho A v^2$$

ρ is the density of the liquid/gas, A is the area of cross section of the body, v is the velocity, and c_w is the so called dimensionless form factor. For convenience we put: $F_{visc} = cv^2$

An estimate of c_w can be found in a standard table of physical constants, where one can also look up the kinematics viscosity v and the dynamic viscosity η.
The connection between the two viscosities is $\eta = v\rho$.

c_w depends on the shape of the body, and the Reynold's number is defined as: $R = \frac{v \cdot D}{v}$.

Where v in the numerator denotes the speed, and v i the denominator is the kinematics viscosity. D is the linear extension of the body.

As we have already seen in section 3, the equations of motion can be solved, if we apply Stoke's law for the drag force $F_{stoke} = 6\pi\eta rv$, but the result for bodies with a diameter larger than a few centimetres and having a weight more than a 100 grams does not yield results in accordance with experience.

If the body moves upwards, as a consequence of the buoyancy, it will be influenced by the buoyancy, the gravity and the viscous resistance, the two latter having the same direction.

$$(3.9) \qquad F_{res} = F_T + F_{up} + F_{visc} \quad \Leftrightarrow \quad ma = -mg + m_v g - cv^2$$

In contrast, if the body sinks it is influenced by the same thee forces, but now the gravity and the drag force have opposite directions.

$$(3.10) \qquad F_{res} = F_T + F_{op} + F_{visc} \quad \Leftrightarrow \quad ma = -mg + m_v g + cv^2$$

m is the mass of the body and m_v is the mass of the displaced liquid according to Archimedes law.

3.2.1 Upward movement

$$(3.11) \qquad a = \frac{dv}{dt} = \frac{m_v - m}{m} g - \frac{c}{m} v^2$$

We put: $\mu = \dfrac{m_v - m}{m}$ in (3.11), and the equation is simplified to: $\quad a = \dfrac{dv}{dt} = \mu g - \dfrac{c}{m} v^2 \quad \Leftrightarrow$

$$(3.12) \qquad \frac{dv}{dt} = \mu g \left(1 - \frac{c}{\mu g m} v^2 \right)$$

The equation can be solved in a usual manner by separating the variables v and t followed by doing some integrals and rearranging the terms, but it is easier to notice that $(\tanh x)' = 1 - \tanh^2 x$

If we put: $k^2 = \dfrac{c}{\mu g m}$ the equation takes the form.

$$(3.13) \qquad \frac{dv}{dt} = \mu g (1 - (kv)^2)$$

And it is seen to have the solution

$$(3.14) \qquad v = \frac{1}{k} \tanh(\mu g k t)$$

Or, when the expressions for k and μ are reinserted.

$$(3.15) \qquad v = \sqrt{\frac{(m_v - m)g}{c}} \tanh \sqrt{\frac{c(m_v - m)g}{m^2}} t$$

tanh approaches rather quickly asymptotic to 1, e.g. *tanh*(1) =0.76 og *tanh*(2) =0.96.

The end velocity is seen to be:

(3.16) $$v_\infty = \frac{1}{k} = \sqrt{\frac{(m_v - m)g}{c}} \ ,$$

as can also be inferred directly by putting: $\dfrac{dv}{dt} = 0$ in (3.13), implying:

$$\mu g(1 - (kv)^2) = 0 \quad \Leftrightarrow \quad v = \frac{1}{k}$$

We may then apply the results to a beach ball, with a diameter of 0.30 m, and estimate how high it will jump, when it is hold under water and released.
You may find the form factor in a table of physical constants, and for a ball it is: $c_w = 0.2$.
For the ball in consideration it gives the value $c = 7.07$ kg/m, in the formula: $F_{visc} = \frac{1}{2} c_w \rho A v^2 = cv^2$.

If we in the formula for the velocity:

$$v = \sqrt{\frac{(m_v - m)g}{c}} \tanh \sqrt{\frac{c(m_v - m)g}{m^2}} t$$

We solve the equation $\sqrt{\dfrac{c(m_v - m)g}{m^2}} t = 2$, corresponding to 96% of the end velocity, we se that

the ball will reach this value in fractions of a second, so we may safely use the end velocity in the calculations.

$$v_{end} = \sqrt{\frac{(m_v - m)g}{c}} = 4.4 \ m/s \ .$$

So when a beach ball is held under water and released, the calculations show that it will jump to:

$$h = \frac{v^2}{2g} = 0.98 \ m$$

For a ping pong ball with a radius 2 cm, and the mass 3.0 g the calculations goes as follows:
The Reynold's number: $R = \dfrac{v \cdot D}{v} = \dfrac{4 \cdot 0{,}04}{1.0 \cdot 10^{-6}} = 1.6 \ 10^7$ gives the form factor $c_w = 0.2$
$A = \pi \ (0.02)^2 \ m^2 = 1.26 \ 10^{-3} \ m^2$, $\rho = 10^3 \ kg/m^3$.
$c = \frac{1}{2} c_w \rho A = 0.1 \cdot 10^3 \cdot 1.26 \ 10^{-3} \ kg/m = 0.126 \ g/m$. $m_v = \frac{4}{3} \pi r^3 \rho = 0.0335 \ kg$, which gives the end velocity:

$$v = \sqrt{\frac{(m_v - m)g}{c}} = \sqrt{\frac{0.0305 \cdot 9.82}{0.126}} \ m/s \ = \ 1.54 \ m/s$$

With this velocity the ping pong ball will, however, only jump: $h = \dfrac{v^2}{2g} = 0.12 \ m$

3.2.2 Downward movement:

We shall look into the case of a body sinking in water. The equation of motion is.

