The special theory of Relativity

Chapter 10 of the textbook Elementary Physics 2

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Acknowledgment

This paper is a digitalized translation from Danish of a chapter in a textbook on physics written in 1976, that was conceived and written for the second year of the Danish 3-year high school (called gymnasium, for 16 -19 year-olds).

Rereading it, I have found that it is still an elementary yet rigorous presentation of the special theory of relativity, and it covers most of the aspects of the theory. Therefore it can still be used as an undergraduate text on the subject of special relativity.

Even in 1970's it was somewhat above the standard level both in mathematics and in abstract comprehension for use in the second year of teaching physics, and today it would be entirely out of question to try to use it in the Danish gymnasium.

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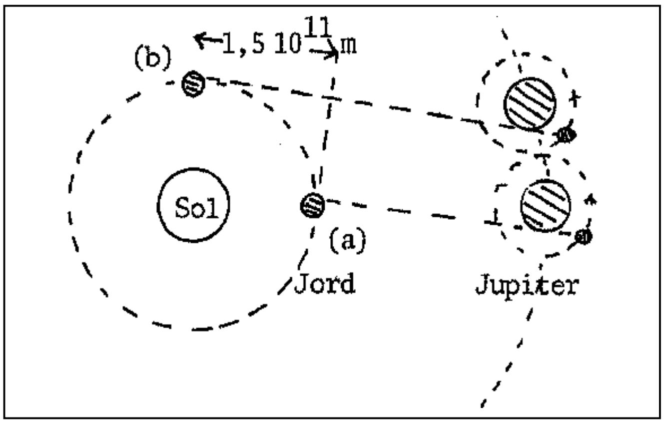
1. Determination of the speed of light

It was the Danish scientist and astronomer Ole Rømer, who in 1676 was the first to discover that light has a finite speed of propagation. Originally Rømer published his discovery as the “hesitation of light”

Rømer had himself designed and developed most of his instruments, and possessed at that period some of the best and most accurate instrument suitable for astronomical observations.

While studying the planet Jupiter, he had discovered that it was surrounded by several moons.



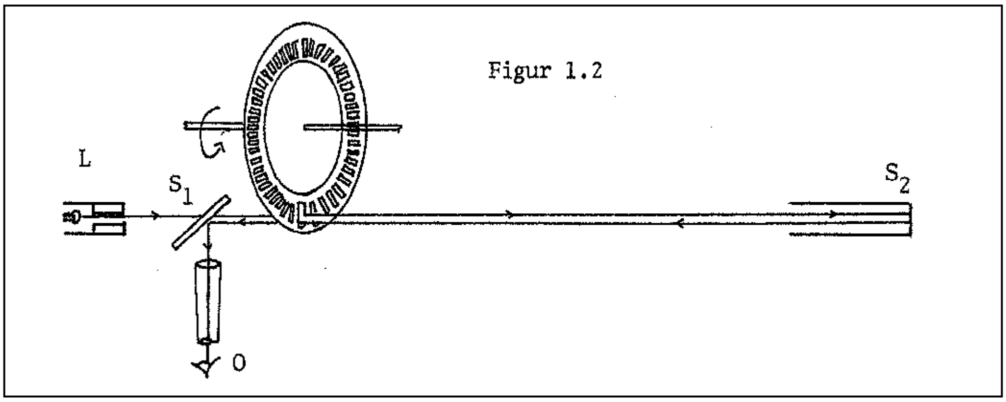
In an attempt to determine the orbital period of one of Jupiter’s moons, he discovered that the moment where the moon disappeared to the back of the planet apparently was dependent of the earth’s position relative to Jupiter.

The figure shows two such positions (a) and (b). In position (a) Rømer determined the orbital period of the moon. Under the assumption that the moon had a constant orbital period, he could calculate the exact moment where the moon would disappear behind the planet, when the earth was in position (b). However Rømer found that the observed time interval to be about 11 minutes delayed from what he had expected.

Rømer correctly explained this discrepancy in time as the amount of time used by the light to reach the earth from the two positions. The difference in distance between the earth and Jupiter, Rømer estimated to be the same as the radii in the earth’s movement around the sun, about $1.5\ 10^{11}$ *m*. By dividing this distance by 11 *min* = 666 *s*, Rømer found a value for the speed of light to be $2.3\ 10^{8}$ *m/s*. Today the accepted value for the speed of light is $c = 2.997929\ 10^{8}$ *m/s*.

The large magnitude of the speed of light makes a direct (mechanical) determination of c extremely difficult.

The most well known experiment was conducted by the two Frenchmen Fizeau and Foucault in 1849. In principle the experimental setup was as shown below.



L represents the source of light, from which the light propagates through the semi transparent mirror S_1. The light then passes a rotating disc, equipped with a circular row of equidistant holes in a manner, so that the width of the hole is equal to the separation between them.

After the light has been reflected from the mirror S_2 it again passes the holes, and the reflected light can be observed at *O*. However, when the disc rotates, it has moved a tiny bit during the time interval the light travels from S_1 to S_2 and back. With a suitable high frequency of rotation, it is possible that the light hits the space between two holes, at the light will disappear at *O*.
If the frequency of the discs rotation and the distance between two neighbouring holes are known, the time it takes the light to travel forth and back to S_2 can be determined, and this time interval can be divided into $2|S_1S_2|$ to obtain the speed of light.

Example 1.1
To illustrate how difficult the experiment actually was to perform, this numerical calculation will serve.
The Fizeau Foucault experiment was performed with a distance $|S_1S_2| = 9.0\ km$. The time interval Δt for traversing $2|S_1S_2|$ was therefore $2|S_1S_2|$ divided by c, where c is the speed of light. $\Delta t = 18\ 10^3\ m/3.0\ 10^8\ m/s = 6.0\ 10^{-5}$ s.
The wheel had a radii 0.38 *m*, and the distance between two neighbouring holes were 1.7 *mm*.
In the stated time interval the wheel must turn an angle $\Delta\varphi = 1.7/380$ radian $= 0.0044$. From this the angular velocity can be determined: $\omega = \frac{\Delta\varphi}{\Delta t} = \frac{0{,}0044}{6{,}0\,10^{-5}} = 73\ rad/s \Rightarrow\ \nu = \frac{\omega}{2\pi} = 11.6\ rps$

Even with modern technology it is quite impressive that the two Frenchmen actually succeeded to measure a value for the speed of light quite close to the correct value.

Nowadays (1977) the speed of light can be determined by radar waves, which are electromagnetic waves, but with a wavelength about 1 *cm*. Or one can use a laser which radiates completely coherent light : that is,with only one well defined wavelength. In both cases the speed of light can be determined by studying the interference pattern from the emitted and reflected light. Namely the interference pattern reveals the time interval between departure and arrival of the reflected light.

2. The Galileo transformation

The law of inertia was formulated by Newton, as his first law.

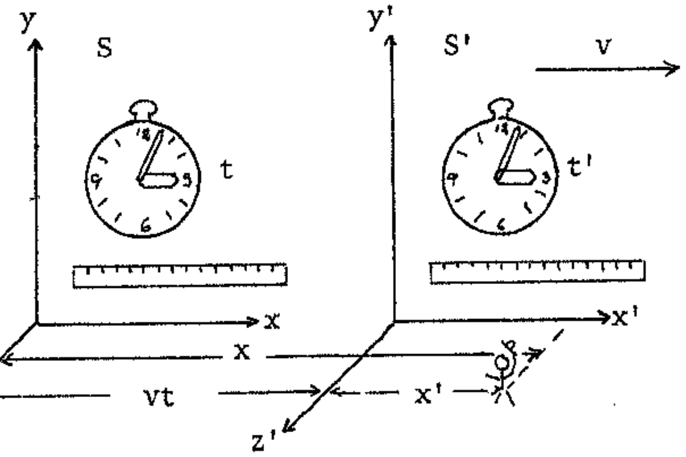
A body which is not influenced by forces will either be at rest or move (forever) with a constant velocity. No force is necessary to maintain a steady motion.

An inertial frame of reference (or inertial system) is a coordinate system where the law of inertia is valid. The law of inertia is thus a precondition for Newton’s second and third law.

All inertial frames move uniformly relative to each other. There exist no physical experiment that can determine whether you are “at rest” or move uniformly.

We shall characterize an event by the place (x, y, z) where it happened, and the time t when it happened.
We then wish to find the connection between the coordinates for the same event as observed from two observes in two inertial frames S and S', where S' moves with the relative velocity v along the x-axis. For simplicity we shall assume that the two coordinate systems have parallel axis, and that they coincide at time $t = t' = 0$.
Both observers situated in the inertial frames S and S' are equipped with identical clocks, which are synchronized, when they pass each other.
Finally and we shall also make two seemingly trivial assumptions.

Figur 2.1



1. The two clocks will forthright show identical times *t = t'* for the same event, observed from *S* and *S'*, but understood that they correct for the time it takes the light to travel from the event in question to the observer.

2. If the two observers are provided with identical rulers, they will find the same result of measuring a length, made from the two inertial systems *S* and *S'*. Especially they will agree that the two identical rulers still have the same length, independent of their mutual motion.

The reason why these two (trivial) assessments are emphasized is that neither of them are valid in the Special Theory of relativity – in any case when the relative speed of the two observes approaches the speed of light.

However, if we hold on to the two assumptions, it is immediately obvious from the figure, that the following relations holds between the coordinates *(x, y, z)* and *t* in *S* and *(x', y, z)* and *t'* in *S'*.

(2.1) $$x' = x - vt \ , \ y = y' \ , \quad z = z' \quad and \quad t = t'$$

The relations (2.1) illustrates the connection between the coordinates for the *same* event, as seen from the two observes *S* and *S'*.
The formulas expresses the transition from *S* to *S'*, but they can easily be transformed to reflect the transition from *S'* to *S*, by replacing the marked coordinates with the unmarked, and *v* by –*v*.

(2.2) $$x' = x + vt \ , \ y = y' \ , \quad z = z' \quad and \quad t = t'$$

Thus there is a complete symmetry between the observations of *S* and *S'*, and there are no means to decide who is moving, and who is "at rest".

If we want to find the connection between velocities and accelerations of a particle observed by *S* and *S'* it can be done by differentiating with respect to *t*. If a particle has the velocity $u_x = dx/dt$ in *S* and the velocity $u_x' = dx'/dt$ in *S'*, we get immediately from (2.1).

