Vertical motion in liquids and gasses

This is an article from my home page: www.olewitthansen.dk

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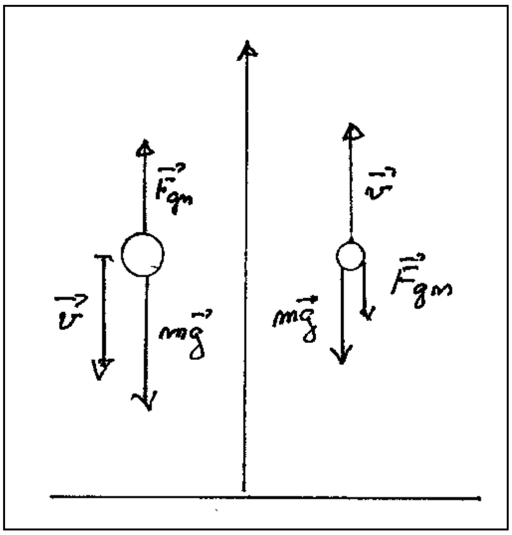
1. Introduction to the issue

The students in the Danish 9-12 grade high school must hand in a project during the last semester in school. The subject of the project is mainly chosen by the student himself.
This may result in subjects where the teacher has no knowledge whatsoever, and neither has the student. Search in the Internet gives only rarely a tangible theoretical analysis, and since the teacher has the responsibility of formulate the project, this situation often results in that the teacher must himself provide the necessary notes, that the student then can copy more or less.
The article below is an example of the present rather awkward state of the educational system in Denmark.
Some projects are chosen to be concerned with viscous motion in liquids or motion with air drag. Nowadays the theoretical aspect in the Danish high school is almost absent, apart perhaps from establishing the equations of motion (but not solving them by hand).
Instead the projects are focused on making videos with high speed cameras followed by analyzing the videos with specialized computer programs that have appeared during the last fifteen years.

With the present (very low) theoretical level in mathematics and physics, presenting an analytical solution is not realistic, but nevertheless it may be interesting to make such an analysis, which may be compared to the computer graphs, coming from the high speed cameras.

Searching the Internet, I have not found any theoretical treatment to this kind of problems, and therefore I had (as many times before in similar situations) to make my own analysis, which is, however, not founded on scientific computer programs, but rather on my studies on mathematics and physics at the university of Copenhagen in the sixties.

2. Vertical motion in liquids



We shall first consider a body, which moves vertically in a liquid or gas, only influenced by gravity, the buoyancy, and the viscous force.
For liquids, we shall separate the equation of motion whether the density of the body is greater or less than the density of the liquid. In the first case the body will sink, and in the second case it will move up to the surface.
The viscous force is always directed against the motion, while the gravitational force is always directed downwards.
We shall here apply the semi-empiric expression for the viscous force.

(2.1) $$F_{visc} = \tfrac{1}{2} c_w \rho A v^2$$

ρ is the density of the liquid/gas, A is the (largest) cross section of the moving body, v is the velocity of the body, and c_w is the so called (dimensionless) form factor.

To ease the calculations we put: $F_{visc} = cv^2$.

An approximate values for c_w may be found in a handbook of physical tables, where you may find both the kinematical viscosity ν and the dynamical viscosity η.

The relation between the two entities is that: η = νρ. The value of c_w is stated for different designs of the moving body, and the Reynold's number is defined as:

$$\mathrm{Re} = \frac{v \cdot D}{\nu}$$

Where v (in the numerator) is the velocity, while D represents the linearly dimension of the body.

It should be noted, the equation of motion may be solved[1] if one applies Stokes law, assuming that the viscous force is linearly dependent of the velocity, which is the case for non turbulent motion. For a spherical body Stokes law reads:

(2.2) $$F_{stoke} = 6\pi\eta r v$$

Here r is the radius of the spherical body, v is the velocity and η is the drag coefficient.
For bodies with a diameter lager than 1 *cm*, having a mass larger than 50 *g*, Stokes law is generally not in accordance with reality.
When the body moves upward, caused by the buoyancy, it will also be influenced by gravity and the viscous force. Then the gravity and the viscous force have the same direction, and the resulting force on the moving body is therefore:

(2.3) $$F_{res} = F_G + F_{Buo} + F_{visc} \quad \Leftrightarrow \quad ma = -mg + m_v g - cv^2$$

Here m is the mass of the body, and m_v is the mass of the displaced amount of liquid, according to Archimedes law.
On the other hand, if the body sinks, then the body is now influenced by the buoyancy, the gravity and the viscous force, which are now opposite directed.