(3.17) $\qquad F_{res} = F_T + F_{op} + F_{visc} \iff ma = -mg + m_v g + cv^2$

The difference from above is, that the density of the body is larger than the density of water, so that $m > m_v$ i.e. the mass of the body is larger than the mass of the displaced water.
The equations of motion are otherwise the same, apart from a minus sign. The acceleration is:

$$a = \frac{dv}{dt} = \frac{m_v - m}{m}g + \frac{c}{m}v^2 = -\frac{m - m_v}{m}g + \frac{c}{m}v^2$$

We put: $\mu = \dfrac{m - m_v}{m}$.

(3.18) $\qquad a = \dfrac{dv}{dt} = -\mu g + \dfrac{c}{m}v^2 \iff \dfrac{dv}{dt} = -\mu g(1 - \dfrac{c}{\mu g m}v^2)$

And we get the same solution as in (3.15), apart from a change in sign.

As before we put: $k^2 = \dfrac{c}{\mu g m}$, which gives the equation:

(3.19) $\qquad \dfrac{dv}{dt} = -\mu g(1 - (kv)^2)$

Having the solution:

$$v = -\frac{1}{k}\tanh(\mu g k t) \qquad \iff \qquad v = -\sqrt{\frac{(m - m_v)g}{c}}\tanh\sqrt{\frac{c(m_v - m)g}{m^2}}t$$

If we for example look at an iron ball with radius 5 *cm* , and density $\rho = 7.8 \ 10^3 \ kg/m^3$,
the constant $c = \frac{1}{2}c_w\rho A = 0.79$ (SI-units), $m_v = \rho_{water}\, V_{ball} = 1.0 \ 10^3 \cdot 4/3(5 \ 10^{-2})^3 \ kg = 0.524 \ kg$,
$m = \rho_{iron}\, V_{ball} = 7.8 \ 10^3 \ 4/3(5 \ 10^{-2})^3 \ kg = 4{,}1 \ kg$. From which we get the end velocity.

$$v_\infty = -\sqrt{\frac{(m - m_v)g}{c}} = -6{,}7 \ m/s = -24 \ km/h$$

3.2.3 Vertical motion in air

For motion of a body in air, we need not to be concerned with the buoyancy, since it is vanishing compared to the gravity and drag forces. The equations of motion are therefore.

Upward: $\qquad\qquad F_{res} = F_T + F_{luft} \iff ma = -mg - cv^2$
(3.20)
Downward: $\qquad\quad F_{res} = F_T + F_{luft} \iff ma = -mg + cv^2$

First we solve for the upward movement:

$$a = \frac{dv}{dt} = -g - \frac{c}{m}v^2 \quad \Leftrightarrow \quad \frac{dv}{dt} = -g(1 + \frac{c}{mg}v^2)$$

Setting $k = \sqrt{\frac{c}{mg}}$ gives:

(3.21)
$$\frac{dv}{dt} = -g(1 + (kv)^2)$$

As mentioned earlier, (3.21) can be solved by separating the variables followed by integration, but it is easier to notice that $(\tan x)' = 1 + \tan^2 x$, (which correspond to the right hand side), and then try with a parametric solution:

$$v = -a \tan bt$$

Differentiating we find:

$$\frac{dv}{dt} = -a(1 + \tan^2 bt)b$$

Which we compare this to: $\frac{dv}{dt} = -g(1 + (kv)^2)$

we see that

$$\tan bt \quad \Rightarrow \quad v = \frac{1}{k}\tan bt$$

so

$$a = -\frac{1}{k} \text{ and consequently } ab = -g \quad \Rightarrow \quad b = kg .$$

The solution then becomes:

(3.22)
$$v - v_0 = -\frac{1}{k}\tan(kgt)$$

where $\quad k = \sqrt{\frac{c}{mg}} \quad$ and $\quad c = \frac{1}{2}c_w \rho A$

If $kgt \ll 1$ then $\tan(kgt) \approx kgt$, and the formula goes into $v = v_0 - gt$, as it should.

If we have a ball with radius $r = 0.05\ m$ and mass $m = 250\ g$, $c = 2.0\ 10^{-3}\ kg/m$ and $k = 0.0285\ s/m$, And if the ball has an initial velocity 5,0 m/s, we can determine the max height, by solving the equation

$$v = 0 \quad \Leftrightarrow \quad \tan gkt = kv_0 , \text{ which gives: } t = 0.51\ s.$$

If we want to find the max height, we must integrate (3.22) to give

(3.23)
$$s - s_0 = -\frac{1}{k^2 g}\ln(\cos(gkt))$$

When we calculate the distance, with $v_0 = 5.0$ *m/s*, it is only a correction on the second decimal compared to a vertical throw without air resistance.

We shall then deal with a free fall with air resistance.

$$F_{res} = F_T + F_{luft} \iff ma = -mg + cv^2 \iff$$

$$-\frac{dv}{dt} = -g + \frac{c}{m}v^2 \iff \frac{dv}{dt} = -g(1 - \frac{c}{mg}v^2) \iff \frac{dv}{dt} = -g(1 - (kv)^2)$$

Where we have put: $k^2 = \dfrac{c}{mg}$.

The last equation has the same solution as (3.15) and (3.19).