(2.3) $$u_x' = \frac{dx'}{dt'} = \frac{d}{dt}(x - vt) = u_x - v \quad and \quad a_x' = \frac{d}{dt}(u_x - v) = a_x$$

The velocities and accelerations in the y and z directions are evidently the same according to (2.1). The last result in (2.3) ensures that Newton's second law is form invariant (covariant) under the Galileo transformation, but actually this requires one last assumption:

3. *That the mass of a body does not change with velocity.*

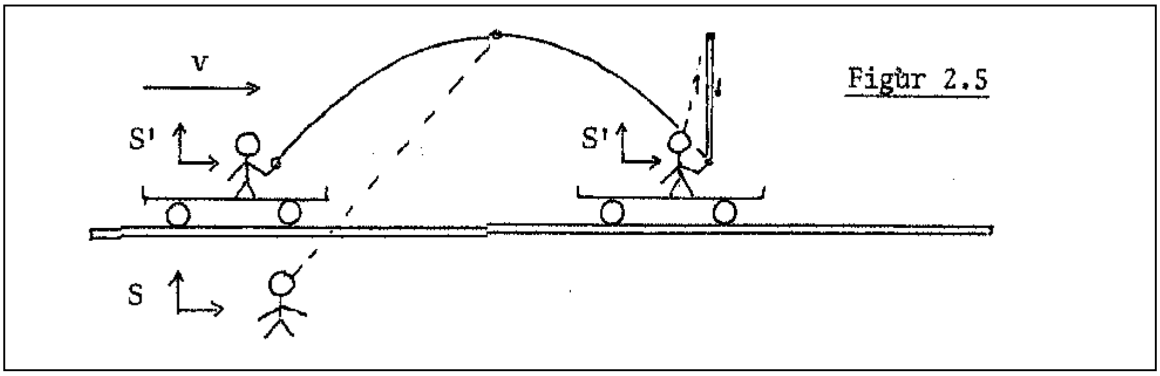
Namely from (2.3) we have:

(2.4) $F'=m'a_x' = ma_x = F \quad => \quad F=ma_x$ in S is identical to $F'=m'a_x'$ in S'

Although seemingly trivial, we have emphasized the three conditions for the validity of the Galileo transformation (2.1), because neither of them is valid in the special theory of relativity.
The equations above expresses the fact that Newton's laws are *form invariant* under the Galileo transformation, and for that reason the two inertial frames are completely equal.
There exits no experiment, which can decide, which of the two frames is moving, and which one is "at rest".
For this reason one may state that the *principle of relativity* holds.

Although Newton's second law has the same form in the two coordinate systems, it does not mean that the motion of a body will look the same, when viewed from S or S'.

2.5 Example
The same movement of a body, will certainly look different from the two observes S and S', moving uniformly with respect to each other.
As and example we shall look into, how a vertical throw performed in S' will look for an observer in S.



Suppose that the observer in S' throws a body with an initial velocity u_0 along the y' axis.
Below are established the equation of motion in the two inertial systems S and S' respectively.

S'	S
$a_x'=0$	$a_x=0$
$a_y'=-g$	$a_y=-g$
$v_x'=0$	$v_x=0$
$v_y'=u_0-gt'$	$v_y=u_0-gt$
$x'=0$	$x=vt$
$y'=u_0t'-\frac{1}{2}gt'^2$	$y=u_0t-\frac{1}{2}gt^2$

In *S'* the throw is a vertical motion with initial velocity u_0, whereas the trajectory in *S* can be found by eliminating *t* from the two last equations. Putting $t = x/v$ into the expression for *y*, we find:

$$y = -\frac{g}{2v^2}x^2 + \frac{u_0}{v}x \quad \Rightarrow \quad y = ax^2 + bx$$

Where we recognize the latter expression as the familiar equation for a parabola.

3. The principle of relativity and the invariance of the speed of light

From the time when Newton published his Principia (1689) and to the beginning of the twenties century, Newton's famous three laws and his law of gravitation combined with the Galileo transformation formed the foundation of the global view of the world.
The gravitational attraction could account for the physics on the earth, the orbital motion of the moon, as well as for the planets orbital motion around the sun.
However from the end of the twenties century both theoretical and experimental evidence were brought about which in one way or another seemed to be in contradiction to the traditional view of the world.

About 1870 the Englishman Maxwell succeeded in combining the laws of electricity and magnetism in the four (so called) Maxwell equations. The equations were complete in the sense that nothing could be added to describe electromagnetism, and Maxwell's equations have never been revised up till nowadays.
As a consequence of his equations (which is one of the most important theoretical achievements in modern physics) Maxwell explained the nature of electromagnetic waves and could conclude that light was the propagation of rapidly alternating electric and magnetic fields i.e. electromagnetic waves, and furthermore that all electromagnetic waves propagated with velocity *c* the "speed of light".
Maxwell combined the two constants of nature ε_0, from Coulombs low of electrostatics, and μ_0 from Ampere's law of magnetism to yield an expression for the speed of propagation of electromagnetic waves.

$$c = \sqrt{\varepsilon_0 \mu_0}$$

This value was in excellent accordance with the previous attempts to determine *c* from direct experiments.

Coulombs low of electrostatics is: $F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r^2}$, which is the force between two charges.

Ampere's law of magnetism is: $B = \frac{\mu_0}{2\pi}\frac{I}{r}$ which gives the magnetic field from a straight wire of current.

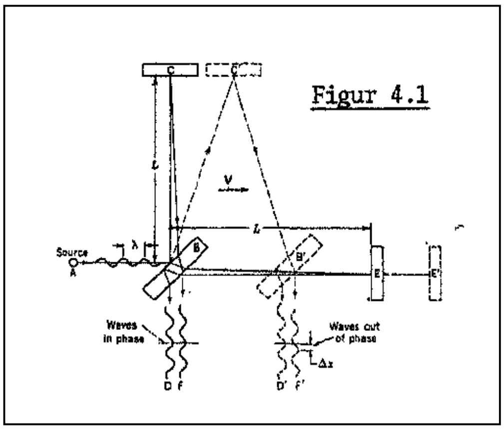
Maxwell's equations seem altogether to be in accordance with the behaviour of nature apart from one theoretical obstacle. The equations were *not* form invariant under the Galileo transformation. In contrast to Newton's laws, the Maxwell equations were not the same in different inertial frames, and consequently it should be theoretically possible to determine, in which inertial frame one was situated in.
However this was in direct conflict with the principle of relativity. But Maxwell's equations had already proved its worth, so the alternative would be to discard the validity of the Galileo transformation. For some time the physical world sought for a compromise.

The theory that followed, the ether theory, was strictly speaking a breach with the principle of relativity but in a couch of words, which could conserve both the Galileo transformation and the Maxwell equations.
In the ether theory light is conceptually considered as the same phenomena as propagation of sound. Sound propagates in air with a constant velocity, independently of the speed of the source. The speed is however dependent of the constituents of the air. This corresponds to, that light has a different speed in various transparent media, as revealed e.g. in the refraction of light when passing from one material to another.
In an inertial system, at rest relative to the air, sound propagates with “the speed of sound”, but in any other moving inertial frame, it propagates with a speed, that can be found by performing the Galileo transformation.
Using this analogy with the propagation of sound, it was proposed that light (electromagnetic waves) propagated with c the “the speed of light” in a media called the *ether*, which flew through the universe, and that the speed of light in any other inertial system moving relative to the ether could be found by the Galileo transformation. Furthermore it was assumed, that the speed of light was independent of the speed of the source.

The theory did not seem to be entirely unjust, having no obvious flaws, and the main obstacle seem to be the non observable existence of the “ether”. (This in contrast to the atmospheric air, which is physical observable, so its properties can be an object of experiments).
Non observable physical objects, that have a disturbing influence on physical phenomena (and are invented to explain unexpected results of physical experiments) have always been (philosophical) problematic and have otherwise been discarded since the middle of the twenties century.
As long as such phenomena are a matter of “belief”, they have never lead to a truer recognizing of nature.

4. The Michelson Morley experiment

The ether theory designated one unique inertial frame, namely the frame where the ether was at rest and the speed of light was c.
The ether theory could, at least partly be established if one was able to measure the speed of the earth relative to the ether – the so called ether wind.
The most spectacular experiment was conducted by the two Americans Michelson and Morley in 1887. The experimental setup is sketched below.



4.1 Experimental setup
Light from the source A hits the semi transparent mirror at B.
Some of the light is reflected to the mirror C where it is reflected back to B'.
Some of the light passes B, where it is reflected from the mirror E.
If the light reflected from C and E is not in phase, one should observe an interference pattern at B'.
We assume that the experimental setup is fabricated so that the length BC is equal to BE. This length is L.
If the apparatus is at rest relative to the ether, then the light, which is reflected from the mirrors C and E would be in phase after the reflection, and one should observe no interference at B'.
On the other hand we can show that one must observe interference, if the apparatus, as shown in the figure, moves with the velocity v relative to the ether.

We begin by finding the two time intervals t_1 and t_2 it takes the light to run through the distance $L = BE$ and back. (In the figure marked as $B'E'$).
The calculation of the velocities is done relative to the ether, where the light has the velocity c.
In the time interval t_1 the light move a distance ct_1, but in the same interval the apparatus has moved vt_1 (namely from E to E'), so the following relation must apply. $ct_1 = L + vt_1$.
In the time interval t_2 the light travels the distance ct_2, while the apparatus moves a distance vt_2.
But since B moves towards the light $ct_2 = L - vt_2$ must apply. From the two equations it follows:

$$(4.2) \qquad t_1 = \frac{L}{c-v} \qquad and \qquad t_2 = \frac{L}{c+v}$$

These expressions can also be noted directly by observing that the light moving out and back will have velocities $c + v$ and $c - v$ relative to the apparatus (according to the Galileo transformation), and that in both cases must travel the length L. From (4.2) we then get:

$$(4.3) \qquad t_1 + t_2 = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2Lc}{c^2 - v^2} = \frac{2L/c}{1 - v^2/c^2}$$

To determine the time interval for the second reflection, we notice that the light in the ether system will follow the dashed line B-C'-B'. The time interval it takes the light to travel through BC' we shall denote t_3. In this time interval the apparatus moves the distance vt_3, while the light moves the distance BC' which is ct_3.
As the pieces L, vt_3 and ct_3 are sides in a right angle triangle, the following equation must hold.

$$(4.4) \qquad (ct_3)^2 = (vt_3)^2 + L^2 \qquad \Rightarrow \qquad t_3 = \frac{L}{\sqrt{c^2 - v^2}}$$

The time it takes the light to run through the distance B-C'-B' is $2t_3$,

$$(4.5) \qquad 2t_3 = \frac{2L}{\sqrt{c^2 - v^2}} = \frac{2L/c}{\sqrt{1 - v^2/c^2}}$$

We notice that according to (4.3) and (4.5) $t_1 + t_2 \neq 2t_3$, so the reflected light for $v \neq 0$ would be out of phase at $D'F'$. Consequently one should observe an interference pattern at this place.
From the interference pattern, it should in principle be possible to estimate the difference $t_1 + t_2 - 2t_3$, and eventually determine the ether wind velocity v.

