Theory of Rotation

Chapter 8+9 of the textbook
Elementary Physics 2

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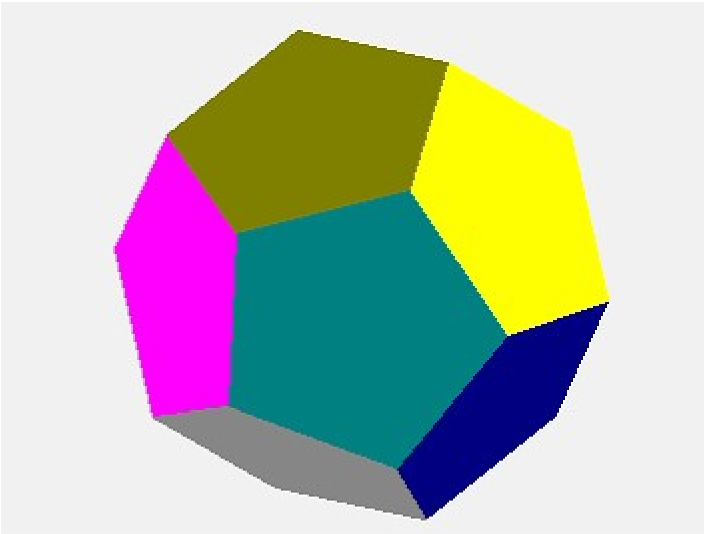

Ole Witt-Hansen 1977 (2016)

Acknowledgment

This paper is a digitalized translation from Danish of a chapter in a textbook on physics written in 1976, that was conceived and written for the second year of the Danish 3-year high school (called the gymnasium, for 16 -19 year-olds).

Rereading it, I have found that it is still an elementary yet rigorous presentation of the theory of rotation, and it covers most of the aspects of the theory. Therefore it can still be used as an undergraduate text on the subject of mechanics.

Also I have not found a similar straight and thorough presentation on the Internet, free from glittering pictures and with the contemporary call for using educational mathematical programs, instead of doing the analytical calculations by hand.

In the last 10 years, however, there has in Demark emerged a rising demand for this kind of textbook of physics, solidly founded on mathematics, and where the basic understanding is the issue. Whether it is also the case in other parts of the world, I do not know, but I can gather it from the hitherto interest in my home page: www.olewitthansen.dk

Even in 1970’s it was somewhat above the standard level both in mathematics and in abstract comprehension for use in the second year of teaching physics, and today it would be entirely out of question to try to use it in the Danish gymnasium.

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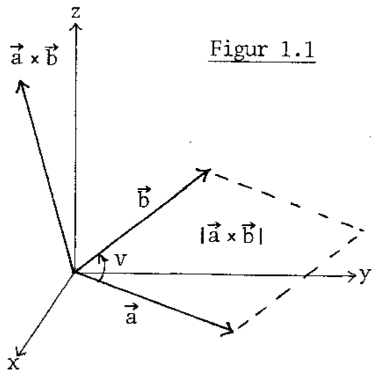
1. Cross product of two vectors

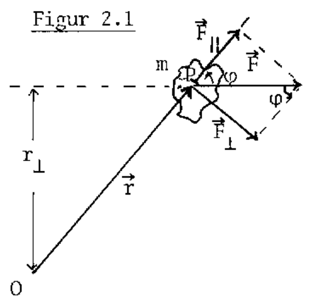
In order to define the physical quantities Angular Momentum and Torque, it is necessary to introduce the concept of cross product from vector algebra.



If $\vec{a}$ *and* $\vec{b}$ are two vectors in space the cross product (or the vector product) $\vec{a}\times\vec{b}$ is defined as a vector, which has length: $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin v$, (where v is the numeric least angle between the vectors $\vec{a}$ *and* $\vec{b}$), and where the direction is perpendicular to $\vec{a}$ as well as $\vec{b}$, so that $(\vec{a},\vec{b},\vec{a}\times\vec{b})$ form a *right hand screw.*
The latter part of the definition requires perhaps a bit of explanation.
There are evidently two vectors with the given length which are perpendicular to $\vec{a}$ and $\vec{b}$. To distinguish between them the right hand screw rule is invented.
With your right hand and your fingers pointing in the direction of $\vec{a}$ you make a twist to $\vec{b}$ in the positive direction. The direction of $\vec{a}\times\vec{b}$ shall then be in the direction of your thumb.
You should notice that, with the given definition then the length of the cross product: $|\vec{a}\times\vec{b}|$ is equal to the numerical value of the determinant of the pair of vectors $(\vec{a},\vec{b})$, since $\det(\vec{a},\vec{b})=|\vec{a}||\vec{b}|\sin v$, and from this, it follows that $|\vec{a}\times\vec{b}|$ is the area of the parallelogram spanned by the two vectors $\vec{a}$ *and* $\vec{b}$.
From the definition follows that: $\vec{a}\times\vec{b}=\vec{0}$ when $\vec{a}\parallel\vec{b}$, ($\vec{a}$ is parallel to $\vec{b}$). From the right hand rule, it further follows that the cross product changes sign, when $\vec{a}$ and $\vec{b}$ are permuted, (as it is also the case for the determinant).
Finally we mention that the distributive law is valid for the cross product, such that

$$\vec{c}\times(\vec{a}\times\vec{b})=\vec{c}\times\vec{a}+\vec{c}\times\vec{b} \qquad \text{(The order of the factors preserved)}$$

2. Torque. Moment of force



In the figure is shown a particle with mass *m*, where its position is given by the position vector $\vec{r}=\overrightarrow{OP}$. The particle is affected by the force $\vec{F}$. We dissolve the force in a component $\vec{F}_{\|}$ parallel to $\vec{r}$, and a component $\vec{F}_{\perp}$ perpendicular to $\vec{r}$.
Clearly $\vec{F}_{\|}$ will seek to draw the particle radially away from the point *O,* while $\vec{F}_{\perp}$ seeks to turn the particle around an axis through *O*.

How much the particle will be turned depends of course of $\vec{F}_{\perp}$, but also on $r=|\vec{r}|$.
Such considerations lead to the concept of *torque*, also denoted as *the moment of force.*

The torque *H,* with respect to the point *O* (in a two dimensional context), is defined by the expression:

(2.2) $$H = rF_{\perp} \quad or \quad H = rF\sin\varphi$$

Where φ is the numerically least angle between the position vector $\vec{r}$ and the force $\vec{F}$.
Thus:

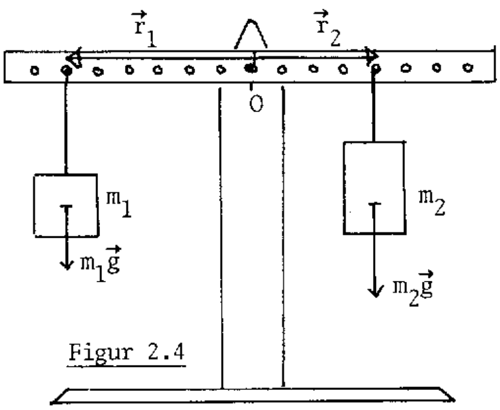
$$F_{\parallel} = F\cos\varphi \quad and \quad F_{\perp} = F\sin\varphi$$

The torque is a *vector*. That the torque must have a direction, you may understand, when you have in mind that to a twist there always belong an axis. The direction of the torque $\vec{H}$ should therefore be along that axis. A twist induces, however, also an orientation, corresponding to a left turn or a right turn. If we express the torque as a cross product of vectors, it complies with all these properties.

