The physics of induction stoves

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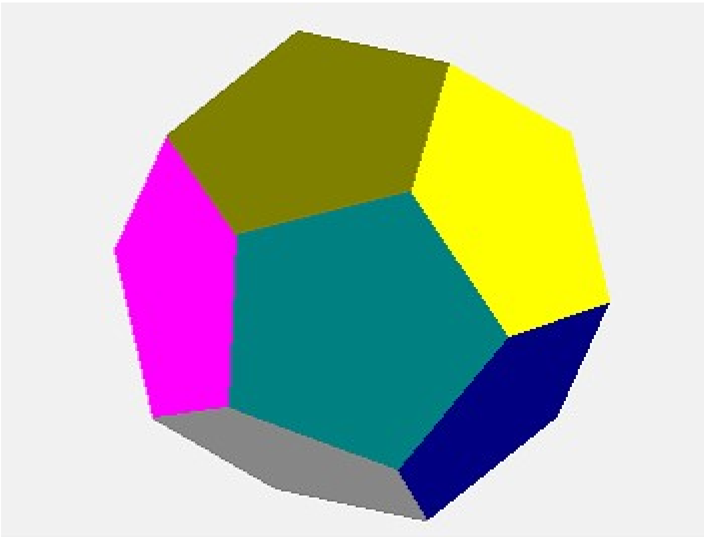

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1. What is an induction stove

When I was teaching physics and mathematics in the Danish 9 – 12 grade high school in 2009, a teacher (with humanistic profession) who had acquired a induction stove, approached me and asked, why it did not work with kitchen pots and pans which were not magnetic.
At that time, I actually knew nothing specific about induction hobs besides, more generally, that they were based on Maxwell's second equation, when written in integral form, is better known as Faraday's law.

$$\vec{\nabla}\, x\, \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

A time dependent magnetic flux (magnetic field times the area) through a closed curve, will induce an *emf* (electromagnetic force) around the curve.

Integrating both sides, using Stokes law, we have:

$$emf : \varepsilon_{ind} = \oint \vec{E}\cdot d\vec{s} = \int_{area} \vec{\nabla}\, x\, \vec{E}\cdot d\vec{A} = -\frac{\partial}{\partial t}\int_{area} \vec{B}\cdot d\vec{A} = -\frac{\partial \Phi_B}{\partial t}$$

$$\varepsilon_{ind} = -\frac{\partial \Phi_B}{\partial t} = -\frac{\partial}{\partial t}(Magnetic\ flux)$$

So my first answer was, that the efficiency of an induction hub, does not depend whether the material is magnetic, as long as it is conducting, so that the induced *emf* from an alternating magnetic flux can drive a current and heats the pan.
From this point of view a material with high conductivity i.e. cobber, should be most efficient.
A crude calculation showed that the heat generated in fact should be proportional to the specific conductivity of the material.

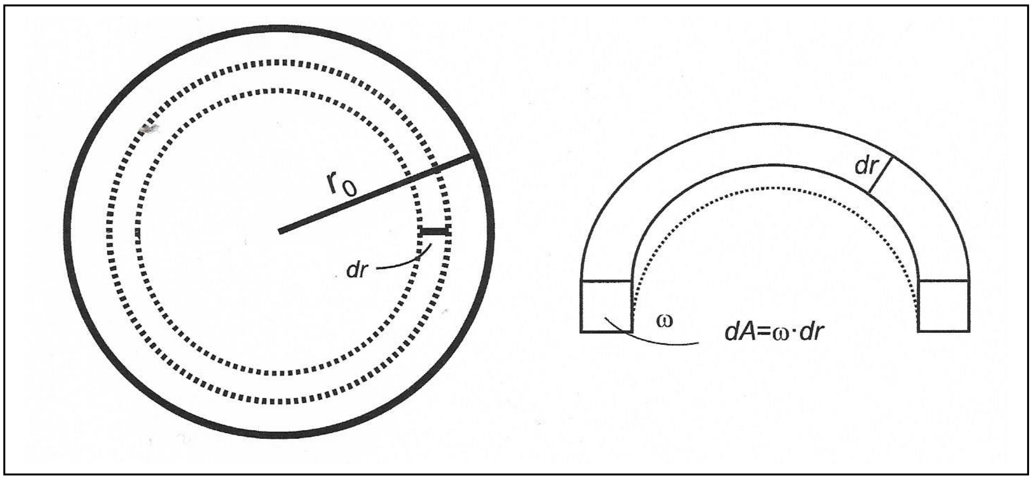
This seems from a physical point of view to be entirely plausible, but when I shortly afterwards (as a cheep offer) bought an induction hub, and tried it with cobber and aluminium pots and pans it did not work at all. (What I could also have read in the user guide!).