In practice it would hardly be possible to obtain $BC = BE$ with an accuracy of about 100 nm, corresponding to the wavelength of light. Instead Michelson and Morley mounted the experiment, so they could rotate the apparatus 90^0. In this way the two light beams switched place, parallel and perpendicular to v and one would expect for $v \neq 0$, a slight displacement of the interference stripes.

Michelson and Morley conducted the experiment in periods, where the velocity from the earth's rotation around its axis, and the earth's velocity from around the sun were parallel with one of the beams in the apparatus, and in the same direction.

The experiment was performed, but showed no displacement whatsoever of the interference stripes.
And all later experiments to determine the velocity of the earth relative to the ether have failed. What the Michelson Morley experiment demonstrated was unquestionably that the ether hypothesis was probably wrong, and the principle of relativity had unrestricted validity.

It was a tremendous blow to the physical society that accepting the rigor of the Maxwell equations, inevitable lead to the conclusion that the Galileo transformation, and consequently Newton's laws had to be revised, at least in cases where the speed of light was the issue.

In the beginning of the twenties century, it became obvious that a rather comprehensive revision of the view of the world was necessary before one could maintain consistency i.e. free of contradiction in the laws of nature.

The revision was launched by Albert Einstein with his revolutionary paper on The Special Theory of Relativity published in 1905.

One might add however, that the negative result of the Michelson Morley experiment can be explained, if you assume that a body moving with velocity v is shortened by a factor: $\sqrt{1-v^2/c^2}$ in the direction of the movement. If you insert $L_{||} = L\sqrt{1-v^2/c^2}$ in (4.3) you find that $2t_3 = t_1 + t_2$.
The shortening of length by the factor above is however one of the bizarre consequences of The Special Theory of Relativity, but it is the correct explanation for the negative outcome of the Michelson Morley experiment.

In more popular accounts of how Einstein came to his theory, it is represented as it came out of the blue in a small office in Zürich, but in reality it was experimental evidence that guided him, since the Michelson Morley experiment pointed unambiguously towards the fact that the speed of light propagated with speed c independent of the inertial frame.

Also the Dutch physicist H. A. Lorentz had in 1904 shown, that although the Maxwell equations were not form invariant under the Galileo transformation, they were invariant under a more general transformation, which is now known as the Lorentz transformation. This transformation therefore becomes the core of the Special Theory of Relativity.

5. The foundation of relativity. Einstein's train

Although there had been several indications that something was wrong with the traditional view of the physical world based on Newton's laws and the Galileo transformation, (and the unspoken assertions of the invariance of length, time and mass from one inertial frame to another), Einstein was the first to revise it and to put it in a consistent theoretical framework without contradictions, as it is required by a physical theory.
The two main ingredients, namely that light propagates with the same speed in all inertial systems (from the Michel Morley experiment), and that the Maxwell equations were form invariant to the Lorentz transformation, were the corner stones on which he based his theory.
So Einstein built his theory on these two fundamental assertions, which he considered evident, on the ground of the experimental results.

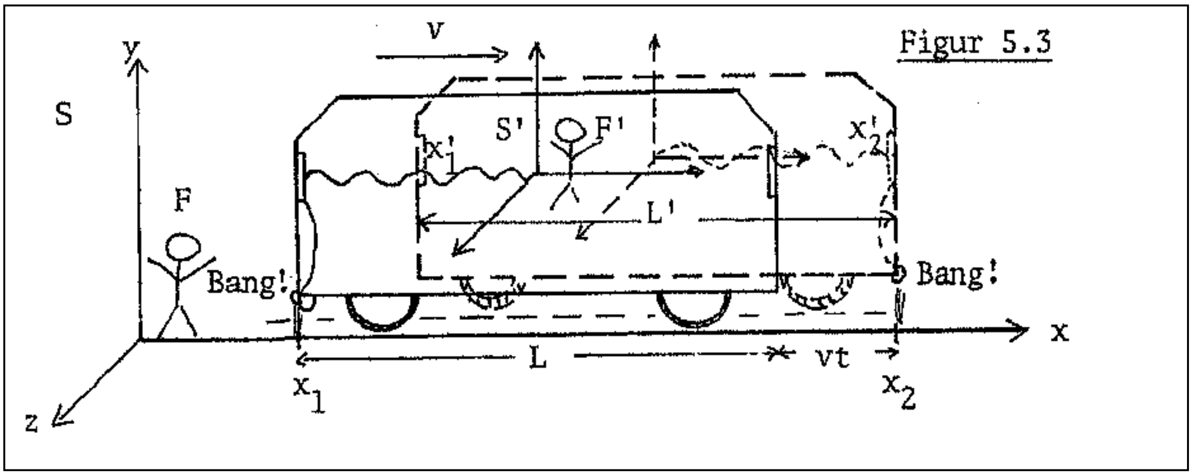
(5.1) <u>The principle of relativity</u>.
All inertial frames are on equal foot. There exist no experiment, which can decide wheather you are at rest, or move with constant velocity.
As a consequence the physical laws of nature must be form invariant, (i.e. they look the same) when transforming from one inertial frame to another.

(5.2) <u>The invariance of the speed of light.</u>
Light propagates with the same speed c in all inertial frames. Furthermore the speed of light is the same in all directions, and is independent of the speed of the source.
Velocities being greater than the speed of light are impossible.

Although in accordance with experiments, these two assumptions are not entirely obvious from common everyday experiences.

However we shall draw some essential consequences from the two assertions, normally known as the Special theory of Relativity. It turns out that both Newton's laws and the Galileo transformation are only approximately correct when the velocities in question are much smaller than the speed of light, which is 3,0 10^8 *m/s*.
Already at speeds which are below 1/100 *c*, the relativistic effects, are hardly measurable.

Einstein conducted some so called thought experiments to enlighten the stringent logic in his theory.
Below, we shall analyze his probably most famous thought experiment, called Einstein's train.



Assume that the train, corresponding to the inertial frame *S'* moves with velocity *v* along a set of rails. The rails correspond to inertial the frame *S*. In the two inertial frames *S* and *S'* are placed physicists *F* and *F'* equipped with identical rulers.

F' wishes to communicate to *F* what is the length of the wagon.
He can measure it himself with the ruler he has brought with him. Let us assume that he measures the length of the wagon to *L'*.
F and *F'* now agree to an experiment, which allows *S* to measure the length of the wagon with his proper ruler. The result of this measurement we shall denote *L*.

F' places warheads in both ends of the train, and the warheads are released when a photocell at the end wall of the wagon is lighten.
From the explosions two marks are left on the rails and subsequently they can me measured by *F*. To ensure, that the explosions take place simultaneous in *S'*, *F'* emits a light signal in both directions from the middle of the wagon. In *S'* the emitted light will hit the walls simultaneous, and consequently the explosions will take place simultaneous, and therefore the marks on the rail, will be set simultaneous is *S'*. He will subsequently claim that the distance between the marks on the rails is the length of the wagon *L'*.

F agrees, according to (5.2) that the speed of light is the same in both directions, even if the light source is moving with velocity v, but in the elapse of time for the light to reach the end walls of the wagon, the wagon has moved, so that the rear wall has moved towards the light, and the impact will be a little earlier, whereas the front wall has moved away from the light, and the impact will be a little later.
F claims that the rear photocell was activated before the front photocell, and therefore the two explosions did not happen simultaneous in the inertial system *S*. And consequently the distance between the marks on the rails is not the length of the wagon in the inertial system *S*.

In the frame *S* the time t_1 , where the light hits the rear wall can be determined, by noticing that the light has moved a distance ct_1, the train has moved a distance vt_1, and half the length of the wagon is $\frac{1}{2}L$.
Therefore: $ct_1 = \frac{1}{2}L - vt_1$. Using a similar argument *F* will claim that the time t_2 where the light reaches the front wall is given by: $ct_2 = \frac{1}{2}L + vt_2$. We may then establish the equations:

$$ct_1 = \tfrac{1}{2}L - vt_1 \quad \textit{and} \quad ct_2 = \tfrac{1}{2}L + vt_2 \quad \Leftrightarrow$$
$$t_1 = \frac{\frac{1}{2}L}{c+v} \quad \textit{and} \quad t_2 = \frac{\frac{1}{2}L}{c-v} \tag{5.4}$$

We notice that $t_1 \neq t_2$ for any value of $v \neq 0$.
From this example we conclude that two events, which are simultaneous in one inertial system *S'* are normally not simultaneous in another inertial system *S*. Consequently the intervals between events must also differ in two different inertial frames.

This is referred to as the relativity of time.

And it differs radically from the Newtonian concept.

But neither do the two physicists *F* and *F'* agree on the measurement of length.
As the explosions occurred simultaneous in *S'*, *F'* will claim that the length between the marks on the rails corresponds exactly to the wagon's length *L'* measured by a ruler in *S'*. On the other hand *F* will claim that the rear explosion came prior to the front explosion, and that the train moved a little bit before the last explosion.
Therefore *F* claims that the wagon is shorter than the length between the marks on the rails, meaning that: $L < L'$.

The conclusion is that in general you may *not* find the same result of a length measurement (in the direction of movement) of two different inertial frames.

This is referred to as the relativity of length.

Which is also absent in the Newtonian description.

6. The Lorentz transformation

We shall now apply the results from the analysis of the moving train to derive a relation between the coordinates and times in the two inertial systems *S* and *S'* , maintaining the two fundamental assertions (5.1) and (5.2).
To obtain the relations, we shall further do some necessary assumptions.

Firstly the new transformations must confirm to the Galileo transformation when velocities are much lower than the speed of light. (that is: $v \ll c$), so the transformations are in some manner bound to resemble the Galileo transformation.