The torque $\vec{H}$, with respect to the origin *O*, on a particle, having the position vector $\vec{r} = \overrightarrow{OP}$, and affected by the force $\vec{F}$ is given by the cross product of $\vec{r}$ with $\vec{F}$.

(2.3) $$\vec{H} = \vec{r} \times \vec{F}$$

If *r* is called the "arm", then the torque can be loosely formulated as: "*Force times arm*".
With the definition (2.3), and the definition of the cross product, one can realize that the torque gets the right value, according to (2.2), as well as the correct direction, since it is directed along the axis of the twist, is complying with the right hand rule.
From (2.2) we have seen that the torque can be found as the "arm" *r* times $F_{\perp}$, the component of the force perpendicular to *r*. But the torque may as well be found as the force *F* times $r_{\perp}$ being the component of the "arm" perpendicular to the direction of the force, as shown in figure (2.1).



Figur 2.4

The figure to the right shows a lever, being able to tip around *O*, and two weights, that can be suspended at different distances from the axis of rotation of the lever. In the present positions of the weights m_1 and m_2, the torque with respect to *O* is zero.

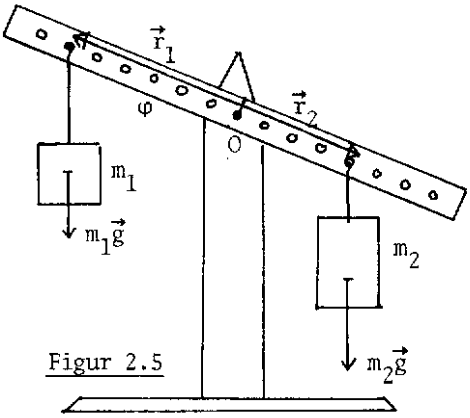
$$\vec{H}_O = \vec{H}_1 + \vec{H}_2 = \vec{r}_1 \times m_1\vec{g} + \vec{r}_2 \times m_2\vec{g}$$

As the figure shows, both torques are directed along the same axis of rotation, but having opposite directions. So we have *H = 0.*

Giving the condition for equilibrium:

$$H = 0 \quad \Leftrightarrow \quad r_1 m_1 g = r_2 m_2 g \ ,$$

Expressing that "the arm times the force", should be the same on both sides of the tilting point. This is (in a more popular context) known as the lever rule.



Figur 2.5

If the lever is tilted, having an angle φ with the horizontal, the vector expression of the torque remain the same as before.

$$\vec{H}_O = \vec{H}_1 + \vec{H}_2 = \vec{r}_1 \times m_1\vec{g} + \vec{r}_2 \times m_2\vec{g}$$

And taking the length:

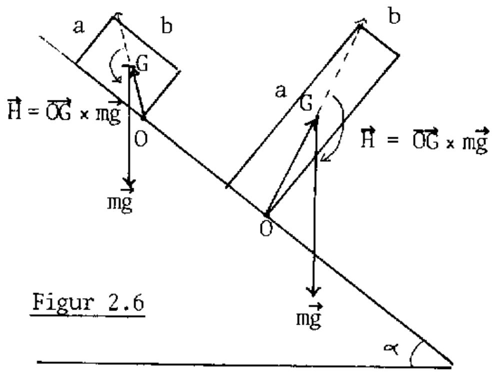
$$H_O = r_1 m_1 g \sin\varphi - r_2 m_2 g \sin\varphi = 0$$

It is seen, that the condition for equilibrium remains the same.

If $r_1 m_1 g \sin\varphi - r_2 m_2 g \sin\varphi \neq 0$, so that the lever is unbalanced, it can only be in balance with the lever in vertical position, since then:

$$\vec{H}_1 = \vec{H}_2 = 0, \text{ because } \vec{r}_1 \parallel \vec{g} \;\; and \;\; \vec{r}_2 \parallel \vec{g}$$

2.6 Example. Equilibrium condition for a box on a slope



Figur 2.6

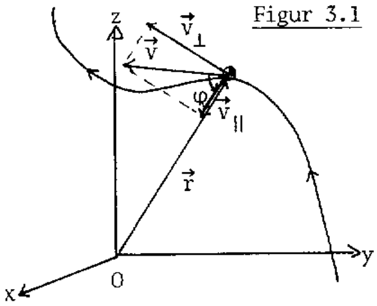
The figure shows two boxes placed on a slope, having a friction that prevents them in sliding down the slope. An observer will (by experience) probably claim that the one box will remain where it is, but that the second box will be overthrown.

We shall now apply the definition of the torques acting on the boxes to determine the condition that the box tilts or not.

The gravity is acting at *G,* the centre of mass (geometric centre) of the box. At the first box it is seen from the figure that the torque from gravity, $\vec{H} = \overrightarrow{OG} \times m\vec{g}$ will turn the box counter clockwise, towards the underlay, while the same torque on the second box, will turn the box clockwise and overthrow it.

The limit between the two cases, is of course when $\vec{H} = 0 \;\Leftrightarrow\; \overrightarrow{OG} \times m\vec{g} = 0 \;\Leftrightarrow\; \overrightarrow{OG} \parallel m\vec{g}$.
In this case it is seen, that the inclination of the slope is equal to the angle between *OG* and the front side of the box. This angle may however be calculated from: $\tan\alpha = b/a$.
We thus find that the condition that the box is overthrown is that: $\tan\alpha > b/a$.

3. Angular momentum. The equation of rotational motion for a particle



Figur 3.1

We shall now proceed to define a new fundamental concept relating to the theory of rotations, namely *angular momentum*, the importance of which will be clarified in the following.

In the figure is shown the trajectory for a particle with mass *m*. The particle is determined from the position vector $\overrightarrow{OG}$, as well as its velocity $\vec{v}$.

From the momentum vector $\vec{p} = m\vec{v}$, we now define the angular momentum vector as:

(3.1)	$\vec{L} = \vec{r} \times \vec{p}$ or $\vec{L} = \vec{r} \times m\vec{v}$

The velocity $\vec{v}$ of the particle can be resolved in two components, one $\vec{v}_{\|}$ parallel with $\vec{r}$, and one $\vec{v}_{\perp}$ perpendicular to $\vec{r}$. If φ denotes the least angle between $\vec{r}$ and $\vec{v}$, we have: $v_{\|} = v\cos\varphi$ and $v_{\perp} = v\sin\varphi$. Then we are able to express the length of the angular momentum.

(3.3) $L = rp\sin\varphi = mvr\sin\varphi$ or $L = rp_{\perp} = mv_{\perp}r$

From the definition: $\vec{L}$ is a vector, which is perpendicular to $\vec{r}$ and to $\vec{v}$, and we notice that

(3.2) $$\vec{L} = \vec{0} \quad \Leftrightarrow \quad \vec{r} \parallel \vec{p} \quad \Leftrightarrow \quad \vec{v} = \vec{v}_{\|}$$

On the other hand, if $\vec{r} \perp \vec{p}$ then $L = rp = mvr$.

Similar to Newton's second law for the motion of a particle, we shall demonstrate that, regarding rotation a corresponding law is valid, if the force $\vec{F}$ is replaced by the torque $\vec{H}$, and the momentum $\vec{p}$ is replaced with the angular momentum $\vec{L}$.

When differentiating the cross product, the same rules apply, as to differentiating the product of two functions, provided that the orders of the factors are kept.