(3.24) $$v = -\frac{1}{k}\tanh(gkt)$$

With the end velocity: $$v_\infty = -\frac{1}{k} = -\sqrt{\frac{mg}{c}} \ .$$

Inserting $c = 8.1 \ 10^{-3}$, corresponding to a ball with radius $r = 0.10$ m, and density $1.0 \ 10^3$ *kg/m^3*, we find: $v_{end} = 226$ *m/s*.

The distance completed can be calculated by integrating (3.24) to give: $s - s_0 = \dfrac{1}{gk^2}\ln(\cosh(gkt))$

The end velocity occurs when: $gkt = 2$, which gives: $t = 1/gk = 46$ s, and this corresponds to the distance: $s - s_0 = 5200$ *m*.

I will not be held responsible, whether these results are in accordance with reality. Firstly the v^2 dependence is not necessarily correct, and the form factor is only fixed by a factor of 2.

4. The motion of a projectile

We next consider the trajectory of a projectile, fired at an angle θ, and the initial speed v_0.
First we shall review the trajectory for motion of a particle in the gravitational field near the surface of the earth. The projectile is thought to move in the x –y plane only influenced by gravity, so the equation of motion is:

4.1 The motion of a projectile without drag

(4.1) $\vec{F}_{res} = m\vec{g}$,   where $\vec{g} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$ and $\vec{v}_0 = \begin{pmatrix} v_0 \cos\theta \\ v_0 \sin\theta \end{pmatrix}$ (The initial velocity)

The motion is with constant acceleration, and the solution is:

(4.2) $\vec{v} = \vec{a}t + \vec{v}_0$ and $\vec{r} = \frac{1}{2}\vec{a}t^2 + \vec{v}_0 t + \vec{r}_0$

If we put $\vec{r}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ it is found by straightforward insertion:

(4.3) $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_0 \cos \theta \\ v_0 \sin \theta - gt \end{pmatrix}$ and $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v_0 \cos \theta \, t \\ v_0 \sin \theta \, t - \frac{1}{2} gt^2 \end{pmatrix}$

The max height of the trajectory can be found by putting $v_y = 0 \iff t = \dfrac{v_0 \sin \theta}{g}$ and inserting in y,

gives: $y_{max} = \dfrac{(v_0 \sin \theta)^2}{2g}$

The width (in the x direction) of the trajectory, can be found by setting

$y = 0 \iff v_0 \sin \theta \, t - \frac{1}{2} gt^2 = 0 \iff t = 0 \quad \vee \quad t = \dfrac{2 v_0 \sin \theta}{g}$

The maximum width (length of the throw), is then determined by inserting the second value for t into the expression for $x(t)$. The result it can be reduced to:

(4.4) $x_{max} = \dfrac{v_0^{\,2} \sin 2\theta}{g}$

The longest throw is obtained when $\sin 2\theta = 1 \iff \theta = 45^0$, as is well known from elementary physics.
The trajectory is a parabola, by the way, since eliminating t en the expression for x and y leads to:

$$y = x \tan \theta - \frac{\frac{1}{2} g}{(v_0 \cos \theta)^2} x^2$$

4.2 The motion of a projectile with drag (air resistance)

We shall now consider the same motion as above, in the gravitational field of the earth, but this time paying to the inevitable resistance caused by the air.
Here we shall however only be concerned with *laminar flow*, with is the same as saying that the drag force is proportional to and directed opposite to the velocity.
For motion in the air, this is hardly applicable if the speed exceeds about 5.0 *m/s*, where the flow becomes turbulent, and the equations of motion do not have an analytic solution.

With turbulent flow, the drag force can empirically be represented by $F_{drag} = v^{\beta}$, where $1 < \beta \le 2$.
But we shall preliminary only be concerned with *laminar* flow.
When the motion takes place in gasses, we can safely discard the buoyancy. So in that case:

(4.5) $F_{drag} = \alpha \, |\vec{v}| \quad and \quad \vec{F}_{drag} = -\alpha \vec{v}$.

The equation of motion becomes:

$$\frac{d\vec{v}}{dt} = \vec{g} - \frac{\alpha}{m}\vec{v} \qquad \Leftrightarrow$$

(4.6)

$$\frac{dv_x}{dt} = -\frac{\alpha}{m}v_x \quad \wedge \quad \frac{dv_y}{dt} = -g - \frac{\alpha}{m}v_y$$

The differential equation (4.1) separates with respect to the x-direction and the y-direction.
But we have already solved such an equation for a linear motion in (3.2) to (3.5).
If the initial velocity is $\vec{v} = (v_0\cos\theta, v_0\sin\theta)$, we can simply copy the solution from (3.5).

(4.7) $v_x = v_0\cos\theta \cdot e^{-\frac{\alpha}{m}\cdot t}$ and $v_y = v_0\sin\theta \cdot e^{-\frac{\alpha}{m}\cdot t} - \frac{mg}{\alpha}(1 - e^{-\frac{\alpha}{m}\cdot t})$

If $\frac{\alpha}{m}\cdot t \ll 1$, that is, if the resistance of the air is small, we may apply the approximation $e^x \approx 1 + x$

to the factor $e^{-\frac{\alpha}{m}\cdot t}$ to obtain.

$$v_x = v_0\cos\theta\cdot(1 - \frac{\alpha}{m}\cdot t) \quad and \quad v_y = v_0\sin\theta\cdot(1 - \frac{\alpha}{m}\cdot t) - \frac{mg}{\alpha}(1 - (1 - \frac{\alpha}{m}\cdot t))$$

Dropping all terms proportional to α, we retrieve the formulas (4.3) for the motion without drag.