If *S'* moves with a velocity *v* relative to *S*, we may try with the Galileo transformation provided with a factor γ, which is dependent of *v* and *c*: $x' = \gamma(x - vt)$. γ must be positive, independent of the coordinates, and $\gamma \approx 1$ when $v \ll c$, because the transformations in this limit has to confirm with the Galileo transformation.
According to the principle of relativity the two transition formulas between *S'* to *S* must be identical, if one replaces the marked coordinates with the unmarked, and replaces *v* with –*v*.
So we may try with the transitions formulas.

$$(6.1) \qquad x' = \gamma(x - vt) \qquad \text{and} \qquad x = \gamma(x' - vt')$$

Using these equations we now proceed with a quantitative analyze of the thought experiment with the train.
The wagon represents the inertial frame *S'*, which moves with a velocity *v* relative to *S*.
S is the frame where the rails are at rest. We assume further that *S* and *S'* coincides at $t = t' = 0$.
As the two explosions occurred simultaneously in *S'*, we must have: $t_1' = t_2'$, which correspond to the coordinates $x_1' = -\frac{1}{2}L'$ and $x_2' = \frac{1}{2}L'$, while in *S*, the explosions occurred at times t_1 and t_2 given by (5.4).

$$(5.4) \qquad t_1 = \frac{\frac{1}{2}L}{c+v} \qquad \text{and} \qquad t_2 = \frac{\frac{1}{2}L}{c-v}$$

In the following we shall use the symbol Δ to denote usually an small interval.
i.e. $\Delta x = x_2 - x_1$ and so forth and so on. Inserting these expressions in (6.1) gives:

In *S'*: $\quad \Delta x' = x_2' - x_1' = \gamma(x_2 - vt_2) - \gamma(x_1 - vt_1) = \gamma(\Delta x - v\Delta t)$

In *S*: $\quad \Delta x = x_2 - x_1 = \gamma(x_2' - vt_2') - \gamma(x_1' - vt_1') = \gamma(\Delta x' - v\Delta t') = \gamma\Delta x'$ (Since $t_1' = t_2'$)

The time interval $\Delta t = t_2 - t_1$ can be found from (5.4)

$$(6.4) \qquad \Delta t = t_2 - t_1 = \frac{\frac{1}{2}L}{c-v} - \frac{\frac{1}{2}L}{c+v} = \frac{Lv}{c^2 - v^2} = \frac{Lv/c^2}{1 - v^2/c^2}$$

F claims that $\Delta x = x_2 - x_1$ equals the length of the wagon *L* plus the part $v\Delta t$ the train has moved in the time interval Δt.

(6.5) $$\Delta x = L + v\Delta t$$

Inserting the expression for Δt from (6.4) in (6.5), one finds:

(6.6) $$\Delta x = L + v\Delta t = L + \frac{Lv^2/c^2}{1-v^2/c^2} = L(1+\frac{v^2/c^2}{1-v^2/c^2}) = \frac{L}{1-v^2/c^2}$$

In (6.2) and (6.3) we insert $\Delta x' = L'$. Further (6.5) is inserted in (6.2), and (6.6) is inserted in (6.3). The outcome is two equations.

(6.7) $$L' = \gamma L \quad \text{and} \quad \frac{L}{1-v^2/c^2} = \gamma L'$$

Inserting the first equation into last and reducing with *L*, we find:

(6.8) $$\frac{1}{1-v^2/c^2} = \gamma^2 \quad \Rightarrow \quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

The expression obtained for γ is then substituted back in the two expressions (6.1) in the modified Galileo transformation to yield the sought transformation formulas between the two inertial frames *S* and *S'*.

(6.9) $$\boxed{x' = \frac{x-vt}{\sqrt{1-v^2/c^2}} \quad \text{and} \quad x = \frac{x'+vt'}{\sqrt{1-v^2/c^2}}}$$

To find the relation between the measured times in *S* and *S'*, we assume that at the moment $t = t' = 0$ a light signal is sent along the $x = x'$ axis originated from the common origin of the two inertial frames.
As the light propagates with *c* in both inertial frames (the assertion (5.2)) it then follows that for the same event (*x, t)* in *S* and (*x', t*) in *S'* the equalities below must hold.

(6.10) $$x = ct \iff t = \frac{x}{c} \quad \text{and} \quad x' = ct' \iff t' = \frac{x'}{c}$$

Introducing this in (6.9) and dividing with *c* on both sides, reveals the wanted formulas for transformation between *t* and *t'*.

(6.11) $$\boxed{t' = \frac{t - \frac{xv}{c^2}}{\sqrt{1-v^2/c^2}} \quad \text{and} \quad t = \frac{t' + \frac{x'v}{c^2}}{\sqrt{1-v^2/c^2}}}$$

The formulas (6.9) and (6.11) are together called the Lorentz transformations, and they constitute the foundation of The Special Theory of Relativity.

As previously remarked, H.A. Lorenz discovered in 1904 that the Maxwell equations were form invariant under these transformations.

We have deducted the Lorenz transformations from the modified Galileo transformation, and Einstein's two suppositions (5.1) and (5.2).
One should notice that $\gamma \approx 1$ when $v << c$ (v much less than c), and in this limit The Lorenz transformations become equal to the Galileo transformation, which is known to be true, as long as the velocities are much less than the speed of light.

7. Consequences of applying the Lorenz transformation

From the equation (6.7) $L' = \gamma L$ one can derive the connection between the lengths of the wagon as measured by the physicist *F'* who follows the train and *F* from whom the train passes with velocity *v*. Inserting (6.8) for γ, one finds:

$$(7.1) \qquad L' = \frac{L}{\sqrt{1 - v^2 / c^2}} \quad \Leftrightarrow \quad L = L'\sqrt{1 - v^2 / c^2}$$

L' is the length measured by someone at rest relative to the object measure, and therefore it is denoted the rest length, and is usually denoted L_0. Using this notation we get.

$$(7.2) \qquad L = L_0\sqrt{1 - v^2 / c^2}$$

From (7.2) we infer that a body which moves relative to an observer will appear shorter with a factor $\sqrt{1 - v^2 / c^2}$ in the direction of motion. This does not only goes for the wagon, where *F* will claim it is shorter than measured by *F'*.
Since *F'* has measured the wagon with his proper ruler, *F* will claim that everything which is found in the inertial frame *S'* is shortened by the same factor in the direction of motion.

Let us then assume that *F* and *F'* want to make measurements in the inertial frame *S*.
If *F'* intend to make a correct measurement in *S*, he must ensure that the two coordinates x_1' and x_2' are measured simultaneous, so that $t_1' = t_2'$. In *S* however it doesn't matter, whether the coordinates x_1 and x_2 measured simultaneously or not, since the are fixed in *S*.
To find the distances between the two points in *S'*, we use the second formula (6.9) with $t_1' = t_2'$.

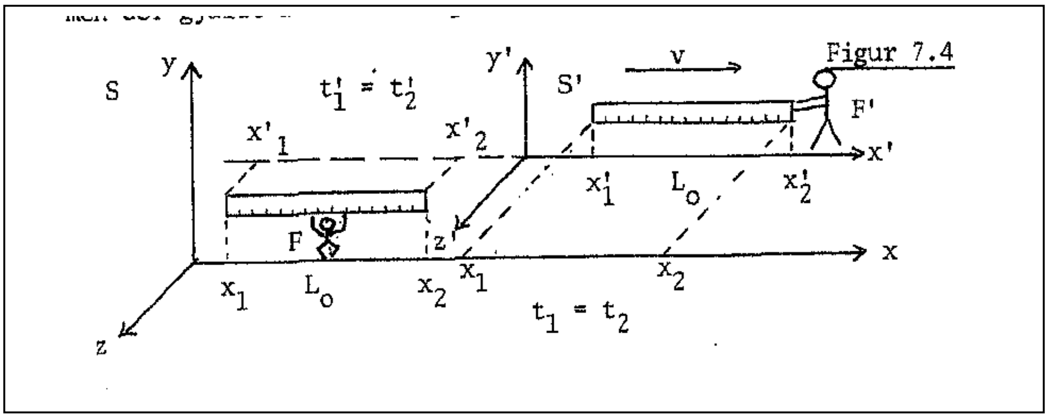
$$(7.3) \qquad \Delta x = x_2 - x_1 = \frac{x_2' + vt_2'}{\sqrt{1 - v^2 / c^2}} - \frac{x_1' + vt_1'}{\sqrt{1 - v^2 / c^2}} = \frac{\Delta x'}{\sqrt{1 - v^2 / c^2}} \quad \Leftrightarrow$$

$$\Delta x' = \Delta x\sqrt{1 - v^2 / c^2}$$

This is called the Lorenz contraction.

The last equation shows that by the measurements made by *F'* a length in *S* is shortened by a factor $\sqrt{1 - v^2 / c^2}$ compared to what is measured by *S*.

Consequently *F'* will claim that the rulers in *S* are shortened by this factor. At a glance this might appear as the opposite of what we found in (7.2), but it is not, because at that time, it was related to simultaneous measurements in *S'*.



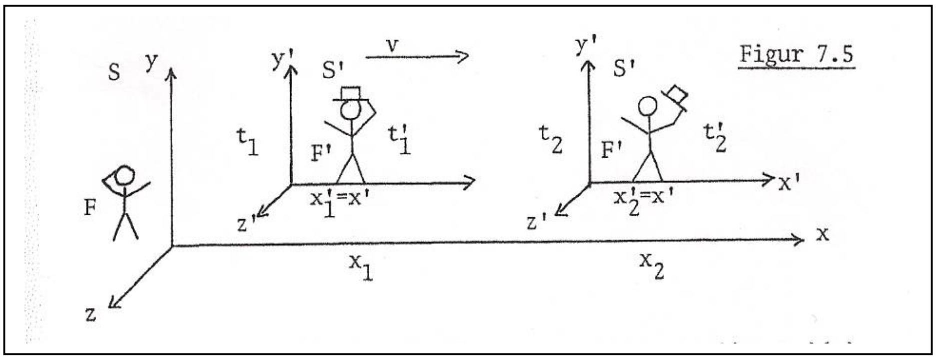
At first it may come as a surprising and even contradictory conclusion that both physicists claim that the other uses a shortened scale on their rulers.
But it is certainly right, and if you think about it, it must necessarily as a consequence of the principle of relativity.
If the two physicists differed in their conclusion about the other observes ruler, it would in principle be possible to decide, which inertial frame was moving, and which one was at rest.