(3.3) $$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{0} + \vec{r} \times \vec{F} = \vec{H}$$

The first of the two terms becomes zero, because $\frac{d\vec{r}}{dt} = \vec{v}$ and $\vec{p} = m\vec{v}$, so that $\frac{d\vec{r}}{dt} \parallel \vec{p}$, and thus:

(3.4)	$\vec{H} = \frac{d\vec{L}}{dt}$

Especially it applies that, when the torque is zero, the angular momentum is constant.

(3.5)	$\vec{H} = \vec{0} \quad \Leftrightarrow \quad \frac{d\vec{L}}{dt} = \vec{0} \quad \Leftrightarrow \quad \vec{L} = \vec{r} \times \vec{p} =$ constant vector

The content of equation (3.7) is of equal importance as the conservation of momentum and is called:

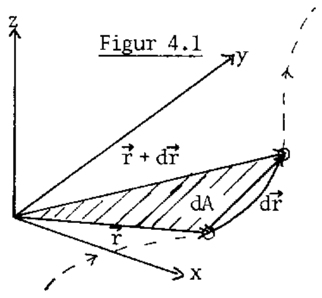
The conservation of angular momentum for a particle.

We notice further that for $\vec{r} \neq \vec{0}$ applies:

$$\vec{H} = \vec{r} \times \vec{F} = \vec{0} \quad \Leftrightarrow \quad \vec{F} = \vec{0} \quad \vee \quad \vec{r} \parallel \vec{F}$$

The angular momentum is constant, either if the force on the particle is zero or the force is directed radially.

4. Area-velocity and Kepler's second law



The figure shows a particle, which performs a motion in the $x-y$ plane. We wish to determine an expression for the *area-velocity*, that is, the area swept by the particle per time unit. The area *dA* that the position vector sweeps in the time *dt*, is equal to half the area of the parallelogram expanded by the vectors $\vec r = \vec r(t)$ and $\vec r + d\vec r = \vec r(t+dt)$.

$$dA = \tfrac{1}{2}|\det(\vec r, \vec r + d\vec r)| = \tfrac{1}{2}|(\vec r \times (\vec r + d\vec r)|$$

$$dA = \tfrac{1}{2}|(\vec r \times \vec r + \vec r \times d\vec r)| = \tfrac{1}{2}|\vec r \times d\vec r| \qquad (\vec r \times \vec r = \vec 0)$$

(4.2) $$\frac{dA}{dt} = \tfrac{1}{2}|\vec r \times \frac{d\vec r}{dt}| = \tfrac{1}{2}|\vec r \times \vec v| \qquad \text{(= Area velocity)}$$

From (4.2) it appears natural to invent an *area velocity vector*, by:

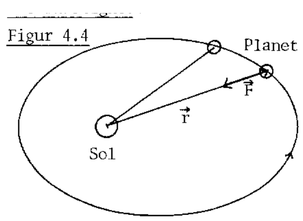
(4.3) $$\frac{d\vec A}{dt} = \tfrac{1}{2}\vec r \times \vec v \quad \text{where} \quad \frac{dA}{dt} = \tfrac{1}{2}|\vec r \times \vec v| \quad \text{is the area velocity.}$$

Holding (4.3) together with the definition of the angular momentum $\vec L = \vec r \times m\vec v$, we see that

$$\vec L = 2m\frac{d\vec A}{dt},$$

from which follows the magnificent theorem:

If the torque $\vec H = \vec r \times \vec F = \vec 0$, either because *H = 0,* or because the particle is moving in a central field, such that $\vec r \parallel \vec F$, then the angular momentum $\vec L$ is a constant vector, and the particle performs a plane motion with constant area-velocity.



For the motion of a particle en a central field e.g. the motion of a planet around the sun, the gravity force $\vec F$ is always directed opposite to the position vector $\vec r = \overrightarrow{OP}$, and therefore, (as demonstrated above) $\vec H = \vec r \times \vec F = \vec 0$, and the angular momentum of the particle (the planet) is constant, and the particle will perform a motion with constant area-velocity.
This is the content of the second law of Kepler, which then trivially follows from conservation of angular momentum.

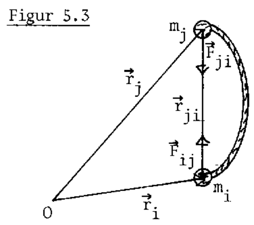
5. Torque and angular momentum for a system of particles

For a system of particles or a solid body, the angular momentum is defined as the vector sum of the angular momentum of the individual particles.

(5.1) $$\vec L = \sum \vec L_i = \sum \vec r_i \times \vec p_i$$

$\vec r_i$ is the position vector for the *i*'th particle, having the momentum $\vec p_i$. In the same manner the torque is as the vector sum of torques acting on the individual particles. If $\vec F_i$ denotes the torque on the *i*'th particle the total torque is:

(5.2) $$\vec H = \sum \vec H_i = \sum \vec r_i \times \vec F_i$$



As we have shown when we calculated the resulting force on a system of particles, it is only the *external torques* that delivers a contribution to the resulting torque.
Let $\vec{F}_{ij}$ be the force on which the *i*'th particle acts on the *j*'th particle. $\vec{r}_{ij}$ being the vector from (*j*) to (*i*). With the notation in the figure we then have $\vec{r}_i = \vec{r}_j + \vec{r}_{ji}$. The torque on which the *j*'th particle acts on the *i*'th particle is then calculated as:

(5.3) $$\vec{H}_{ij} = \vec{r}_i \times \vec{F}_{ij} = (\vec{r}_j + \vec{r}_{ji}) \times (-\vec{F}_{ji}) = -\vec{r}_j \times \vec{F}_{ji} - \vec{r}_{ji} \times \vec{F}_{ji}$$

We have applied Newton's 3. law $\vec{F}_{ij} = -\vec{F}_{ji}$, and the last term becomes zero because $\vec{r}_{ji} \,||\, \vec{F}_{ji}$.
The net result is:

(5.4) $$\vec{H}_{ij} = \vec{r}_i \times \vec{F}_{ij} = -\vec{r}_j \times \vec{F}_{ji} = -\vec{H}_{ji}$$

The terms in the sum (5.2), coming from internal forces thus cancel each other in pairs, and only the external forces gives a contribution to the resulting torque. We are then able to formulate the equation of motion for rotation for a system of particles:

(5.5) $$\vec{H}_{ext} = \sum \vec{H}_i = \sum \frac{d\vec{L}_i}{dt} = \frac{d}{dt}\sum \vec{L}_i = \frac{d\vec{L}}{dt}$$

The relation (5.5) expresses the fact that the external force is equal to the differential quotient of the total angular momentum. Especially:

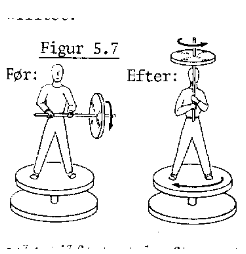
(5.6) $$\vec{H}_{ext} = \vec{0} \quad \Leftrightarrow \quad \frac{d\vec{L}}{dt} = \vec{0} \quad \Leftrightarrow \quad \vec{L} = \sum \vec{L}_i = constant\ vector$$

This is the contents of the law of conservation of angular momentum for a system of particles.

For an isolated system of particles the angular momentum of the system is conserved.