(2.4) $$F_{res} = F_G + F_{Buo} + F_{visc} \quad \Leftrightarrow \quad ma = -mg + m_v g + cv^2$$

2.1 Upward motion

From (2.3) we get:

$$a = \frac{dv}{dt} = \frac{m_v - m}{m} g - \frac{c}{m} v^2$$

We put $\mu = \dfrac{m_v - m}{m}$ and get: $a = \dfrac{dv}{dt} = \mu g - \dfrac{c}{m} v^2 \quad \Leftrightarrow$

(2.5)
$$a = \frac{dv}{dt} = \mu g - \frac{c}{m} v^2 \quad \Leftrightarrow$$
$$\frac{dv}{dt} = \mu g \left(1 - \frac{c}{\mu g m} v^2\right)$$

[1] www.olewitthansen.dk: Non trivial differential equations of physics

The equation may be solved by standard methods, by separating the variables v and t and integrating both sides, but it is more conveniently solved, if we notice that: (tanh x)' = 1 – tanh^2 x .

If we put $k^2 = \frac{c}{\mu g m}$, the equation takes the form

(2.6) $$\frac{dv}{dt} = \mu g(1-(kv)^2)$$

And it is seen to have the solution:

(2.7) $$v = \frac{1}{k}\tanh(\mu g k t)$$

Or written with the original parameters, we find:

(2.8) $$v = \sqrt{\frac{(m_v - m)g}{c}}\tanh\left(\sqrt{\frac{c(m_v - m)g}{m^2}}t\right)$$

tanh approaches quickly asymptotically to 1, e.g. tanh(1) =0.76 and tanh(2) =0.96.

The final velocity is seen to be: $v = \frac{1}{k} = \sqrt{\frac{(m_v - m)g}{c}}$, which can also be seen, by putting $\frac{dv}{dt} = 0$ in (2.6) which gives: $\mu g(1-(kv)^2) = 0 \Leftrightarrow v = \frac{1}{k}$.

2.2 Example

If we look at a beach ball, having a diameter 0.30 m, then we shall calculate the speed in which it leaves the surface, when it is held under water and released.

The form factor c_w may be found in a table of physical constants, and here we find that for a sphere it is around 0.2 .

For the beach ball in question, this results in the value for c = 7.07 kg/m in the formula $F_{visc} = \frac{1}{2}c_w \rho A v^2 = cv^2$. If we solve for the argument of tanh in (8.2):

$$\sqrt{\frac{c(m_v - m)g}{m^2}}t = 2,$$

which as mentioned corresponds to 96% of the final velocity, we can see that it takes only fractions of a second before the final velocity is reached. So we may safely in the examples below assume that the ball has the final velocity, when it reaches the surface, even if it is released less than a meter blow the surface.

We thus find: $$v = \sqrt{\frac{(m_v - m)g}{c}} = 4.4\ m/s.$$

So if we release a beach ball, which has been held under water, it will jump to a height:

$$h = \frac{v^2}{2g} = 0.98\, m\,.$$

For a ping pong ball, the calculations give a rather different result, which is obvious since the buoyancy is proportional to the volume, whereas the mass is proportional to the surface area.

For a ping pong ball having radius 2 *cm*, and mass 3.0 *g* the calculations goes as follows: The Reynolds number: $\mathrm{Re} = \frac{v \cdot D}{\nu} = \frac{4 \cdot 0.04}{1.0 \cdot 10^{-6}} = 1{,}6\,10^7$, which results in a form factor $c_w = 0.2$. The surface area is $A = \pi\,(0.02)^2\ m^2 = 1.26\ 10^{-3}$, and the density of water: $\rho = 10^3\ kg/m^3$.
From which we find:

$c = \frac{1}{2} c_w \rho A =$ $0.1 \cdot 10^3 \cdot 1.26\ 10^{-3}\ kg/m = 0.126\ g/m$, and the mass of displaced water $m_v = \frac{4}{3}\pi r^3 \rho = 0.0335\ kg$.

The final velocity then becomes: $v = \sqrt{\frac{(m_v - m)g}{c}} = \sqrt{\frac{0.0305 \cdot 9.82}{0.126}}\ m/s\ =\ 1.4\ m/s$

Achieving this velocity the pin pong ball will jump only to a height: $h = \frac{v^2}{2g} = 0.12\ m$

2.3 Vertical downward motion

We shall then consider a body sinking in water. The equation of motion becomes:

$$F_{res} = F_G + F_{Buo} + F_{visc} \Leftrightarrow ma = -mg + m_v g + cv^2 \tag{2.9}$$

The difference from before is only that $m > m_v$, that is, the mass of the body is larger than the mass of the displaced water, (because the density is higher).
The equation of motion is the same apart from a minus sign.

$$a = \frac{dv}{dt} = \frac{m_v - m}{m} g + \frac{c}{m} v^2 = -\frac{m - m_v}{m} g + \frac{c}{m} v^2 \tag{2.10}$$

We put $\mu = \frac{m - m_v}{m}$, we find:

$$a = \frac{dv}{dt} = -\mu g + \frac{c}{m} v^2 \quad \Leftrightarrow \quad \frac{dv}{dt} = -\mu g\left(1 - \frac{c}{\mu g m} v^2\right)$$