We shall now look into the time intervals that the two physicists *F* and *F'* measure at the same position in *S'*. We therefore insert $x_1' = x_2' = x'$ in the last of the formulas (6.11), seeking the connection between $\Delta t = t_2 - t_1$ and $\Delta t' = t_2' - t_1'$

$$t_2 - t_1 = \frac{t_2' + \frac{x'v}{c^2}}{\sqrt{1-v^2/c^2}} - \frac{t_1' + \frac{x'v}{c^2}}{\sqrt{1-v^2/c^2}} = \frac{t_2' - t_1'}{\sqrt{1-v^2/c^2}}$$

(7.4)
$$\Delta t = \frac{\Delta t'}{\sqrt{1-v^2/c^2}}$$

The relation (7.4) establishes that the time interval between two events in the same point of space in *S'* , it will appear longer for an observer in *S*, than for an observer in *S'*.
For this reason, *F* will claim that, the clocks in *S'* are slowed down by a factor $1/\sqrt{1-v^2/c^2}$



Let us nest assume that the two physicists will compare measurements of time intervals in the same point of space in *S*.
Inserting $x_1 = x_1 = x$ in the first equation (6.11) we find in the same manner as before.

(7.6) $$\Delta t' = \frac{\Delta t}{\sqrt{1 - v^2 / c^2}}$$

The content of (7.4) and (7.6) is called the time dilation.

The time interval measured by two events in *S* is longer for an observer in *S'*.
F' will therefore claim that the clocks in *S* run too slow. At a glance this seem contradictory to what we found previously, but it is not, since it concerned measurement of a time interval in *S'*.

So both physicists claim, that the other clocks run too slow, but this is in accordance with the principle of relativity. Actually it must be so to maintain this principle.

Especially the relativity of time has caused many objections to the theory of relativity.
The most well known is the so called twin paradox. Let us suppose that one of the twins is sent away from the earth in a space ship, which travels with a speed comparable to the speed of light. According to (7.4) and (7.6) both twins will claim that the other clock runs to slow, and therefore both twins will recognize that the other twin will age slower than himself. Drawn to the extreme, both twins will claim that they will die before the other.
To decide which one is right, the twin in the spaceship turns it around and return to earth, where they eventually can compare their ages by direct confrontation.
The slowing down of the clocks are the same on the way out and back, since the velocity enters as v^2 in the formulas (7.4) and (7.6).
After returning the two twins can directly decide, which one is the oldest. By symmetry they must have the same age, which at first hand seem to be a paradox.
The point is however that the principle of relativity can not be applied when the twin in the spaceship turns it around, since he is no longer situated in an inertial frame, but in a frame undergoing enormous accelerations.
Therefore the Lorentz transformations do not apply.
But are two frames not just accelerating relative to each other? No in an acceleration frame Newton's laws do not apply, and one will be inflicted by fictitious forces, like an outward centrifugal force and a sidewise Corioli's force.
When one frame is accelerating, the two frames are not on equal footing and the principle of relativity does not apply any longer.
So in principle, there is nothing inconsistent by the two twins having a different biological age. But can we decide which one is the oldest? Yes the twin on the earth is the oldest. From his point of view namely the other twin is just during the accelerations travelling in a series of instantaneous inertial frames submitted to the time dilation.
How things act in the spaceship, cannot be deducted from the special theory of theory of relativity.
It can however be decided (with some effort) by the general theory of relativity, which deals with accelerated frames.
The general theory of relativity was put forward by Einstein in 1916 eleven years after he published the special theory.
While the special theory of relativity is conceptually simple and mathematically plain, the general theory is conceptually and mathematically incomprehensible and rather complex.

7.7 Exercises
1) In the discussion about Einstein's train, we concluded that the two explosions were not simultaneous in *S*.
Assume that the wagon has the length of 100 *m*, and that it travels with a speed of 120 *km/h*. Calculate how much the split of time will be in *S*.

2) The earth is orbiting around the sun with a speed of about 30 *km/s*.
Estimate how much longer a year will be for an observer in the sun.

3. Estimate the relative speed a meter stick must have to diminish its length by 0.1%.

4) A myon is an unstable particle which, when produces at rest has a lifetime of $2.2\ 10^{-6}$ s.

a) Estimate the lifetime in the laboratory, when it is created at a speed of 0.99 *c*. *c* is the speed of light.
b) How far can a myon reach at this speed?
c) Estimate how long this distance will be compared to an inertial frame following the myon.

8. Transformation of velocities

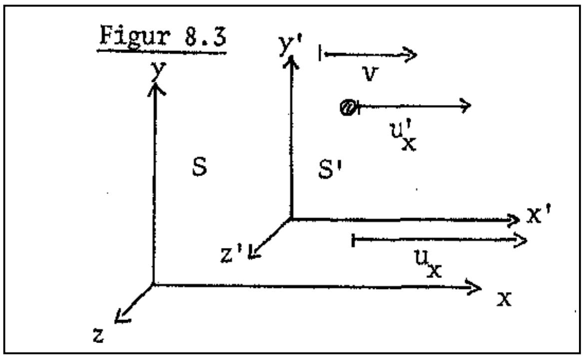
Before deriving the transformation formulas for velocities, we shall notice that a measurement of length perpendicular to the relative velocity, will give the same result for the two observers in *S* and *S'*. This should be evident, since the two observers directly can compare measure sticks simultaneously. The physicist *F* can use his ruler to draw a line parallel to *x = x' axis* in a height of say 1 *m* up the *y* axis. The physicist *F'* takes his 1 *m* ruler and sees if it matches.

To find the transformation formulas for velocities it is however necessary to write the Lorentz transformations using increments (deltas)

$$(8.1)\qquad \Delta x = \frac{\Delta x' + v\Delta t'}{\sqrt{1 - v^2/c^2}} \qquad\qquad \Delta t = \frac{\Delta t' + \dfrac{v\Delta x'}{c^2}}{\sqrt{1 - v^2/c^2}}$$

If follows from the remarks above that: $\Delta y = \Delta y'$, $\Delta z = \Delta z'$.
Let us assume that the motion of the same particle is observed from two inertial frames *S* and *S'*. I *S* the physicist *F* measures the velocities u_x and u_y, while the physicist *F'* in *S'* measures the velocities u_x' and u_y'. These velocities are calculated from the definition of velocity:

$$(8.2)\qquad u_x = \frac{\Delta x}{\Delta t} \qquad u_y = \frac{\Delta y}{\Delta t} \qquad u_x' = \frac{\Delta x'}{\Delta t'} \qquad u_y' = \frac{\Delta y'}{\Delta t'}$$



To find the relation between u_x' and u_x we divide the first of the formulas (8.1) with the second. The square roots cancel, and we get:

$$u_x = \frac{\Delta x}{\Delta t} = \frac{\Delta x' + v\Delta t}{\Delta t' + v\Delta x'/c^2} = \frac{\dfrac{\Delta x'}{\Delta t'} + v}{1 + v/c^2\,\dfrac{\Delta x'}{\Delta t'}}$$

From which it follows by using (8.2)

$$(8.4)\qquad u_x = \frac{u_x' + v}{1 + vu'/c^2}$$

(8.4) is Einstein's famous formula for addition of velocities in the direction of motion.

8.5 Example
Suppose we have a radioactive particle which moves with the speed ½*c* in the laboratory frame. The particle decays and emits another particle also having the velocity ½*c* relative to the mother particle and in the same direction of motion.
According to the Galileo transformation (and common sense notions) the emitted particle should have a velocity ½*c*+½c = c relative to the laboratory, but let us now calculate the velocity according to (8.4).
The mother particle corresponds to the frame *S'*, which moves with the velocity ½*c* relative to the laboratory, which is the frame *S*. The velocity of the emitted particle is therefore u_x' in *S'*.

$$u_x = \frac{\tfrac{1}{2}c + \tfrac{1}{2}c}{1 + \frac{\tfrac{1}{2}c\tfrac{1}{2}c}{c^2}} = \frac{c}{1+\frac{1}{4}} = \tfrac{4}{5}c$$

For the relativistic addition of velocities, ½ + ½ does not give 1 but 4/5.

Example
Let us then assume that a radioactive particle moves with velocity v, and that it emits a gamma quant in its direction of motion. A gamma quant is electromagnetic radiation, and it moves with velocity c. The velocity of the gamma in the laboratory can be determined by the formula (8.4) with $u_x' = c$.

$$u_x = \frac{c+v}{1+\frac{vc}{c^2}} = c\frac{c+v}{c+v} = c$$

It is therefore a consequence from the relativistic addition of velocities that "light" will have the same speed c in all inertial frames, and in all directions, in accordance with the basic assumption (5.2).

We shall now turn to the formulas for transformation of the velocities in the direction of the y-axis. We therefore calculate $u_y = \Delta y/\Delta t$, by applying (8.1) with the supposition $\Delta y' = \Delta y$.

$$u_y = \frac{\Delta y}{\Delta t} = \frac{\Delta y'}{\frac{\Delta t' + v/c^2 \Delta x'}{\sqrt{1-v^2/c^2}}} = \frac{\frac{\Delta y'}{\Delta t'}\sqrt{1-v^2/c^2}}{1+\frac{\Delta x'}{\Delta t'}v/c^2}$$

(8.6) $$u_y = \frac{u_y'\sqrt{1-v^2/c^2}}{1+vu_x'/c^2}$$

It is noticed that u_y does not only depend on u_y' and v but also on u_x'. Often the formula is applied with $u_x' = 0$ and it then takes the form.

(8.7) $$u_y = u_y'\sqrt{1-v^2/c^2}$$ (Valid only when $u_x' = 0$)

The formulas (8.4), (8.6) and (8.7) are the relativistic formulas for transformation of velocities between two inertial frames, which move with velocity v relative to each other.
These formulas will be applied to obtain one of the main results of relativistic mechanics.

9. Relativistic Mechanics

It is obvious that classical mechanics based on Newton's had to be revised following the acknowledgement of The Special Theory of Relativity.
We just need to consider a particle in a constant field of force. The particle will have a constant acceleration, and according to the formula $v = at$ the speed will inevitable exceed the speed of light. According to The Special Theory of Relativity, this is impossible.