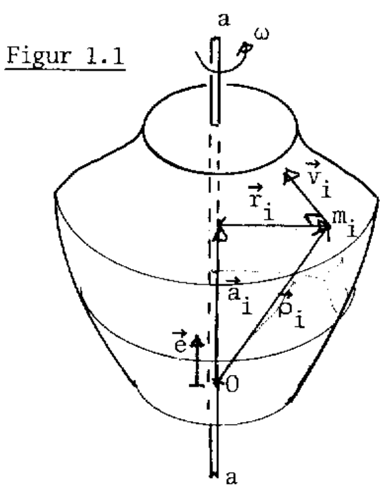
One should emphasize the vector character of the angular momentum. It is rather obvious that there must be a torque to initiate the rotation of a body, but perhaps less obvious that it also needs a torque to change the direction of the rotational axis.
However, this is in full accordance with the fact, that a force is needed to change the direction of the momentum, even if its numerical value is unchanged. (For example the centripetal force in a uniform circular motion).

Experience tells us, however the former, for example when riding a bike. The angular momentum of the wheels is directed along the axis of rotation, and it helps us to keep the balance. To keep the balance on a bike at rest for a longer period, is a skill reserved for artists in a circus.



Example 5.7 The figure shows an example of angular momentum conservation.
The person, holding a heavy rotating wheel, is standing on a disc, being able to rotate without friction along a vertical axis, such that there is no external torque along this axis. When the person changes the axis of the rotating wheel from horizontal to vertical, he must supply a torque (directed along a vertical axis, not a horizontal one), and from Newton's law of action and reaction, he will receive an opposite torque, which makes him turn the opposite way of the wheel.
The angular velocity of the wheel will not change, but his rotation will have the effect, that the angular momentum is conserved to zero along the vertical axis.

6. Torque and angular momentum with respect to an axis of rotation



When a rigid body is fixed to rotate along an axis, it is most suitable to define the torque and angular momentum with respect to that axis of rotation.
In the figure the rigid body rotates along an axis *a*.
For an arbitrary mass element of the body m_i , its momentum: $\vec{p}_i = m_i\vec{v}_i$ is perpendicular to the axis *a*.

Let $\vec{L}$ and $\vec{H}$ be the angular momentum and the torque of the rotating body with respect to a point *O* on the axis.
If $\vec{e}$ denotes a unit vector along the axis of rotation, the components of $\vec{L}$ and $\vec{H}$ along the axis may be written as $L_a = \vec{L}\cdot\vec{e}$ and $H_a = \vec{H}\cdot\vec{e}$. Furthermore let $\vec{\rho}_i$ be the position vector from *O* to the mass element m_i, and let r_i be the distance from m_i to the axis.
With the notations in the figure it follows that: $\vec{\rho}_i = \vec{a}_i + \vec{r}_i$, where $\vec{a}_i$ is directed along the axis.
We shall then evaluate L_a , the angular momentum with respect to the axis *a*.

$$\begin{aligned} L_a &= (\sum_i \vec{\rho}_i \times \vec{p}_i)\cdot\vec{e} = (\sum_i (\vec{a}_i + \vec{r}_i)\times\vec{p}_i)\cdot\vec{e} \\ &= (\sum_i \vec{a}_i \times \vec{p}_i)\cdot\vec{e} + (\sum_i \vec{r}_i\times\vec{p}_i)\cdot\vec{e} \\ &= \sum_i r_i p_i = \sum_i r_i p_i = \sum_i m_i r_i v_i \end{aligned} \tag{6.1}$$

In the second equation the first term vanishes, because $\vec{a}_i \times \vec{p}_i \perp \vec{a}_i$ and therefore: $\vec{a}_i \times \vec{p}_i \perp \vec{e}$.
The second term can be written as the product of the lengths of the vectors, because $\vec{r}_i \perp \vec{p}_i$ and therefore $\vec{r}_i \times \vec{p}_i \parallel \vec{e}$.

For the torque we have a similar situation, since it is only forces $\vec{F}_{i\parallel}$ parallel to $\vec{v}_i$ which contributes to the rotation. The other components are cancelled by the reactive forces from the axis.

$$H_a = (\sum_i \vec{r}_i \times \vec{F}_i)\cdot\vec{e} = \sum_i r_i F_{i\parallel} \tag{6.2}$$

From Newton's second law for rotation: $\vec{H} = \frac{d\vec{L}}{dt}$ it immediately follows, when multiplying this equation by $\vec{e}$, bearing in mind that $\vec{e}$ is a constant vector.

$$\vec{H}\cdot\vec{e} = \frac{d\vec{L}}{dt}\cdot\vec{e} = \frac{d}{dt}(\vec{L}\cdot\vec{e}) \quad\Rightarrow\quad H_a = \frac{dL_a}{dt} \tag{6.3}$$

So the equation of motion for rotation

$$\vec{H} = \frac{d\vec{L}}{dt}$$

remains valid, when the torque and the angular momentum is evaluated with respect to a fixed axis.

7. Rotation about an axis. Moment of inertia

When rotating about a fixed axis, as show in the figure on the preceding page, all of the bodies particles perform a circular motion with the same angular velocity ω. Inserting $v_i = \omega r_i$ in the expression (6.1) for the angular momentum with respect to an axis, we find:

(7.1) $$L_a = \sum_i m_i r_i v_i = \sum_i m_i \omega r_i r_i = \omega \sum_i m_i r_i^2 = \omega I_a$$

The sum: $$I_a = \sum_i m_i r_i^2$$

Where the sum is extended to all parts of the body is called the *moment of inertia* of the body with respect to the axis *a*. The SI unit of the moment of inertia is *kg* m^2.

The moment of inertia depends only on the mass, the shape of the body, and the position of the axis within or without the body.
So for rotation along a fixed axis the relation: $L_a = \omega I_a$ applies. If the axis is understood, one can drop the index. Further the rotation angle φ is often used instead of ω, where $\omega = d\varphi/dt$.
So we may in general write:

(7.2) $$L = I\omega \quad \text{or} \quad L = I\frac{d\varphi}{dt} \quad \text{where} \quad I = \sum_i m_i r_i^2$$

Using this notation the equation of motion for rotation takes the form:

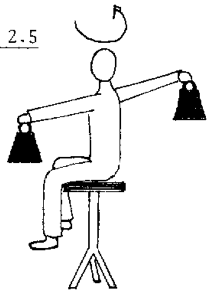
(7.3) $$H = \frac{dL}{dt} \quad \text{or} \quad H = I\frac{d\omega}{dt} \quad \text{or} \quad H = I\frac{d^2\varphi}{dt^2}$$

The two latter expressions are only valid for a rigid body, where the moment of inertia is constant in time. If the torque with respect to the axis is zero, the angular moment is constant.

(7.4) $$H = 0 \quad \Leftrightarrow \quad L = const \quad \Leftrightarrow \quad I\omega = const$$

For a rigid body (7.4) means that it in lack of any external torque, it rotates with constant angular velocity. However, the moment of inertia can be changed by internal forces and since the angular momentum is constant, this will change the angular velocity.
For example a "skate princess" preparing a "pirouette", starts out with a moderate rotation speed with her arms stretched, next when she pulls her arms in, gaining angular velocity because her moment of inertia is diminished. The fast rotation stops, as she pulls out her arms (or even a leg) .

Figur 2.5



7.5 Example

The figure shows a test person who has been “persuaded” to take a seat in a chair. The seat can rotate freely about a vertical axis. The person is told to hold two 5 kg weights out in stretched arms at a distance 0.75 *m* and (rather irresponsible) the teacher puts him in a slow rotation.