(4.8) $v_x = v_0\cos\theta \quad og \quad v_y = v_0\sin\theta - g\cdot t$

We may also find the position $x(t), y(t))$, by integrating (4.7) with respect to t.
Choosing $(x_0, y_0) = (0,0)$, we get:

$$x = v_0\cos\theta\int_0^t e^{-\frac{\alpha}{m}\cdot t}dt \quad and \quad y = v_0\sin\theta\int_0^t e^{-\frac{\alpha}{m}\cdot t}dt - \frac{mg}{\alpha}\int_0^t(1 - e^{-\frac{\alpha}{m}\cdot t})dt$$

(4.9) $x = v_0\cos\theta\cdot\frac{m}{\alpha}(1 - e^{-\frac{\alpha}{m}\cdot t}) \quad and \quad y = v_0\sin\theta\cdot\frac{m}{\alpha}(1 - e^{-\frac{\alpha}{m}\cdot t}) - \frac{mg}{\alpha}(t - \frac{m}{\alpha}\cdot(1 - e^{-\frac{\alpha}{m}\cdot t}))$

Again if $\frac{\alpha}{m}t \ll 1$, we may apply the approximation $e^x \approx 1 + x + \frac{1}{2}x^2$ with $x = -\frac{\alpha}{m}\cdot t$.

Dropping all terms, proportional to α, we retrieve the former expressions (4.3), derived without resistance.

$$x = v_0\cos\theta\cdot t \quad og \quad y = v_0\sin\theta\cdot t - \tfrac{1}{2}g\cdot t^2$$

Neither (4.8) nor (4.9) are particular transparent, when determining the maximum height or the width of the throw. It is actually possible to determine y_{max}, but we cannot determine x_{max}, since the equation $y = 0$ is transcendent.
In a later section we shall however look at numerical solutions to differential equations.

As mentioned, the equation of motion for a projectile can not be solved when assuming turbulent flow i.e. when the drag force is proportional to v^2, $F_{drag} = \alpha v^2$. Below is shown a numerical solution with: $\alpha = 0$ (no air resistance) , $\alpha = 0.0001$, $\alpha = 0.0005$, $\alpha = 0.001$.
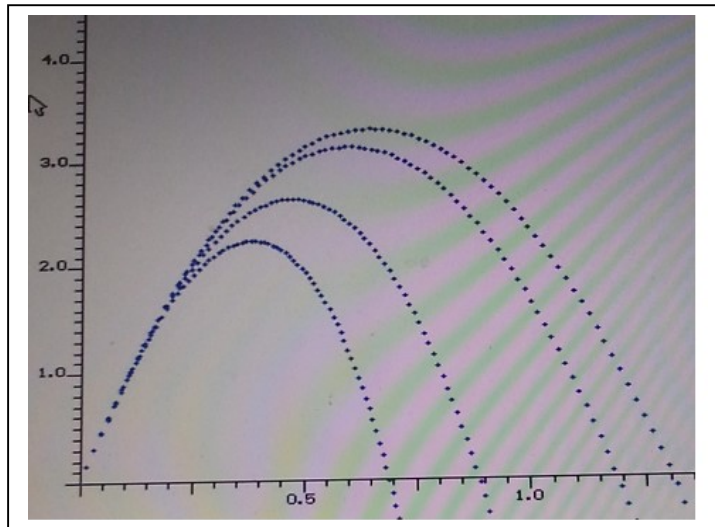
5. Damped harmonic oscillations

A harmonic oscillation is a linear movement (along an axis), where the resulting force is always directed against and proportional to the distance to the position of equilibrium.
If the motion is along the x – axis, then the equation of motion is:

$$(5.1) \qquad F_{res} = -k \cdot x \quad \Leftrightarrow \quad m\frac{d^2x}{dt^2} = -kx \quad \Leftrightarrow \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

If we put $\omega = \sqrt{\dfrac{k}{m}}$ (the cyclic frequency), then the equation becomes:

$$(5.2) \qquad \frac{d^2x}{dt^2} = -\omega^2 x$$

Which has the familiar solution:

$$(5.2) \qquad x = A\cos(\omega \cdot t + \varphi_0)$$

A is the amplitude, ω is the cyclic frequency, and φ_0 is the initial phase.

The period is $\qquad\qquad T = \dfrac{2\pi}{\omega} \quad \Leftrightarrow \quad T = 2\pi\sqrt{\dfrac{m}{k}}$.

In the Math classroom, one usually writes the solution in a slightly different way:

$$(5.3) \qquad x = c_1 \cos\omega t + c_2 \sin\omega t$$

That this is actually the same solution, one may realize, by applying one of the addition formula, mentioned earlier:

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

to the solution (5.2).

$$x = A\cos(\omega \cdot t + \varphi_0) = A\cos(\varphi_0)\cos(\omega \cdot t) - A\sin(\varphi_0)\sin(\omega \cdot t)$$

If we put $c_1 = A\cos(\varphi_0)$ *and* $c_2 = -A\sin(\varphi_0)$, having the solutions: $\tan\varphi_0 = -\dfrac{c_2}{c_1}$, $A = \sqrt{c_1^2 + c_2^2}$

We regain the solution (5.2).

If there is friction or viscous forces (drag forces), another term has to be added to (5.1). We shall first assume that the drag force is proportional to the velocity, and directed opposite to the velocity.