From more than 300 years of experience we know however, that Newton's laws are extremely accurate, it is not a theory that can be improved, as long as the speed of the bodies are much lower than the speed of light.

As it is always the case in theoretical physics, when you develop a new and more comprehensive theory, it must have the existing theory as a limit. There may not be a contradiction to an already accepted theory.
For the extension of the laws of mechanics to comply with Special Theory of Relativity, it is therefore required that we recover Newton's laws, when velocities are much less than the speed of light.

To conduct the necessary revision of Newton's laws, we shall build on the following three assumptions.

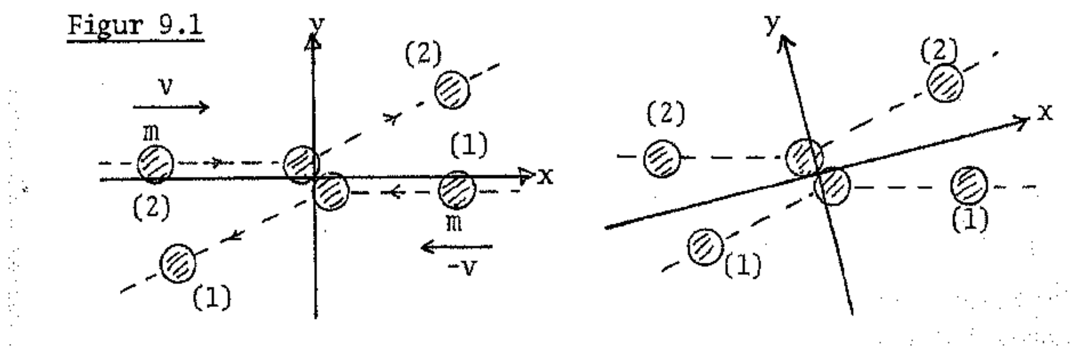
1. The validity of the Lorentz transformation and its consequences.

2. The mass of a body depends on its velocity. Consequently we shall write *m(u)* or m_u
   For the mass of a body, which move with speed *u*.

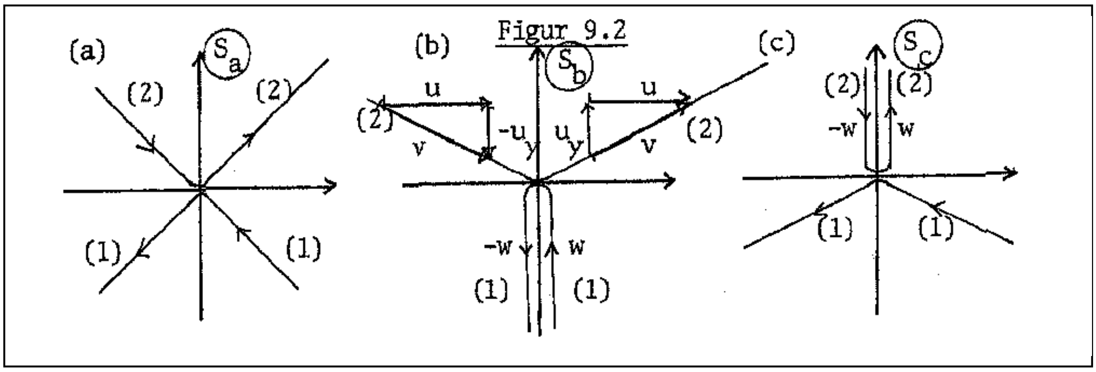
3. The conservation of momentum must $\vec{p} = m_v \vec{v}$ still have unlimited validity, but now with a velocity dependent mass.

Our first task is then to try to determine how the mass m_u depends on the speed *u*. Following Einstein, we shall achieve this by analyzing a simple example of impact of two bodies, and in a cleaver way apply the Lorentz transformation and conservation of momentum.

Figur 9.1



We consider an *elastic* impact between two identical masses, which move towards each other with opposite directed velocities. The situation before and after the impact is shown above.
The overall momentum is zero before the impact, and according to (3), the overall momentum must also be zero after the impact.
After the impact the two masses have equal but opposite directed momentums, both with respect to the *x* direction and the *y* direction. As the two masses are identical the must also have equal but opposite directed velocities.



First we rotate the coordinate system, so that the impact is symmetric with respect to the coordinate axis, as shown at the second figure (9.1). And above figure (9.2a) the impact is shown again schematically. Because of the symmetry of the impact, the velocity of each body is unchanged in the x direction, whereas the velocities in the y direction merely change sign.

What holds for the velocities must also hold for the momentum of the bodies.
Besides the inertial frame S_a, we shall also look at the impact in two other inertial frames S_b and S_c. The inertial frame S_b follows the body (1), so that this body in S_b has the velocity zero in the x direction. This is shown in figure (9.2b).

On the other hand the inertial frame S_c follows the body (2), so that this body in S_c has zero velocity in the x direction. This is shown in figure (9.2c). One should notice, however that because of the symmetry of the figures (b) and (c) are identical but a mirror image of each other.

First we concentrate on S_b. The velocities of the bodies in this inertial frame are shown on the figure. We now express the conservation of momentum in the y direction. (Conservation of momentum must be valid in every inertial frame due to the Principle of Relativity).
By the impact the two bodies get changes of momentum: Δp_{1y} and Δp_{2y}.

(9.3) $$\Delta p_{1y} = m_w(-w) - m_w w = -2m_w w \qquad \Delta p_{2y} = m_v u_y - m_v(-u_y) = 2m_v u_y$$

v denotes the speed of body (2) in S_b, and $\vec{v} = \vec{u} + \vec{u}_y$
To explore the conservation of momentum, we are complied to find a relation between w and u_y. But here we can (and that is the cleaver trick) apply the inertial frame S_c. As $u_y' = 0$ in S_c, this inertial frame must move with the velocity u with respect to S_b. According to the formula (8.7) we can calculate u_y in S_b. The velocity $u_y' = w$ in S_c, and we therefore get:

(9.4) $$u_y = w\sqrt{1-u^2/c^2} \quad \Rightarrow \quad \Delta p_{2y} = 2m_v w\sqrt{1-u^2/c^2}$$

The expression (9.4) is then used to establish the conservation of momentum in the y direction.

(9.5) $$\Delta p_{1y} + \Delta p_{2y} = 0 \quad \Rightarrow \quad -2m_w w + 2m_v w\sqrt{1-u^2/c^2} = 0$$

From which follows:

(9.6) $$m_v = \frac{m_w}{\sqrt{1-u^2/c^2}}$$

(9.6) has been derived for an arbitrary velocity $w \neq 0$. The relation must the therefore also be valid when w approaches 0. From the relation $\vec{v} = \vec{u} + \vec{u}_y$ follows:

$$v^2 = u^2 + u_y{}^2 = u^2 + w^2(1-u^2/c^2)$$

From which it is seen that v approaches u when w approaches 0. If we subsequently take the limit of (9.6) when w approaches 0, we finally arrive at the wanted dependence on mass on velocity.

(9.7) $$m_u = \frac{m_0}{\sqrt{1-u^2/c^2}}$$

m_0 denotes the so called rest mass, meaning the mass of a body, when at rest in an inertial frame. According to The Principle of Relativity m_0 must be the same in all inertial frames.

(9.7) constitutes the fundamental relativistic equation which breaks with the hitherto accepted principle (first established by Lavoiser) of the invariance of mass.

Using (9.7), we are then able to conduct the announced revision of Newtonian mechanics.

Newton's second law: First we notice that the momentum of a body continually should be evaluated as $\vec{p} = m_v\vec{v}$, but where m_v is given by the expression (9.7).

(9.8) $$\vec{p} = \frac{m_0\vec{v}}{\sqrt{1-v^2/c^2}}$$

Newton's second law is then written as:

Force equals the rate of change of the momentum. $\vec{F} = \frac{d\vec{p}}{dt}$

Instead of Force equals mass times acceleration: $\vec{F} = m\vec{a}$

This is due to the fact that the mass now depends on velocity. The relativistic formulation of Newton's second law then becomes.

(9.9) $$\vec{F} = \frac{d}{dt}\frac{m_0\vec{v}}{\sqrt{1-v^2/c^2}}$$

It is evident that (9.9) becomes Newton's non relativistic equation, when v is much lower than c. The acceleration is still evaluated as $\vec{a} = d\vec{v}/dt$. If you want to include the acceleration in (9.9) one must carry out the differentiation. However you must remember that the velocity is in general a vector with three components, and that: $v^2 = v_x{}^2 + v_y{}^2 + v_z{}^2$.

After a lengthy, but straightforward composite differentiation one finds:

(9.10) $$\vec{F} = \frac{m_0}{\sqrt{1-v^2/c^2}}\frac{d\vec{v}}{dt} + \frac{m_0\vec{v}\cdot\dfrac{d\vec{v}}{dt}}{c^2(\sqrt{1-v^2/c^2})^3}\vec{v}$$

From (9.10) we conclude that generally the force does have the same direction as the acceleration, since the second term is along the velocity. The exception is when acceleration and velocity are orthogonal, so that the last tem vanishes.

Only in the case where $\vec{v} \perp d\vec{v}/dt$, we have the more familiar expression: $\vec{F} = m_v\vec{a}$.
A plane movement, where $\vec{v} \perp d\vec{v}/dt$ at all moment is a uniform circular motion, and for such a motion i.e. the moon around the earth, planets motion around the sun, one can still apply $\vec{F} = m_v\vec{a}$, but where the rest mass should be replaced by (9.7).
It is a comforting fact all the familiar formulas we have derived for the uniform circular motion still apply, as long as we replace the rest mass with the relativistic mass.
A rather importing point, if you want to consider relativistic corrections to known formulas.

10. Linear motion in a constant field of force.

For example we may think of a charged particle with charge q and rest mass m_0, which is accelerated in a constant electric field E. Also in the theory of relativity the charge is a conserved entity and independent of the velocity.
The particle will therefore be influenced by a constant force: $F = qE$.
If the particle initially has the velocity 0, it will result in a linear motion, and Newton's second law becomes according to (9.9).