From a signal from the teacher he pulls his arms in, so that the weights are now held at a distance of 0.25 *m*, from the axis (The teacher must be precautious, that he will not get hurt, when he falls down from the wildly rotating chair).

I usually referred this to: “Learning by doing”, for the less clever students.

Assuming that the moment of inertia of the person is 2 kgm^2, and that the initial rotation is 1 round per sec, we shall make an estimate of what happens.

The moments of inertia can be calculated as: $I = I_{person} + I_{weights}$.

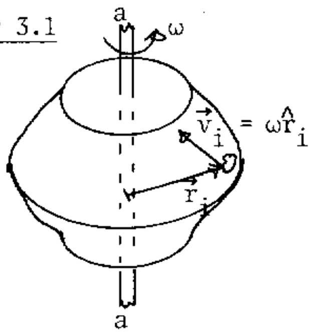
$I_{before} = 2\ kgm^2 + 2\cdot 5\ (0.75\ m)^2 = 7.6\ kgm^2$. and $I_{after} = 2\ kgm^2 + 2\cdot 5\ (0.25\ m)^2 = 2.6\ kgm^2$.

According to the conservation of angular momentum: $I_{before}\,\omega_{before} = I_{after}\,\omega_{after}$, from which we calculate ω_{after}.

$$\omega_{after} = \frac{I_{before}}{I_{after}}\omega_{before} = \frac{7.6}{2.6}2\pi\ rad/\sec = 18.4\ rad/\sec = \frac{18.4}{2\pi} = 2.9\ rps$$

8. Rotational energy. Calculation of moments of inertia

Figur 3.1



We shall then proceed to determine the kinetic energy of a rigid body, rotating with an angular velocity ω along a fixed axis.

For the particle with mass m_i, having the distance r_i from the axis of rotation, its velocity is: $v_i = \omega\, r_i$.

It then follows:

$$E_{kin} = E_{rot} = \sum_i \tfrac{1}{2} m_i v_i^2 = \sum_i \tfrac{1}{2} m_i (\omega r_i)^2$$

$$(8.1) \qquad E_{rot} = \tfrac{1}{2}\omega^2 \sum_i m_i r_i^2 = \tfrac{1}{2} I \omega^2$$

Where *I* is the moment of inertia along the axis.

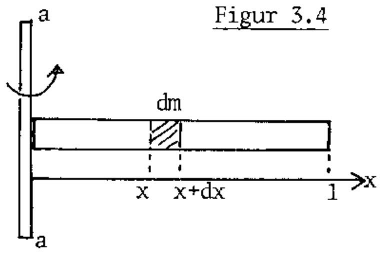
The *moment of inertia* has the same significance for rotation as the *mass* has for a body in a translation. The analogy between a linear motion and rotation about a fixed axis is in fact close and far reaching. Below is a scheme illustrating the analogous’ concepts.

Linear motion		**Rotation about a fixed axis**	
Coordinate	s	Rotation angle	φ
Velocity	$v = \frac{ds}{dt}$	Angular velocity	$\omega = \frac{d\varphi}{dt}$
Mass	m	Moment of inertia	I
Momentum	$p = mv$	Angular momentum	$L = I\omega$
Force	$F = \frac{dp}{dt}$	Torque (Moment of force)	$H = \frac{dL}{dt}$
Kinetic energy	$E_{kin} = \frac{1}{2}mv^2$	Rotational energy	$E_{rot} = \frac{1}{2}I\omega^2$

For applying the theory of rotation to the physical world it is of course imperative to be able to calculate the moments of inertia of various rigid bodies.
For extended bodies, the moment of inertia may in some cases be evaluated by integration.
If the body is homogenous with density ρ, the mass *dm*, being in the volume element:
$dV = dxdydz$ is given by: $dm = \rho\, dxdydz$. The moment of inertia is then calculated by integration, when replacing the mass element m_i by *dm*, and replacing summation by integration.

(8.3) $$I = \int r^2 dm = \rho \int r^2 dV$$

8.4 Example. The moment of inertia of a homogenous rod



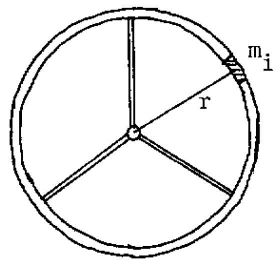
We shall first consider a rod rotating about an axis, which is perpendicular to the rod at its end point. The rod has the length *l* and mass *m*.
The mass pr unit length is *m/l*, and the mass located at *dx*, in the distance *x* from the axis is therefore: $dm = \frac{m}{l}dx$. The moment of inertia is then calculated from (8.3).

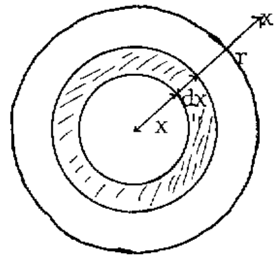
(8.4.1) $$I = \int_0^l x^2 dm = \frac{m}{l}\int_0^l x^2 dx = \frac{m}{l}\left[\tfrac{1}{3}x^3\right]_0^l = \tfrac{1}{3}ml^3$$

If the axis goes through the centre (of mass) of the rod, the moment of inertia is found, changing the limits of integration.

(8.4.2) $$I_G = \frac{m}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 dx = \frac{m}{l}\left[\tfrac{1}{3}x^3\right]_{-\frac{l}{2}}^{\frac{l}{2}} = \tfrac{1}{12}ml^3$$

8.5 Example. The moment of inertia of a wheel with mass *m* along the perimeter, and that of a disc.





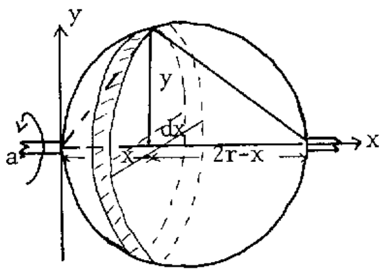
We shall then calculate the moment of inertia from a (bike) wheel, where the axis is through the centre of the wheel perpendicular to the wheel. Since all of the mass is located at *r* (the radius of the wheel), the moment of inertia is simply:

(8.5.1) $$I_{wheel} = \sum_i m_i r^2 = r^2 \sum_i m_i = mr^2$$

We shall next calculate the moment of inertia of a disc or a cylinder, where the axis is through the centre, perpendicular on the disc. We do this by dividing the disc into rings with radius *r* and thickness *dx*. We then have:

$$dm = \frac{m}{\pi r^2} 2\pi x dx = \frac{2m}{r^2} x dx$$

(8.5.2) $$I_{disc} = \frac{2m}{r^2}\int_0^x x^3 dx = \frac{2m}{r^2}\left[\tfrac{1}{4}x^4\right]_0^r = \tfrac{1}{2}mr^2$$



We shall then calculate the moment of inertia of a solid ball with respect to an axis through its centre. The ball is assumed to have the mass *m* and the radius *r*. The calculation is done by dividing the ball in discs with radius *y* and thickness *dx*. If such a disc has the mass *dm*, the moment of inertia of the disc is $dI = ½dm\, y^2$, as we found in the previous example.
A subordinate theorem from elementary plane geometry states that the square of the height in a right angle triangle is equal to the product of the sides in which it divides the base line. $y^2=x(2r-x)$.
We may then find an expression for *dm* expressed by *x*.

$$dm = \rho dV_{disc} = \frac{m}{\frac{4}{3}\pi r^3}\pi y^2 dx = \tfrac{3}{4}\frac{m}{r^3}y^2 dx = \tfrac{3}{4}\frac{m}{r^3}x(2r-x)dx$$