The coefficient of proportionality depends on the shape of the body, and the nature of the medium (air, liquid) the body moves in.

(5.4) $$F_{visc} = -\alpha v \quad \Rightarrow \quad F_{visc} = -\alpha \frac{dx}{dt} \; .$$

Hereafter the differential equation for the damped harmonic oscillator becomes.

$$F_{res} = -k \cdot x + F_{visc} \quad \Leftrightarrow$$

(5.5) $$m\frac{d^2x}{dt^2} = -\alpha \frac{dx}{dt} - kx \quad \Leftrightarrow$$

$$\frac{d^2x}{dt^2} + \frac{\alpha}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

It is however associated with a bit more ingenuity to solve (5.5), than to solve (5.1).
First we simplify the equation a bit, with the aim of having lesser constants.

(5.6) $$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c \cdot x = 0$$

Where we have put $b = \dfrac{\alpha}{m}$ *og* $c = \dfrac{k}{m}$

(5.6) is now a 2. order linear homogenous differential equation with the two constants b and c.
It is linear, because all terms containing x appears in first order, and homogenous, because there are no terms which depend explicit on t.

5.1 Solution to the differential equation using complex numbers.

The solution of linearly homogenous differential equations with constant coefficients, can always be reduces to finding the complex roots in the characteristic polynomial of the same order as the differential equation
To solve (5.6), we put $x = e^{z \cdot t}$ where z is a complex number to be determined. It then follows:

$$\frac{dx}{dt} = z \cdot e^{z \cdot t} \quad og \quad \frac{d^2 x}{dt^2} = z^2 \cdot e^{z \cdot t}$$

Inserting in (5.6) and dividing by $e^{z \cdot t}$, we get a complex quadratic equation:

$$z^2 + b \cdot z + c = 0$$

The discriminant is: $d = b^2 - 4 \cdot c$. The discriminant is real, so if $d > 0$ then quadratic equation has two real solutions.

$$(5.7) \qquad z = -\frac{b}{2} + \frac{\sqrt{b^2 - 4 \cdot c}}{2} \quad \vee \quad z = -\frac{b}{2} - \frac{\sqrt{b^2 - 4 \cdot c}}{2}$$

Returning to the original equation, we notice that $b > 0$ and $c = k/m > 0$, so both solutions (5.7) are negative. The case $d = 0$ reduces to one solution.

If $d < 0$ then the quadratic equation has no real solutions, but rather two complex solutions.

$$(5.8) \qquad z = -\frac{b}{2} + i\frac{\sqrt{4 \cdot c - b^2}}{2} \quad \vee \quad z = -\frac{b}{2} - i\frac{\sqrt{4 \cdot c - b^2}}{2}$$

Here i is the complex unit. $i^2 = -1$.

In the theory of complex numbers one of the most important formulas is Eulers formula.
Actually one of the most important formulas in the mathematical analysis at all.
If $z = x + i \cdot y$ is a complex number, where x and y are real, Euler's formula reads:

$$(5.9) \qquad e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$$

We are only interested in the real part of the solution – naturally!
Futhermore, we notice that when we made the substitution $x = e^{z \cdot t}$, we might as well have written $x = A e^{z \cdot t + i \varphi_0}$, having two integration constants. necessary for the complete solution of a second order differential equation.

$$(5.10) \qquad x(t) = A e^{-\frac{b}{2}t} \cos(\omega t + \varphi_0)$$

What we see is that the solution is a harmonic oscillation, but with an amplitude which decreases exponentially with time. This is called damped harmonic oscillations.
If the original values for the constants for b and c: $b = \frac{\alpha}{m}$ and $c = \frac{k}{m}$, are substituted back, we

get $\omega = \sqrt{\frac{k}{m} - \frac{\alpha^2}{4m^2}}$. Inserted in (5.10) gives:

$$(5.11) \qquad x(t) = A e^{-\frac{\alpha}{2m} \cdot t} \cos(\sqrt{\frac{k}{m} - \frac{\alpha^2}{4m^2}} \cdot t + \varphi_0)$$

α is the coefficient of viscosity, defined by the equation: F_{visc} = -α·v, and k is "constant of the spring".

The condition for the validity of the solution (5.11) is that the expression under the square root is positive. Otherwise the oscillating system will never perform one period, and the system will approach the equilibrium exponentially.

5.2 Traditional solution of the same differential equation

We shall again look at the differential equation

(5.12) $$\frac{d^2x}{dt^2} + \frac{\alpha}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

For convenience, as above, we make some abbreviation.

(5.13) $$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c \cdot x = 0 \quad , \text{hvor } b = \frac{\alpha}{m} \quad og \quad c = \frac{k}{m}$$

We solved earlier the equation by resorting to complex numbers, but here we shall apply a more traditional method, resembling the method used to solve a general linear first order equations.

The method is to introduce an aiding function, equipped to rewrite the differential equation to one, we can solve, that is, the equation for the harmonic oscillator.