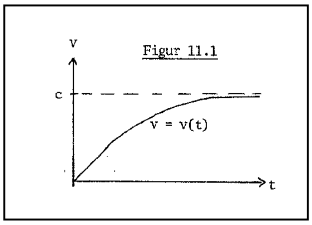
(10.1) $$F = qE = \frac{d}{dt}\frac{m_0 v}{\sqrt{1-v^2/c^2}} \quad \Leftrightarrow \quad \frac{qE}{m_0} = \frac{d}{dt}\frac{v}{\sqrt{1-v^2/c^2}}$$

We put $a = \dfrac{qE}{m_0}$, where a is the acceleration in the non relativistic description.
Since $v(0) = 0$, (10.1) is easily integrated to give:

(10.2) $$\frac{qE}{m_0}t = \frac{v}{\sqrt{1-v^2/c^2}} \quad \Leftrightarrow \quad \frac{v}{\sqrt{1-v^2/c^2}} = at$$

By taking the square of (10.2) and solving with respect to v, one finds.

(10.3) $$v = \frac{at}{\sqrt{1+(at/c)^2}} \quad \Leftrightarrow \quad v = \frac{at}{\sqrt{c^2+(at)^2}}c$$



The formula (10.3) demonstrates how the velocity of a massive particle increases with time in the case of a constant force. The v -. t graph is sketched on the figure to the left. One should notice that the acceleration is not constant, (since the $v - t$ graph is not linear) although the force is. The graph approaches asymptotically to c as t goes to infinity. To analyze the formula (10.3), we shall consider two limits.

1) $$\frac{at}{c} << 1 \quad \Leftrightarrow \quad t << \frac{c}{a}.$$

In this case one can neglect $(\frac{at}{c})^2$ compared to 1 in the square root in the denominator of (10.1).
The result is simply: $v = at$, as is the case of a non relativistic description.
This is in accordance with the claim, that a more comprehensive theory, must give the same results as "the old theory", in some limit.

2) $v << c$ $\qquad \frac{at}{c} >> 1 \quad \Leftrightarrow \quad (at)^2 >> c^2$

In this case we can neglect c^2 compared to $(at)^2$ in the second expression (10.3), and the result is $v = c$. The particle moves with constant velocity equal to the speed of light. We notice that since the denominator in the second expression of (10.3) is always greater than the nominator the speed of the particle will never exceed the speed of light.
This may also be understood in another way, since the mass increases dramatically when the speed approaches c, so it will get only a minor acceleration from the constant force.

The formula (10.3) can also be integrated to yield the distance $s(t)$ the particle has moved.
We use $v = ds/dt$, and assume that $s(0) = 0$.

(10.5)
$$s = \int_0^t \frac{acdt}{\sqrt{c^2 + (at)^2}} = \frac{c}{a}\int_0^t \frac{d(at)^2}{2\sqrt{c^2 + (at)^2}} = \frac{c}{a}\Big[\sqrt{c^2 + (at)^2}\Big]_0^t$$
$$s = \frac{c}{a}\sqrt{c^2 + (at)^2} - \frac{c^2}{a} = \frac{c^2}{a}\sqrt{1 + (\frac{at}{c})^2} - 1$$

As previously, we shall investigate the non relativistic limit, which occurs when $at << c$.
In this limit the second term in the square root is much less than 1 and we may expand the square root, according to $\sqrt{1+x} \approx 1 + \frac{1}{2}x$.

(10.6)
$$s = \frac{c^2}{a}(1 + \tfrac{1}{2}(\frac{a}{c})^2 t^2 - 1) = \tfrac{1}{2}at^2$$

Again we recover the well known Galileo formula for the distance done by a particle in a movement with constant acceleration.

10.7 Example

To explore under which circumstances one can reach relativistic velocities, we shall consider an electron moving in a constant electric field $E = 10^5$ *V/m*. (This can for example be realized with a capacitor with 1 *cm* between the plates and induced with a voltage of 1000 *V*. From the two cases on the foregoing page we see that the relativistic case occurs, when $at = c$. Inserting this in the last formula (10.3), it is easy to see that $v = \frac{\sqrt{2}}{2}c = 0{,}71c$.
To investigate how much time elapses before the electron has reached this speed, we insert $a = qE/m_0$ in the equation $c = at$ and solve for t.

$$c = \frac{m_0 c}{qE} = \frac{9.11 \cdot 10^{-31} kg\ 3.0 \cdot 10^8\ m/s}{1.6 \cdot 10^{-19}\ C\ 10^5\ V/m} = 1.7 \cdot 10^{-8} s$$

Further we can apply the formula (10.5) for the distance s covered by the electron in the time interval above. Inserting $at = c$ in (10.5), one finds:

$$s = \tfrac{c^2}{a}(\sqrt{2} - 1) = ct(\sqrt{2} - 1) = 3.0\,10^8\ 1.7\,10^{-8}\ 0.41\, m = 2.1\, m$$

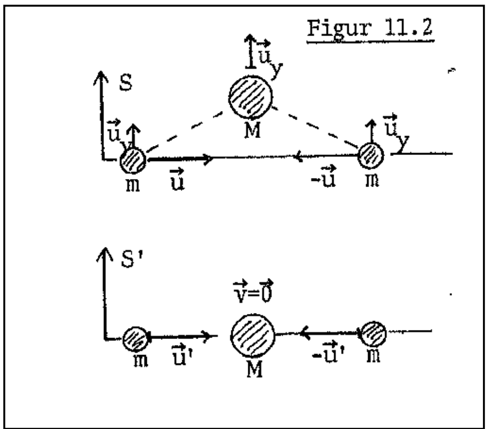
11. The equivalence between mass and energy

We shall then look into one of the most excessive conceptual consequence of special relativity, which has actually turned into an accepted logo for Einstein's Theory of Relativity.
It is of course the equation:

$$E = mc^2$$

It is a logic (but not trivial) consequence from the theory that the energy of a body can always be expressed as its relativistic mass times the square of the speed of light.

(11.1) $$E = m_v c^2 \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}$$



Figur 11.2

Following Einstein's original paper, we shall now make a simple but ingenious argument for this assertion.
We therefore consider an impact between to identical bodies (same rest mass), and with equal but opposite directed velocities in the x-direction.
To take advantage of the conservation of momentum, we shall first assume that the two bodies also have the same velocity component u_y in the y - direction.
We shall also assume that the collision is completely inelastic, which means that the two bodies, each with the velocity u_v stay together as a composite body with mass M after the collision, and thereafter move with a common velocity.It is obvious, that M must also have the velocity u_y in the y- direction after the collision.

Namely, if we look at the collision in an inertial frame, which moves with the velocity u_y in the y – direction, relative to the first frame, then both incoming bodies have the velocity $u_y' = 0$ in the new frame. As the two incoming bodies after the collision constitutes one body with mass M, then for the reason of conservation of momentum, this body must also have $u_y' = 0$. Consequently the composite the mass M must have the velocity u_y in the original frame.

We begin by applying the law of conservation of momentum in the y-direction, where we must remember to use the relativistic masses m_v, where $v^2 = u^2 + u_y^2$.

(11.3) $$m_v u_y + m_v u_y = M u_y \qquad \Rightarrow \qquad M = \frac{2m_0}{\sqrt{1 - v^2/c^2}}$$

The relation (11.3) holds good for all $u_y \neq 0$. We may therefore let u_y tend to zero. It follows then that v approaches u, and since M is at rest, M will tend to its rest mass M_0. The relation (11.3) then reads.

(11.4) $$M_0 = \frac{2m_0}{\sqrt{1 - v^2/c^2}}$$

The very important point is that from (11.4), one sees that the composite body has a rest mass, which is the sum of the two incoming bodies' relativistic masses.

This is entirely not trivial, since M is at rest. From this we can namely conclude that, there must have been an increase of rest mass for the composite system.
At the same time it is obvious that the composite system has lost kinetic energy.
In we insist on upholding a notion of conservation of energy (and we do so) it is natural to interpret the increase of mass as a compensation for the loss of kinetic energy.
But this must necessarily lead to a conception that mass itself is a form of energy

We are then ready to give a more formal proof to substantiate the fundamental conceptual equivalence between mass and energy.

We remind you of the definition of kinetic energy of a body as the work of the resulting force (that is the force that goes into the second law of Newton) on a body, is equal to the (signed) change of kinetic energy.

$$W_{res} = F_{res} s = \tfrac{1}{2} m v^2$$

Valid for a constant force, and m initially at rest, or generally:

$$\int_{s_0}^{s} \vec{F}_{res} d\vec{s} = \tfrac{1}{2} m v^2 - \tfrac{1}{2} m {v_0}^2$$

If we hold on to this theorem (and we do), we can evaluate the work done on a body with mass m, but using the formulas from the relativistic mechanics.

$$E_{kin} = \int_0^s \vec{F}_{res} \cdot d\vec{s} = \int_0^s \frac{d}{dt}\left(\frac{m_0 \vec{u}}{\sqrt{1 - u^2/c^2}}\right) \cdot d\vec{s}$$

$$E_{kin} = \int_0^s d\left(\frac{m_0 \vec{u}}{\sqrt{1 - u^2/c^2}}\right) \cdot \frac{d\vec{s}}{dt} = \int_0^v \vec{u} \cdot d\left(\frac{m_0 \vec{u}}{\sqrt{1 - u^2/c^2}}\right)$$

The last integral is written, so we can apply the rule of partial integration: $\int f dg = fg - \int g df$, where the scalar product can be treated as an ordinary product.

$$E_{kin} = \left[\frac{m_0 u^2}{\sqrt{1 - u^2/c^2}}\right]_0^v - \int_0^v \frac{m_0 \vec{u} \cdot d\vec{u}}{\sqrt{1 - u^2/c^2}}$$

$$E_{kin} = \frac{m_0 v^2}{\sqrt{1 - v^2/c^2}} - \left[- m_0 c^2 \sqrt{1 - u^2/c^2}\right]_0^v$$

$$E_{kin} = \frac{m_0 v^2}{\sqrt{1 - v^2/c^2}} + m_0 c^2 \sqrt{1 - v^2/c^2} - m_0 c^2$$

$$E_{kin} = \frac{m_0 v^2 + m_0 c^2 (1 - v^2 / c^2)}{\sqrt{1 - v^2 / c^2}} - m_0 c^2$$

(11.5) $$E_{kin} = \frac{m_0 c^2}{\sqrt{1 - v^2 / c^2}} - m_0 c^2 = m_v c^2 - m_0 c^2$$

The last expression above can be regarded as a definition of kinetic energy in relativistic mechanics. In the previous example we saw that kinetic energy apparently can be converted to an increase in mass, such that rest mass just another form of energy.