And we may perform the integration over x.

$$I = \int_0^{2r}\tfrac{1}{2}y^2 dm = \tfrac{3}{8}\frac{m}{r^3}\int_0^{2r}x^2(2r-x)^2 dx$$

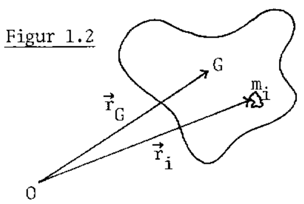
$$I = \int_0^{2r}\tfrac{1}{2}y^2 dm = \tfrac{3}{8}\frac{m}{r^3}\left(4r^2\int_0^{2r}x^2 dx + \int_0^{2r}x^4 dx - 4r\int_0^{2r}x^3 dx\right)$$

$$I_{ball} = \tfrac{3}{8}\frac{m}{r^3}\left(\frac{32}{5}r^5 + \frac{32}{5}r^5 - 16r^5\right) = \tfrac{2}{5}mr^5$$

9. Theorems concerning the centre of mass, moment of inertia and rotational energy

The centre of mass is actually defined in another chapter of the textbook in which this chapter belongs to, so I shall repeat it here.

9.1 Centre of mass



The figure shows a system of particles or a massive body. The single element of the system is described by the position vector:

$\vec{OP_i} = \vec{r}_i$ and $m = \sum_i m_i$ is the mass of the system.

We now define a (fictive) point *G*, by the equation:

(9.1) $$\vec{r}_G = \vec{OG} = \frac{1}{m}\sum_i m_i\vec{r}_i$$

G is called *the centre of mass* of the system, and it is of vital importance, when solving the equations of motion for the system. Often (for massive bodies) we shall replace the summation by integration.

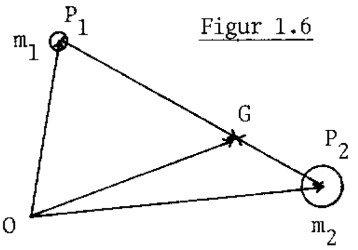
(9.2) $$\vec{r}_G = \frac{1}{m}\int \vec{r}dm = \frac{1}{m}\int \vec{r}\rho dV$$

where $\vec{r} = (dx, dy, dz)$, ρ is the density, and $dV = dxdydz$ is the volume element.
The three coordinates of the centre of mass, are then found by:

(9.5) $x_G = \frac{1}{m}\int \rho x dV \qquad y_G = \frac{1}{m}\int \rho y dV \qquad z_G = \frac{1}{m}\int \rho z dV$

If a body has a symmetry line or a symmetry plane, the centre of mass must lie on that line or in that plane. If the body has two non parallel symmetry planes, the centre of mass lies in their intersecting line. If the body has three non parallel symmetry planes, the centre of mass lies in their common intersecting point.
The centre of mass for a box, therefore lies the intersection of the cross diagonals. The centre of mass of a sphere lies in its centre. The centre of mass of a triangle lies in the common intersecting point of the medians.

9.6 Example. The centre of mass from two bodies having different masses (e.g. earth and moon).



Figur 1.6

As an example we shall determine the position of the centre of mass for a system consisting of two masses m_1 and m_2, placed in the positions P_1 and P_2 According to the definition of the centre of mass G we have.

$$\vec{OG} = \frac{1}{m_1 + m_2}(m_1 \vec{OP_1} + m_2 \vec{OP_2}) \quad \Leftrightarrow$$

$$(m_1 + m_2)\vec{OG} = m_1 \vec{OP_1} + m_2 \vec{OP_2} \quad \Leftrightarrow$$

$$m_1(\vec{OG} - \vec{OP_1}) = -m_2(\vec{OG} - \vec{OP_2}) \quad \Leftrightarrow \quad m_1 \vec{P_1G} = -m_2 \vec{P_2G} \quad \Rightarrow \quad \frac{|P_1G|}{|P_2G|} = \frac{m_2}{m_1}$$

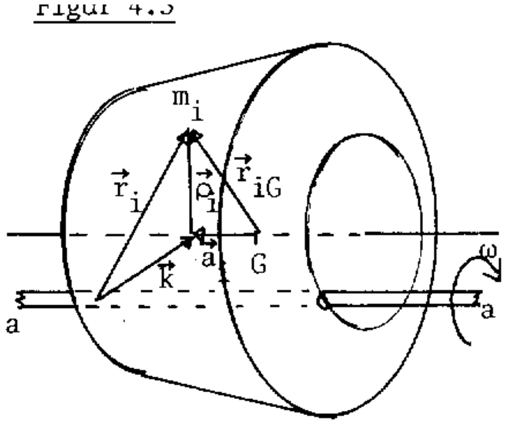
The calculation above shows, that the centre of mass for a system consisting of two particles lies in their connecting line, and divides this line in the inverse ratio of the two masses.
This result can for example be applied on the system consisting of the earth and the moon.
The ratio M_{moon}/M_{earth} = 0.0123, and their distance is roughly $60R$, where $R = R_{earth}$ = 6.370 *km*. If r_G is the distance to the centre of mass from the centre of the earth, then we have:

$$\frac{r_G}{60R - r_G} = \frac{M_{moon}}{M_{earth}} = 0.0123 \quad \Rightarrow \quad r_G = 0.729R$$

The centre of mass lies (surprisingly) within the surface of the earth. As a consequence of the mutual attraction of the earth and moon, both objects move in elliptical orbits, with focus in their common centre of mass.
However, because of Newtons 3. law: $F_2 = -F_1 \Leftrightarrow m_2a_2 = -m_1a_1$ .The ration between their accelerations is also the inverse ration between their masses 0.0123.

9.2 Steiner's theorem

Figur 4.5



Because of the nice symmetry properties of the centre of mass (*CM)*, it often advantageous to calculate the moment of inertia with respect to an axis going the centre of mass *G*.
If the axis is not through the *CM*, then one may apply Steiner's theorem:
The moment of inertia I_a , with respect to an axis *a*, which does not goes through the *CM*, can be calculated as the moment of inertia I_G along an axis through the CM and parallel to the given axis plus mk^2, where *k* is the separation between the two parallel axis.

(9.7) $I_a = I_G + mk^2$

With the notations shown in the figure above we have:

$$I_a = \sum_i m_i \vec{r}_i^{\,2} = \sum_i m_i (\vec{k} + \vec{\rho}_i)^2 = \sum_i m_i \vec{k}^2 + \sum_i m_i \vec{\rho}_i^{\,2} + \sum_i 2 m_i \vec{k} \cdot \vec{\rho}_i$$

$$I_a = k^2 \sum_i m_i + \sum_i m_i \vec{\rho}_i^{\,2} + 2\vec{k} \cdot \sum_i m_i \vec{\rho}_i = mk^2 + I_G + 0$$

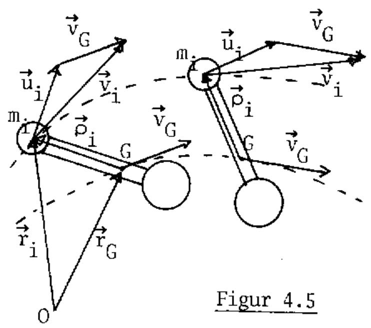
The last term becomes zero for the following reasons. $\vec{k} \cdot \sum_i m_i \vec{\rho}_i = \vec{k} \cdot \sum_i m_i (\vec{\rho}_i + \vec{a}_i)$