(5.14) $$m\frac{d^2y}{dt^2} = -ky \quad \Leftrightarrow \quad \frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \quad \Leftrightarrow \quad \frac{d^2y}{dt^2} + \omega^2 y = 0$$

Having the solution:

(5.15) $$y = A\cos(\omega t + \varphi_0)$$

To obtain this we have put $y = x\,e^{\beta \cdot t}$, where x refers to the solution to the original differential equation (5.13)

(5.16) $$\frac{d^2(xe^{\beta \cdot t})}{dt^2} + \omega^2 xe^{\beta \cdot t} = 0$$

By a suitable choice of β and ω^2, we hope to make (5.16) have the same form as (5.13)

$$\frac{d^2(xe^{\beta \cdot t})}{dt^2} = \frac{d}{dt}(\frac{dx}{dt}e^{\beta \cdot t} + x\beta e^{\beta \cdot t}) = \frac{d^2x}{dt^2}e^{\beta \cdot t} + \frac{dx}{dt}\beta e^{\beta \cdot t} + \frac{dx}{dt}\beta e^{\beta \cdot t} + x\beta^2 e^{\beta \cdot t}$$

$$\frac{d^2(xe^{\beta \cdot t})}{dt^2} = \frac{d^2x}{dt^2}e^{\beta \cdot t} + 2\beta\frac{dx}{dt}e^{\beta \cdot t} + x\beta^2 e^{\beta \cdot t}$$

We add the term $\omega^2 xe^{\beta \cdot t}$ to the second derivative of $xe^{\beta \cdot t}$ and put the result to 0.

(5.17)
$$\frac{d^2(xe^{\beta \cdot t})}{dt^2} + \omega^2 xe^{\beta \cdot t} = 0 \qquad \Leftrightarrow$$

$$\frac{d^2 x}{dt^2} e^{\beta \cdot t} + 2\beta \frac{dx}{dt} e^{\beta \cdot t} + x\beta^2 e^{\beta \cdot t} + \omega^2 xe^{\beta \cdot t} = 0$$

The equation is simplifying by diving by $e^{\beta \cdot t}$.

(5.18)
$$\frac{d^2 x}{dt^2} + 2\beta \cdot \frac{dx}{dt} + (\beta^2 + \omega^2) = 0$$

We then compare (5.18) with the original differential equation:

(5.19)
$$\frac{d^2 x}{dt^2} + b\frac{dx}{dt} + c \cdot x \cdot = 0$$

And we see that the two differential equations are identical if and only if:

$$\beta = \frac{b}{2} = \frac{\alpha}{2m} \quad \text{and} \quad \beta^2 + \omega^2 = c \ \Leftrightarrow \ \omega^2 = c - \frac{b^2}{4} \ \Rightarrow \ \omega^2 = \frac{k}{m} - \frac{\alpha^2}{4m}.$$

However, we can solve (5.17) directly. If we put $y = x \cdot e^{\beta \cdot t}$, the equation takes the form:

(5.20)
$$\frac{d^2 y}{dt^2} = -\omega^2 y \qquad \text{has the solution:} \quad y = A\cos(\omega t + \varphi_0)$$
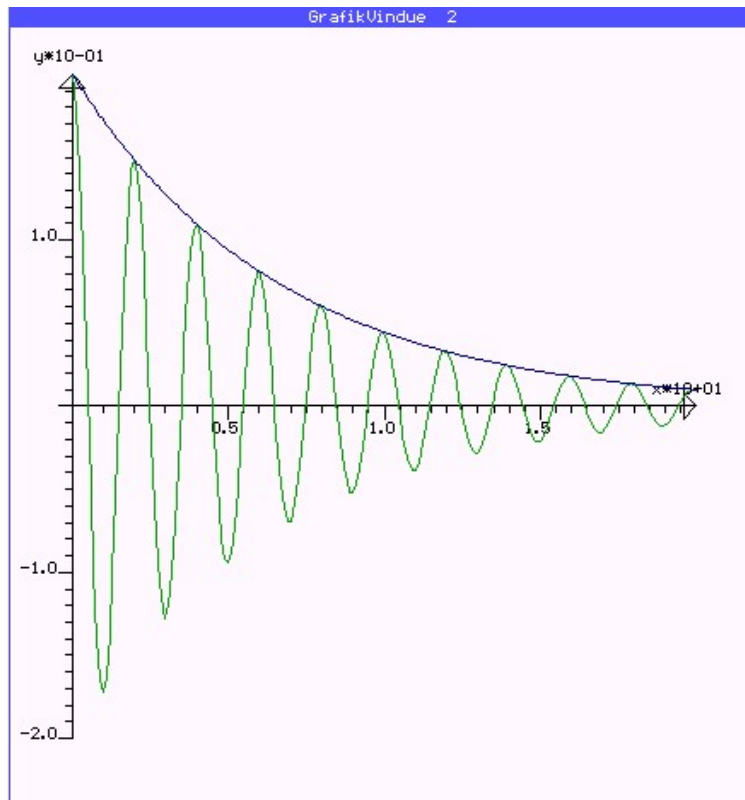
So we find:

(5.21)
$$y = x \cdot e^{\beta \cdot t} = A\cos(\omega t + \varphi_0) \qquad \Leftrightarrow \qquad x = A \cdot e^{-\beta \cdot t}\cos(\omega t + \varphi_0)$$

Inserting the constants: $\beta = \dfrac{\alpha}{2m}$ and $\omega^2 = \dfrac{k}{m} - \dfrac{\alpha^2}{4m}$ we arrive at the solution to (5.12)

(5.22)
$$x(t) = Ae^{-\frac{\alpha}{2m} \cdot t}\cos(\sqrt{\frac{k}{m} - \frac{\alpha^2}{4m^2}} \cdot t + \varphi_0).$$

The solution is a harmonic oscillation with exponential decreasing amplitude.

Below is shown an example of a solution, where the exponential envelope curve is also shown.

Damped harmonic oscillations turn up in many fields of physics, and therefore it is not without interest to be able to solve the associated differential equations.