This line of thought encouraged Einstein to suggest that the total energy of a system always can be expressed as the total relativistic mass times the square of the speed of light.
This is Einstein's famous equivalence between mass and energy.

(11.7) $$\boxed{E = m_v c^2 \quad or \quad E = \frac{m_0 c^2}{\sqrt{1 - v^2 / c^2}}}$$

We must emphasize the generality of Einstein's equivalence between mass and energy.

For a system of particles which are bound together by forces, the mass will be less than the sum of the masses that constitute the system, by the overall potential and kinetic energies divided by c^2.
The difference in mass ΔM (called the mass defect) times c^2 is equal to the sum of the particles kinetic energy and their mutual potential energy. It is called the binding energy.
For a system of particles the relations below will hold in all cases.

(11.8) $$E_{system} = M_{system} c^2 \qquad \text{and} \qquad E_{binding} = -\Delta M c^2 = E_{kin} + E_{pot}$$

When the speed v is zero, it follows from (11.7) that $E = m_0 c^2$, this energy is called the rest energy. The equation (11.6) shows that the total energy equals the rest energy plus the kinetic energy.

$$E_{total} = E_{rest} + E_{kin} \qquad E_{total} = m_0 c^2 + E_{kin}$$

Apparently the expression (11.6) has only little similarity with the mathematical expression for kinetic energy in non relativistic mechanics. $E_{kin} = \frac{1}{2} m v^2$, but we shall now demonstrate that indeed (11.6) has this non relativistic limit.
Form the formula: $f(x_0 + h) \approx f(x_0) + f'(x_0) h$ when h is small it follows: $\frac{1}{\sqrt{1+h}} \approx 1 - \frac{1}{2} h$.
And when the formula is applied to $f(x) = 1/\sqrt{x}$ with $h = -v^2/c^2$ in the expression for the relativistic kinetic energy (11.6), the condition $h << 1$ becomes $v << c$, and we thus find.

$$E_{kin} = \frac{m_0 c^2}{\sqrt{1 - v^2 / c^2}} - m_0 c^2 \approx m_0 c^2 (1 + \tfrac{1}{2} v^2 / c^2) - m_0 c^2$$

(11.9) $$E_{kin} = \tfrac{1}{2} m_0 v^2 \qquad \text{for } v << c.$$

As is the case in the other examples we have presented, we find that the relativistic mechanics is identical to the Newtonian mechanics in the limit, where the velocities are much less than the speed of light.
The Newtonian mechanics is not wrong, but it is merely a limiting case of the relativistic mechanics, when the velocities are much smaller than the speed of light, which is always the case for massive bodies on the earth.

Finally we will state an important formula, which constitute an invariance in relativistic mechanic.

(11.10) $$E^2 - \vec{p}^2 c^2 = m_0{}^2 c^4$$

This is easily proved by inserting the expression (11.7) for E and (9.8) for p and evaluate the left side.

(11.11) $$E^2 - \vec{p}^2 c^2 = \left(\frac{m_0 c^2}{\sqrt{1-v^2/c^2}}\right)^2 - \left(\frac{m_0 \vec{v}}{\sqrt{1-v^2/c^2}}\right)^2 c^2 = \frac{m_0{}^2 c^4 (1-v^2/c^2)}{1-v^2/c^2} = m_0{}^2 c^4$$

Thus (11.11) is valid independently of the inertial frame in question, and is therefore an invariant under Lorentz transformations in relativistic mechanics.

There exits another combination of coordinates and time: that is, invariant under Lorentz transformations, as we shall show that $x^2 - c^2 t^2 = x'^2 - c^2 t'^2$, when coordinate and time are evaluated in the inertial frames S and S' respectively. To do so we apply the formulas (6.9) and (6.11).

$$x'^2 - c^2 t'^2 = \left(\frac{x - vt}{\sqrt{1-v^2/c^2}}\right)^2 - c^2\left(\frac{t - xv/c^2}{\sqrt{1-v^2/c^2}}\right)^2 = \frac{x^2 + v^2 t^2 - 2xvt - c^2(t^2 + x^2 v^2/c^4 - 2xvt/c^2)}{1-v^2/c^2} =$$

(11.12) $$x'^2 - c^2 t'^2 = \frac{x^2(1-v^2/c^2) + c^2 t^2 (1-v^2/c^2)}{1-v^2/c^2} = x^2 - c^2 t^2$$

In the four dimensional Murkowski space, the infinitesimal distance element is defined by

(11.13) $$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$$

And one can see from (11.29) that "distances" in the Minkowski space are independent of the inertial frame - as they should be. Sometimes (11.13) is used with a reversed sign as.

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

It certainly looks strange in Euclidian geometry, but if you divide with dt^2 it makes sense, since

$$(\frac{ds}{dt})^2 = c^2 - (\frac{dx}{dt})^2 - (\frac{dy}{dt})^2 - (\frac{dz}{dt})^2 \qquad \text{or}$$

$$(\frac{ds}{dt})^2 = c^2 - v_x{}^2 - v_y{}^2 - v_z{}^2 \quad \Leftrightarrow \quad (\frac{ds}{dt})^2 = c^2 - v^2$$

For a physical event (a particle with velocity v at time t) the last formulation ensures that the square of the distance element is always positive, since $c^2 - v^2 > 0$.

12. Relativistic Doppler shift

The non relativistic Doppler shift for sound waves is well known.
For a moving source with velocity v, the shifted frequency is given by:

$$\upsilon' = \frac{\upsilon}{1 - u/v}$$, where v is the speed of sound.

We shall now proceed to derive a similar relativistic formula.
The relativistic expression for the Doppler shift of light has had a tremendous significance in astrophysics, determining the velocity of distant celestial objects.
In this connection the Doppler shift is normally referenced to as the red shift, since the most frequent observed spectral line is the red line in the hydrogen spectrum.

Suppose that in the inertial frame S we observe a plane harmonic wave propagating along the x-axis.

(12.1) $$u(x,t) = A\cos(\omega t - kx)$$

In the inertial frame S' which moves with velocity v relative to S, is observed the same plane harmonic wave, but with another frequency ω' and wave number k'.
We consider then the phase $\omega t - kx$, corresponding to the event (x, t) in S.
The corresponding event in S' that has the same phase, is (x', t') and this phase is expressed by means of the Lorentz transformations (6.9) and (6.11).

(12.2) $$\omega t - kx = \omega \frac{t' + \frac{x'v}{c^2}}{\sqrt{1 - v^2/c^2}} - k\frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$$

We shall then try to write this as $\omega' t' - k' x'$, which is the phase in S'. By rearranging the terms in (12.2) , collecting the coefficients of x' and t' one finds:

(12.3) $$\omega t - kx = \frac{\omega - kv}{\sqrt{1 - v^2/c^2}} t' - \frac{k - \omega v/c^2}{\sqrt{1 - v^2/c^2}} x' = \omega' t' - k' x'$$

From which we conclude that:

(12.4) $$\omega' = \frac{\omega - kv}{\sqrt{1 - v^2/c^2}} \quad \textit{and} \quad k' = \frac{k - \omega v/c^2}{\sqrt{1 - v^2/c^2}}$$

One should notice that the transformation formulas for ω and k are almost the same as the Lorentz transformations for x and t. (The only difference is the position of c^2).
(12.4) are then the relativistic expression for the Doppler shift.
The formulas (12.4) have significant appliances for electromagnetic waves, which propagates with velocity c. Setting $k = \omega/c$ in first equation (12.4) we get:

$$\omega' = \frac{\omega - \omega v/c}{\sqrt{1 - v^2/c^2}} = \omega \frac{1 - v/c}{\sqrt{1 - v^2/c^2}} \quad \Rightarrow \quad \omega' = \omega \sqrt{\frac{1 - v/c}{1 + v/c}} \tag{12.5}$$

One notice that the shift in frequency depends on the sign of v, so one can determine whether a distant object is moving away from the observer or towards the observer.

13. Evidence for the reliance of the theory of relativity

The theory of relativity was earlier considered as an abstract even contradictory theory, created in the human mind, without any connection to the real world. This is of course because it breaks with many common everyday experiences. It is an ironic fact that Einstein did not receive the Nobel price neither for the special nor for the general theory of relativity, but instead for his explanation of the photoelectric effect. The reason was presumably that only a few seriously believed that the theory would have any place in the real world.
Another more tangible reason was that at the time of publication no one could really appreciate, how it could possibly be confirmed experimentally.
As always in the history of theoretical physics it is not a matter of philosophical conviction, but experimental evidence that decides the prevailling of a theory.
And a hundred years after its appearance the special theory of relativity is one of the most experimentally soundly supported theories.
Here we shall only consider some examples, which have been known for more than 50 years.

1. In collision between atomic particles with high kinetic energy are often created new particles. The increase in mass of the resulting particles (times c^2) correspond exactly to the loss of kinetic energy.
   Likewise in the decomposition of atomic nuclei, it has been possible to establish Einstein's equivalence of mass and energy.

2. A myon is a kind of heavy electron, when produced (at rest) in a laboratory has a mean life time τ of about 2.2 10^{-6} s. From a non relativistic point of view it can travel no longer than $L = c\,\tau = 660\ m$. In spite of this we receive myons on the earth from the universe that must have travelled a much longer distance. The explanation is the time dilation, given by the formula (7.4) as a consequence of the Lorentz transformation. The lifetime for a particle in an inertial frame, which follows the particle is extended by the factor $1/\sqrt{1 - v^2/c^2}$, which in principle can be as close to infinity as you wish.

3. When producing particles with a life time about 10^{-10} s in large accelerators and studying them in a bubble chamber, on can measure their traces, and directly compare their lifetime with the speed of which they are produced, and hereby confirm the time dilation.

4. From the middle of 1960'ties there have been built accelerators which are able to accelerate particles to velocities near the speed of light. At the 28 GeV proton synchrotron in CERN Switzerland, protons are accelerated so their masses become roughly 28 times their rest masses. To direct these protons in magnetic field, one must of course use the relativistic mass, not the rest mass. In this manner one has verified the dependence of mass with velocity, down to 0.01% .

5. In nuclear power plants mass is converted to energy by fission of U-235, according to Einstein's equivalence of mass and energy. Why it is not possible unrestricted to convert mass to energy, is not a matter of relativity, but of elementary particle physics.

The invention of atomic clocks has made it possible to observe the time dilation of clocks when placed in a satellite in orbit around the earth.