Firstly because $\vec{k} \perp \vec{a}_i$ and secondly because $\vec{r}_{iG} = \vec{\rho}_i + \vec{a}_i$ is the position vector from G to the mass m_i. So that

$$\sum_i m_i \vec{\rho}_i = \sum_i m_i (\vec{\rho}_i + \vec{a}_i) = \sum_i m_i \vec{r}_{iG} = \vec{r}_{GG} = \overrightarrow{GG} = \vec{0} \qquad \text{so}$$

$$I_a = mk^2 + I_G$$

9.3 König's theorem



Figur 4.5

If a body performs a combination of translation and rotation, then the overall kinetic energy can be written as:

(9.6) $$E_{kin} = E_{trans} + E_{rot} = \tfrac{1}{2} v_G^{\ 2} + \tfrac{1}{2} I_G \omega^2$$

Here v_G is the velocity of the *CM*, and the second term is the rotational energy relatively to the *CM.*
In the figure to the right is showed a body in motion described from a reference point O. Let $\vec{r}_i$ be the position vector to the mass particle m_i , and $\vec{r}_G = \overrightarrow{OG}$ the position vector of the *CM*. $\vec{\rho}_i$ is defined by the relation: $\vec{r}_i = \vec{r}_G + \vec{\rho}_i$. $\vec{v}_i = d\vec{r}_i / dt$ is the velocity of the particle with mass m_i , relative to *O*, and $\vec{u}_i = d\vec{\rho}_i / dt$ is the velocity of the particle relative to *G*.
From the relation: $\vec{r}_i = \vec{r}_G + \vec{\rho}_i$ we therefore have: $\vec{v}_i = \vec{v}_G + \vec{u}_i$. $\vec{v}_G = \dfrac{1}{m} \sum m_i \vec{v}_i$ is the velocity of the *CM* relative to *O*. $\sum_i m_i \vec{u}_i = 0$, which follows from differentiating the relation . $\sum_i m_i \vec{\rho}_i = \vec{\rho}_{GG} = 0$.
A calculation now shows:

$$E_{kin} = \sum \tfrac{1}{2} m_i v_i^{\ 2} = \sum \tfrac{1}{2} m_i (\vec{v}_G + \vec{u}_i)^2$$

$$E_{kin} = \tfrac{1}{2} \vec{v}_G^{\ 2} \sum m_i + \sum \tfrac{1}{2} m_i \vec{u}_i^{\ 2} + \vec{v}_G \cdot \sum m_i \vec{u}_i = \tfrac{1}{2} m \vec{v}_G^{\ 2} + \sum \tfrac{1}{2} m_i \vec{u}_i^{\ 2} + 0$$

(9.10) $$E_{kin} = \tfrac{1}{2} m \vec{v}_G^{\ 2} + \sum \tfrac{1}{2} m_i (\vec{\rho}_i \omega)^2 = \tfrac{1}{2} m \vec{v}_G^{\ 2} + \tfrac{1}{2} \omega^2 \sum m_i \vec{\rho}_i^{\ 2} = \tfrac{1}{2} m \vec{v}_G^{\ 2} + \tfrac{1}{2} I_G \omega^2$$

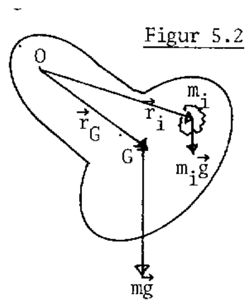
We have used that $\vec{v}_G$ is a constant vector that can be moved in front of the summation.

The first term: is the *CM* energy or the translation energy. It is the kinetic energy relative to other objects, what we hitherto have called the kinetic energy.
The second term: $\sum \frac{1}{2} m_i \vec{u}_i^{\,2}$ is the kinetic energy of the masses m_i relative to G the centre of mass.

It is therefore called the relative energy, internal kinetic energy or the rotational energy.
For a rigid body this motion can always be described as an instant rotation about an axis through G, with a moment of inertia I_G. In the second equation the last term becomes zero because of the relation: $\sum_i m_i \vec{u}_i = 0$. Thus we have proved König's theorem.

(9.11) $$E_{kin} = \tfrac{1}{2} m \vec{v}_G^{\,2} + \tfrac{1}{2} I_G \omega^2$$

10. Centre of gravity equals centre of mass



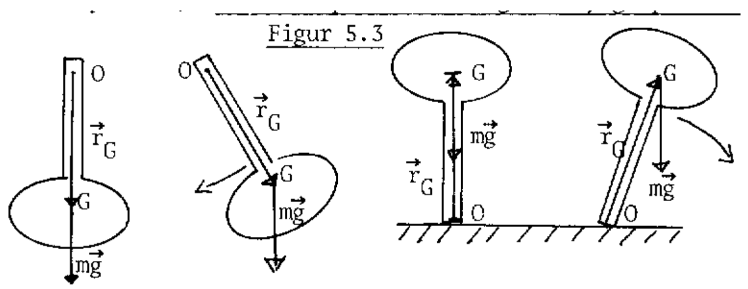
The centre of gravity of a rigid body is usually described as a point of suspension, where the body is balanced (equilibrium), with respect to every displacement from this position.
The condition of equilibrium is obviously that the sum of torques acting on the body is zero.
The figure shows a body suspended in a point O (or on a rod through O) in the gravity field.
The sum of torques from gravity acting on the mass elements m_i situated at $\vec{r}_i$ with respect to O, is calculated from:

(10.1) $$\vec{H} = \sum \vec{r}_i \times m_i \vec{g} = \left(\sum m_i \vec{r}_i\right) \times \vec{g} \quad \Leftrightarrow \quad \vec{H} = m\vec{r}_G \times \vec{g} = \vec{r}_G \times m\vec{g}$$

From (10.1) it appears that the entire torque (moment of force) can be found, as if the whole mass of the body is situated in the *CM* of the body.

Especially if $O = G$, that is, the body is suspended in its centre of mass, $\vec{r}_G = \vec{0}$ and therefore $\vec{H} = 0$, the body will be in equilibrium turned in any position, and this will also hold if the body is suspended on an axis through the CM.

The centre of gravity is the same as the centre of mass.

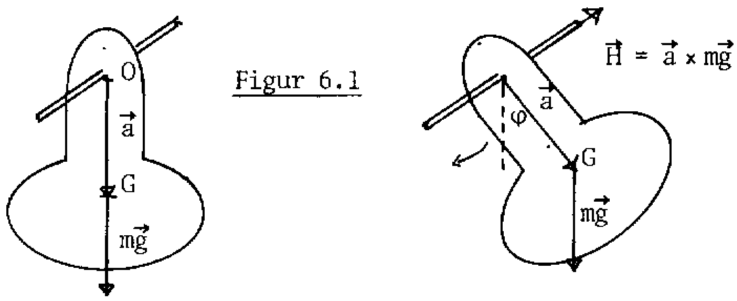


From (10.1) it is seen that a suspended body is in balance, that is, when $\vec{H} = 0$, when $\vec{r}_G$ is parallel with $\vec{g}$.
It therefore follows that the body is balanced, if the point O of suspension or support is right below or above the centre of mass G. If G is below O, the equilibrium is *stable*, since the body will move towards the equilibrium position from any minor displacement. However, if G is vertically above O, the equilibrium is *unstable*, since even a minor displacement from the equilibrium position, will generate a torque, turning (with increasing strength) the body

away from the equilibrium. Well known experience from everyone, including balancing artists. This is illustrated in figure (5.3).