The equation will have the same solution as before, (apart from a minus sign).
If we again put: $k^2 = \frac{c}{\mu g m}$, the equation takes the form;

$$\frac{dv}{dt} = -\mu g(1 - (kv)^2)\,, \tag{2.11}$$

and it is seen to have the solution:

$$v = -\frac{1}{k}\tanh(\mu g k t) \qquad \text{Or written out:}$$

(2.12)
$$v = -\sqrt{\frac{(m-m_v)g}{c}}\tanh\sqrt{\frac{c(m_v-m)g}{m^2}}t$$

If we for example consider a iron ball having the radius 5.0 cm and density ρ=7.8 10^3 kg/m^3, we find for the constant c =0.79 SI-units, $m_v = \rho_{water} V_{sphere}$ = 1.0 10^3 ·4/3(5.0 $10^{-2})^3$ kg = 0.524 kg, and $m = \rho_{iron} V_{sphere}$ = 7.8 10^3 4/3(5.0 $10^{-2})^3$ kg = 4.1 kg. From which we get the final velocity.

$$v = -\sqrt{\frac{(m-m_v)g}{c}} = 6.7\ m/s = 24\ km/h$$

2.3 Vertical motion in air

For motion in air, one may (except for very light substances) discard the buoyancy.
The two equations relating to upward and downward motion then become:

Upward motion: $F_{res} = F_T + F_{luft} \Leftrightarrow ma = -mg - cv^2$

(2.13)

Downward motion: $F_{res} = F_T + F_{luft} \Leftrightarrow ma = -mg + cv^2$

We shall first solve for the upward motion:

$$a = \frac{dv}{dt} = -g - \frac{c}{m}v^2 \Leftrightarrow \frac{dv}{dt} = -g(1+\frac{c}{mg}v^2)$$. We put $k = \sqrt{\frac{c}{mg}}$, and get:

(2.13)
$$\frac{dv}{dt} = -g(1+(kv)^2)$$

As in the previous case, the equation may be solved by separation of the variables,
But it is easier to notice that (tan x)' =1+tan^2x, and then guess at a solution on the form:
$v = -a\tan bt$. Using this expression, we find: $\frac{dv}{dt} = -a(1+\tan^2 bt)b$.

If we compare this expression to: $\frac{dv}{dt} = -g(1+(kv)^2)$ it is seen, that tan bt = kv implies that $v = \frac{1}{k}\tan bt, \quad so \quad a = -\frac{1}{k}$, and subsequently $ab = -g \Rightarrow b = kg$. The solution is thus.

(2.14)
$$v - v_0 = -\frac{1}{k}\tan(kgt) \quad \text{where} \quad k = \sqrt{\frac{c}{mg}} \quad \text{and} \quad c = \tfrac{1}{2}c_w \rho A$$

For $kgt << 1$ $tan(kgt) \approx kgt$, and the formula transform into $v = v_0 - gt$, as it should

For a ball with radius $r = 0.05$ m and mass $m = 250$ g , the constant c =2.0 10^{-3} kg/m and the constant k = 0.0285 s/m.
If this ball is thrown with an initial velocity of 5.0 m/s, we may determine the time where it stops, by solving the equation: $v = 0 \Leftrightarrow \tan gkt = kv_0$, which gives: t =0.51 s.
If we want to determine how high the ball reaches, we must integrate the equation:

$$\frac{ds}{dt} = v - v_0 = -\frac{1}{k}\tan(kgt),$$

which gives.

$$s - s_0 = -\frac{1}{k^2 g}\ln(\cos(gkt))$$

Evaluating this distance, corresponding to a initial velocity v_0 = 5.0 m/s, then only the second decimal in the result reveals the discrepancy to a vertical throw without air drag.

2.4 Free fall with air drag

(2.13) $$F_{res} = F_G + F_{air} \quad \Leftrightarrow \quad ma = -mg + cv^2$$

Which leads to the equation:

(2.14) $$\frac{dv}{dt} = -g + \frac{c}{m}v^2 \quad \Leftrightarrow \quad \frac{dv}{dt} = -g(1 - \frac{c}{mg}v^2) \quad \Leftrightarrow \quad \frac{dv}{dt} = -g(1-(kv)^2)$$

We have put: $k^2 = \frac{c}{mg}$. This equation has (as shown above) the solution:

(2.15) $$v = -\frac{1}{k}\tanh(gkt)$$

The final velocity then becomes $v = -\frac{1}{k} = -\sqrt{\frac{mg}{c}}$
Inserting the values c =8.1 10^{-3}, corresponding to a ball with radius r = 0.10 m, and density 1.0 10^3 kg/m^3 , we obtain: v_{end} = 226 m/s.
The distance that the ball travels is: $s - s_0 = \frac{1}{gk^2}\ln(\cosh(gkt)$
The final velocity is achieved when gkt = 2, which gives t = $1/gk$ = 46 s, which corresponds to a distance : $s - s_0$ = 5200 m.

Whether the estimates above are in accordance with reality is, however, far from certain. The formula for the drag force is semi empirical and dependent on the Reynolds number, and the form factor is only fixed within a factor by 2.