What I realized was, that the Maxwell equations, that I knew of, are actually only valid in vacuum, but also that there are large differences, when for example, there are applied in materials with a magnetic dipole moment as iron or nickel.
The magnetic flux in the metal, comes of course from the *B*-field in the metal, which is proportional to *H,* the field outside the metal times the relative permeability μ_r .

$$B = \mu_r H$$

Then I went ahead with a calculation of the power *P*, which is generated from an alternating magnetic field in the metal.
Since the field lines from the electric field that causes the eddy currents in the metal can be considered circular, we slice the bottom of the metallic disc into ring with thickness *dr*.

The disc itself is assumed to have thickness *w*, and radius r_0. See the figure below.
The resistance in a ring with radius *r* is according to the formula: $R = \rho \frac{l}{A}$ where ρ is the resistivity of the material, *l* is the length and *A* is the cross section area, so we find for *dR*.



$$dR = \rho \frac{2\pi r}{wdr}$$

The flux through the ring is: $\Phi_B = \pi r^2 B = \pi r^2 \mu_r H$. Here μ_r is the relative permeability for the disc, and *H* is the external magnetic field generated by the coil.

In each ring is induced an *emf* (electromagnetic force).

$$\varepsilon_{ind} = -\frac{\partial \Phi_B}{\partial t}$$

Where Φ_B is the magnetic flux through the ring. The power *dP* which is set aside in the ring is:

$$dP = \frac{\varepsilon_{ind}^2}{dR} \quad \text{where} \quad \frac{1}{dR} = \frac{w}{2\pi\rho}\frac{dr}{r}$$

The power can then be found by integration.
The outer magnetic field is assumed to be given by: $H(t) = B_0 e^{i\omega t}$. With

$$\Phi_B = \pi r^2 B = \pi r^2 \mu_r H \text{ , and } \varepsilon_{ind} = -\frac{\partial \Phi_B}{\partial t}$$

Wethen get an expression for the induced *emf* in a ring with radius *r*.

$$\varepsilon_{ind} = -\pi r^2 \mu_r B_0 i\omega e^{i\omega t}$$

As it is the case with an alternation voltage, the effective value is obtained by taking the modulus multiplied by $\frac{\sqrt{2}}{2}$.

$$\varepsilon_{eff}^2 = \tfrac{1}{2}\pi^2 r^4 \mu_r^2 B_0{}^2 \omega^2$$

when inserted in $dP = \frac{\varepsilon^2{}_{ind}}{dR}$, with $\frac{1}{dR} = \frac{w}{2\pi\rho}\frac{dr}{r}$ gives:

$$dP = \frac{\pi}{4}\frac{w\mu_r^2 B_0^2 \omega^2}{\rho} r^3 dr$$

Integrating this expression from 0 to r_0 , (the radius of the disc) we find:

$$P = \frac{\pi}{16}\frac{w\mu_r^2 B_0^2 \omega^2}{\rho} r_0^4$$

You may wonder why we integrate over $1/dR$ instead of integrating over dR, but this is as it should be, since the rings can be considered as resistances in parallel, according to the formula: $1/R = 1/R_1 + 1/R_2$.

In the popular description of induction cooking plates, you can read that they are driven by a high frequency magnetic field, but I have not been able to find a value for the frequency.

To asses whether the calculation above bears any relation to reality, I have put (somewhat arbitrary). The thickness of the plate in the iron pot w = 5mm. The peak value of the external of the inducting magnetic field B_0 = 0.10 Wb/m^2. The cyclic frequency of the field $\omega = 2\pi \cdot 1$kHz, and the radius of the pot r_0 = 10 cm. The relative permeability for iron is $\mu_r = 66$, and the resistivity of iron is $\rho = 8.9\ 10^{-2}\ \Omega m$.

When this is inserted in the expression for the delivered power P, it gives 1887 W, which must be considered as very satisfactory. But the crucial point is of course the magnitude and the frequency of the external magnetic field, which generates the eddy currents, since they enter as the square in the formula for the generated power. So, for example, if the frequency or the generating field is 10 times less, then the generated power is 100 times less.