11. The physical pendulum

A physical pendulum is a rigid body suspended on a horizontal rod, and performing harmonic oscillations in the gravity field.



In the figure to the left the pendulum is at rest. The position vector from the axis of rotation to the centre of mass G is denoted $\vec{a}$. In the second figure the pendulum is displaced an angle φ from the equilibrium position. Since we have proven that the acting torque can be found as if the whole mass of the body is situated in G, the torque is: $\vec{H} = \vec{a} \times m\vec{g}$, seeking to turn the body back to equilibrium position. Taking the lengths of the vector gives:

$$H = -mga\sin\varphi \approx -mga\varphi \text{ (for small deviations } \varphi < 20^0)$$

We then apply Newton's 2. law for rotations with respect to the rotation axis through O

$$H = -I_O \frac{d^2\varphi}{dt^2}$$

Also using Steiner's theorem according to which, the moment of inertia I_O with respect to an axis at O, may be calculated as the sum of the moment of inertia I_G having a parallel axis through G, plus $\frac{1}{2}ma^2$, where a is the distance between the axes.

$$H = I_O \frac{d^2\varphi}{dt^2} \quad \wedge \quad H = -mga\varphi \quad \Rightarrow I_O \frac{d^2\varphi}{dt^2} = -mga\varphi$$

(11.2) $$\frac{d^2\varphi}{dt^2} = -\frac{mga}{I_O}\varphi$$

The differential equation (11.2) is recognized as the equation for a harmonic oscillation.
It has therefore the solution: $\varphi = \varphi_0 \cos(\omega t + \delta_0)$. Namely, when differentiating twice, we get:

(11.3) $$\frac{d^2\varphi}{dt^2} = -\omega^2\varphi_0\cos(\omega t + \delta_0) = \frac{mga}{I_O}\varphi_0\cos(\omega t + \delta_0)$$

Which shows that it is a solution, if and only if:

$$\omega^2 = \frac{mga}{I_O} \quad \Leftrightarrow \quad \omega = \sqrt{\frac{mga}{I_O}} \quad \text{since} \quad \omega = \frac{2\pi}{T}$$

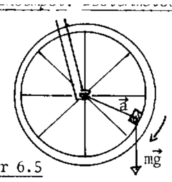
We can then find for the period of oscillation for the physical pendulum:

(11.4)	$T = 2\pi\sqrt{\frac{I_O}{mga}}$

The formula (6.4) is the formula for the period for the physical pendulum. Note that it can also be applied for a mathematical pendulum with a pendulum length $a = l$, since in that case $I_O = ml^2$, and we retrieve the familiar formula:

(11.5) $$T = 2\pi\sqrt{\frac{l}{g}}$$

11.6 Example. The moment of inertia of a bicycle wheel.

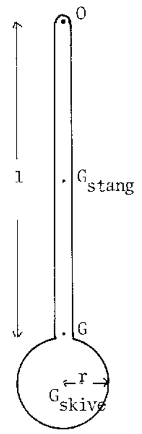


Figur 6.5

The formula for the physical pendulum may for example be applied to determine the moment of inertia of a wheel, which can turn freely around an axis. The figure is suppose to show the front wheel of a bicycle. We shall design an outmost simple experimental determination of its moment of inertia.
On the rim of the wheel at the distance 0.35 *m*, we place a weight *m* with mass of 0.125 *kg*. The wheel is put into oscillations around its horizontal axis, and the period is measured to $T = 3.6$ *s*. From this measurement, it is straightforward to determine the moment of inertia of the wheel.
Since the wheel is symmetric with respect to its axis of rotation the torque on the wheel comes exclusively from the weight. In the formula for the period: *mga* , *a* is the radius in the wheel. The moment of inertia is $I = I_{wheel} + ma^2$, where $ma^2 = 0.125 \cdot 0.35^2\ kgm^2 = 0.0153\ kgm^2$. From the formula for the period of a physical pendulum we find.

$$T = 2\pi\sqrt{\frac{I_O}{mga}} \quad \Leftrightarrow \quad I_O = \left(\frac{T}{2\pi}\right)^2 mga \ = 0.141\ kgm^2 \quad => \quad I_{wheel} = I_O - ma^2 = 0.015\ kgm^2$$

11.7 Example. Clock frequency for an old long case clock. (supposedly 1 sec)



Next we shall try to calculate the theoretical period of a pendulum in a long case clock, as shown in the figure (Danish text). The pendulum consists of a rod with mass $m_{rod} = 0.50\ kg$ and length $l_{rod} = 0.70\ m$, together with a disc with mass $m_{disc} = 1.5\ kg$ and radius $r_{disc} = 0.1 m$. First we determine the position of the CM for the pendulum. According to our previous results, the *CM* can be determined as if the whole mass of the rod and the disc were located in their respective *CM's* G_{rod} and G_{disc}. If the position of the CM is *a* measured from *O,* we have:

$$m_{rod}(a - l) = m_{disc}(r + l - a) \quad \Leftrightarrow \quad a = \frac{m_{disc}(r + l) + m_{rod}(\frac{1}{2}l)}{m_{rod} + m_{disc}} = 0.69\ m$$

The moment of inertia is calculated as the moment of inertia with respect to the axis at *O* using Steiner's theorem.

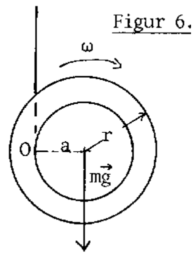
$$I = I_{rod} + I_{disc} + m_{disc}(r + l)^2 = \tfrac{1}{3}m_{rod}l^2 + \tfrac{1}{2}m_{disc}r^2 + m_{disc}(r + l)^2 = 1.05\ kgm^2$$

The moment of force $mga = (m_{disc} + m_{rod})ga = 13.5\ N$, and when inserted in the formula for the period of a physical pendulum it gives:

$$T = 2\pi\sqrt{\frac{I_O}{mga}} = \sqrt{\frac{1.05}{13.5}}\ s = 1.75\ s$$

(But as a physicist it is more important to understand what you are doing, than to obtain the correct result).

11.8 Example. The Yo-yo.



The figure shows a yo-yo. For convenience we shall assume that the yo-yo is a homogenous disc with radius $r = 3.0\ cm$. Further we assume that the cord is wound on a disc with radius $a = 2.0\ cm$. We wish to find the acceleration of the yo-yo, when it is dropped freely, but held on the cord.
We apply the equation of motion for rotation, with respect to an axis through O.
The moment of force with respect to O is $H = mga$. To calculate the moment of inertia with respect to O, we apply Steiner's theorem: $I_O = I_G + ma^2 = \frac{1}{2} mr^2 + ma^2$.
Furthermore: $v = \omega a$, where ω is the angular velocity, and v is the velocity of the CM.
We may then write the equation of motion:

$$H = I_O \frac{d\omega}{dt} \quad \Leftrightarrow \quad mga = (\tfrac{1}{2} mr^2 + ma^2)\frac{d\omega}{dt} \quad \Leftrightarrow \quad \frac{d\omega}{dt} = \frac{mga}{\frac{1}{2} mr^2 + ma^2}$$

$$\frac{dv}{dt} = \frac{mga^2}{\frac{1}{2} mr^2 + ma^2} = \frac{a^2}{\frac{1}{2} r^2 + a^2} g \quad \Rightarrow \quad \frac{dv}{dt} = \tfrac{8}{17} g = 4.62\ m/s^2$$