6. Forced harmonic oscillations without damping

We shall consider a forced oscillation without damping, where the mass m, besides the "spring force" i.e. obeys Hookes law $-kx$, is driven by an external time dependent force $f(t)$.
The results are directly applicable to an electrical circuit consisting of a capacitor and a coil, and driven by an alternating current.

$$F_{res} = -k \cdot x + F_{ext} \quad \Leftrightarrow$$

(6.1)
$$m\frac{d^2x}{dt^2} = -kx + F_{ext}(t) \quad \Leftrightarrow$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{F_{ext}(t)}{m}$$

We shall assume that the external force varies harmonically: $\dfrac{F_{ext}(t)}{m} = \dfrac{f_0}{m}e^{i\omega t}$.

The solution to (6.1) is (as well known from the theory of differential equations) a particular solution to the non homogeneous equation plus the complete solution to the corresponding homogeneous equation:

(6.2)
$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Which has the solution:

$$x = A\cos(\omega_0 t + \varphi) \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

Since the differential equation

(6.3)
$$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{f_0}{m}e^{i\omega t} \quad \Leftrightarrow \quad \frac{d^2x}{dt^2} + \omega_0^2 x = \frac{f_0}{m}e^{i\omega t}$$

of second order with constant coefficients, we may write a particular solution as: $x = Ae^{i\omega t}$ (where ω is the enforced frequency), when inserted in (6.3) gives:

(6.4)
$$-\omega^2 Ae^{i\omega t} + \omega_0^2 Ae^{i\omega t} = \frac{f_0}{m}e^{i\omega t},$$

When solved with respect to A gives: $A = \dfrac{\dfrac{f_0}{m}}{\omega_0^2 - \omega^2}$

The complete solution to (6.3) can thereafter be written as the particular solution plus the solution to the homogeneous equation:

(6.5)
$$x = A_0\cos(\omega_0 t + \varphi) + \frac{\dfrac{f_0}{m}}{\omega_0^2 - \omega^2}\cos(\omega t)$$

Writing this as: $x = A \cdot \cos(\omega_0 t + \varphi) + B \cdot \cos(\omega t)$, we can in the case where $A = B$ apply the first of the logarithmic formulas for addition of two cosine functions:

$$\cos u + \cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2} \quad \text{og} \quad \cos u - \cos v = -2\sin\frac{u+v}{2}\sin\frac{u-v}{2}$$

(6.6)
$$x = 2A\cos(\frac{\omega_0 + \omega}{2}t + \tfrac{1}{2}\varphi)\cos(\frac{\omega_0 - \omega}{2}t + \tfrac{1}{2}\varphi)$$

The system will perform oscillations with a frequency $\dfrac{\omega_0 + \omega}{2}$ with an amplitude

$2A\cos(\dfrac{\omega_0 - \omega}{2}t + \tfrac{1}{2}\varphi)$, which is time dependent and varies between the values -2A and 2A.

A phenomena which is familiar for sound waves, and goes under the name modulation and beats.

The beat frequency is $\dfrac{\omega_0 - \omega}{2}$, and when $\omega_0 - \omega << \omega_0$ the signal will sound like a sirene.

In general the two amplitudes A and B are not equal, but it changes only the situation in the sense that the signal will have two beat frequencies instead of one.

We may namely always determine two numbers C and D, so that $A = C+D$ and $B = C - D$, and solve for C and D:

$$C = \frac{A+B}{2} \quad og \quad D = \frac{A-B}{2}$$

$A \cdot \cos(\omega_0 t + \varphi) + B \cdot \cos(\omega t) = (C+D)\cos(\omega_0 t + \varphi) + (C\text{-}D)\cos(\omega t) =$

$C \cdot \cos(\omega_0 t + \varphi) + C \cdot \cos(\omega t) + D \cdot \cos(\omega_0 t + \varphi) \text{-} D \cdot \cos(\omega t)$

Rewriting the solution using the logarithmic formulas:

$$(6.6) \quad x = 2C\cos(\frac{\omega_0 + \omega}{2}t + \tfrac{1}{2}\varphi)\cos(\frac{\omega_0 - \omega}{2}t + \tfrac{1}{2}\varphi) - 2D\sin(\frac{\omega_0 + \omega}{2}t + \tfrac{1}{2}\varphi)\sin(\frac{\omega_0 - \omega}{2}t + \tfrac{1}{2}\varphi)$$

The result is two modulations with the same frequency, but where the amplitudes (the beats) are $\frac{\pi}{2}$ out of phase.

7. Numerical solution to differential equations

It is actually the fewer differential equations in physics, which have an analytical solution.
Analytical means that the solution is given by mathematical expressions that describes the position and velocity at any time *t*.
The mathematical discipline which deals with numerical, that is, step by step solutions to problems is called numerical analysis.
Theoretically numerically analysis covers a large area of methods, and in contras to what one may think it is developed long before the invention of computers.

However, one should not diminish the importance of analytical solutions, even if the small mathematics computers, which has already been in use in high schools for almost ten years now (2016) are able to do differentiation, evaluate integrals and solve differential equations analytically, that my generation had to do by hand.
But of course the computers can do no more, than (some of us) could do by hand.

The alternative to analytical solutions are numerical solutions, where one roughly speaking replaces infinitesimal entities dx and dy by small changes Δx, Δt, differential quotients $\frac{dx}{dt}$ by

$\frac{\Delta x}{\Delta t}$ and integrals $\int f(t)dt$, by sums $\sum f(t_i)\cdot \Delta t_i$

But the theory of numerical analysis relies heavily on analytic methods of course!

7.1 Taylor's formula

We shall begin by looking at numerical solution to first order differential equations.
When doing things numerically, it is imperative to be able to estimate the accuracy.
To do so, it is strictly necessary to be familiar with Taylor's formula. The formula can be written in numerous ways, but we shall apply the one where a real function $y = f(x)$ is developed around a point x_0, and h is (a small) increment to x_0. Under fairly general circumstances the formula can be written:

$$(7.1) \quad f(x_0 + h) = f(x_0) + \frac{f'(x_0)}{1!}h + \frac{f''(x_0)}{2!}h^2 + \frac{f^{(3)}(x_0)}{3!}h^3 + ... + \frac{f^{(n)}(x_0)}{n!}h^n + \int_0^h \frac{f^{(n+1)}(x_0 + t)}{n!}t^n dt$$

The last term (the rest term) is seen to be proportional to h^{n+1}, which we write as $O(h^n)h$, where the symbol $O(h^n)$ is to be read as "order of h^n ". If we omit the rest term, we get an approximation to $f(x_0+h)$. Depending how many terms we include, we get an 0'th, 1., 2., order approximation.

$$f(x_0 + h) = f(x_0) + O(h^0)h$$
$$f(x_0 + h) \approx f(x_0)$$

$$(7.2) \qquad f(x_0 + h) = f(x_0) + f'(x_0)h + O(h)h$$
$$f(x_0 + h) \approx f(x_0) + f'(x_0)h$$

$$(7.3) \qquad f(x_0 + h) = f(x_0) + f'(x_0)h + \tfrac{1}{2}f''(x_0)h^2 + O(h^2)h$$
$$f(x_0 + h) \approx f(x_0) + f'(x_0)h + \tfrac{1}{2}f''(x_0)h^2$$

(7.4)
$$f(x_0 + h) = f(x_0) + f'(x_0)h + \tfrac{1}{2}f''(x_0)h^2 + \tfrac{1}{6}f^{(3)}(x_0)h^3 + O(h^3)h$$
$$f(x_0 + h) \approx f(x_0) + f'(x_0)h + \tfrac{1}{2}f''(x_0)h^2 + \tfrac{1}{6}f^{(3)}(x_0)h^3$$

7.2 Numerical solution to first order differential equations

If we wish to solve the firs order equation

(7.5)
$$\frac{dy}{dx} = f'(x) = g(x,y)$$

Where we know the initial value (x_0, y_0), then it can be done by applying (7.1), since:

(7.6)
$$f(x_0 + h) \approx f(x_0) + f'(x_0)h = y_0 + g(x_0, y_0)h$$

Where $f(x_0 + h)$ is calculated with an accuracy of order h^2.

$$(x_1,\ y_1) = (x_0{+}h,\ f(x_0{+}h)) = (x_0{+}h,\ f(x_0){+}f'(x_0)h) = (x_1,\ y_0{+}g(x_0,\ y_0))$$

The new value can thereafter be used to calculate:

$$(x_2,\ y_2) = (x_1{+}h,\ f(x_1{+}h)) = (x_1{+}h,\ f(x_1){+}f'(x_1)h) = (x_1{+}h,\ y_1{+}g(x_1,\ y_1)\,h)$$

and so on. The method is called Euler integration.
Euler's formula is however hardly ever used, because the errors accumulate, especially when $f''(x)$ is constant. To obtain a better approximation, one may use Aitken's formula:

(7.5)
$$f'(x_0) \approx \frac{f(x_0 + \tfrac{h}{2}) - f(x_0 - \tfrac{h}{2})}{h} \Leftrightarrow f(x_0 + \tfrac{h}{2}) = f(x_0 - \tfrac{h}{2}) + f'(x_0)h$$

Expanding $f(x_0 + \tfrac{h}{2}) - f(x_0 - \tfrac{h}{2})$ by Taylor's formula, we find this time:

$$f(x_0 + \tfrac{h}{2}) - f(x_0 - \tfrac{h}{2}) = f(x_0) + f'(x_0)\tfrac{h}{2} + \tfrac{1}{2}f''(x_0)\tfrac{h^2}{4} - (f(x_0) + f'(x_0)(-\tfrac{h}{2}) + \tfrac{1}{2}f''(x_0)\tfrac{h^2}{4}) + O(h^2)h$$

(7.6)
$$f(x_0 + \tfrac{h}{2}) - f(x_0 - \tfrac{h}{2}) = f'(x_0)h + O(h^2)h$$

As it is seen this formula is correct until order h^3, in contrast to Euler's h^2 approximation, but using the same first order derivatives.
If $h = \tfrac{1}{10}$, then is the correction (the error) of order $h^3 = \tfrac{1}{1000}$ i stead of the Euler integration, where the correction is of order $h^2 = \tfrac{1}{100}$. Otherwise the procedure is the same as the Euler integration, apart from the initial step, where one must calculate $f(x_0 + \tfrac{h}{2})$, by Euler's method.

(7.7) $$f(x_0 + \frac{h}{2}) = f(x_0 - \frac{h}{2}) + f'(x_0)h = f(x_0 - \frac{h}{2}) + g(x_0, y_0)h$$

To scientific an practical purposes, one applies almost always the method of Runge-Kutta, which is far more complicated to account for, than the methods described above. But it has the definite advantage that the corrections (the errors) is of order h^4.