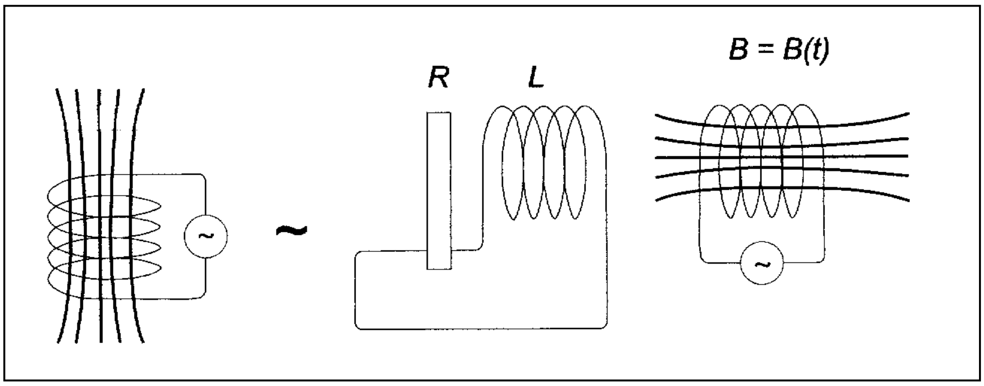
The important point here is that the power generated grows with the square of the B-field, the relative permeability of the metal μ_r , and of the cyclic frequency ω.

The relative permeability for iron $\mu_r = 66$, while for Aluminium it is 2. So the ration of the dissipation of power in an Al plate and a Fe plate is: $(2/66)^2$ $(2/66)^2 \approx 1/1000$.
Since copper is diamagnetic and $\mu_r = -1$, the relative permeability of cobber and Aluminium, serve as an definite explanation why an induction hub only work on cookware made of iron.

This is, however, not the whole story, since we have not yet considered the possible influence of self-inductance, and the possible contribution from the forced oscillations of the magnetic dipole moments of the iron atoms.

2. Including self-inductance

In the calculations above, we chose in the first place to ignore the phenomenon of the self inductance coming from the eddy currents in the iron plate. For completeness, we shall include self-induction in the calculation, although it turns out that it does not change the result significantly.
However, when taking account the self-inductance the calculations progress in a rather different way.



The setup is shown schematically above. The coil can be considered as an ideal coil in series with a resistance. The coil is imposed to a forced oscillation from an external field according to the voltage equation.

$$Ri - L\frac{di}{dt} = \varepsilon_{external}(t) = \frac{\partial \Phi_B}{dt}$$

Where we have put

$$\Phi_B = \Phi_0 e^{i\omega t} \quad \text{and} \quad \Phi_0 = B_0 \pi r^2$$

Then we have:

$$Ri - L\frac{di}{dt} - \frac{\partial \Phi_B}{dt} = 0$$

Trying with a solution: $i = i_0 e^{i\omega t + \varphi}$ gives:

$$Ri_0 e^{i\omega t+\varphi} - i\omega i_0 L e^{i\omega t+\varphi} - \Phi_0 i\omega e^{i\omega t} \quad \Rightarrow$$

$$i_0 = \frac{\Phi_0 i\omega}{R - i\omega L} e^{\varphi} \quad \Rightarrow \quad i_0 = \left|\frac{\Phi_0 i\omega}{R - i\omega L}\right| = \frac{\omega \Phi_0}{\sqrt{R^2 + (\omega L)^2}}$$

The effective value of i_0 is thus: $i_{eff} = \frac{\sqrt{2}}{2} i_0$
To apply this to the induction hub, we must replace *R* with *dR* and the inductance *L*, with the inductance of a single turn. Unfortunately there is no simple formula for the magnetic field from a

single turn, but if we apply the expression for the *B*- field in the centre of a circular conductor with radius *r*, then it is probably not entirely wrong. (Cf. The Helmholtz coils, where magnetic field in between the two coils is almost homogenous). The *B*-field in the centre of a single coil having a current *i* is given by: $B = \mu_0 \frac{i}{2r}$, as can be derived from Biot and Savart's law. The magnetic flux through the circular coil with cross section *A* then becomes: $\Phi_B = BA = \mu_0 \frac{i}{2r} \pi r^2 = \frac{1}{2} \pi \mu r i$.
If we then compare the two expressions for the induced *emf*.

$$\varepsilon_{ind} = -\frac{\partial \Phi_B}{\partial t} = -L\frac{\partial i}{\partial t} \quad \Leftrightarrow \quad \frac{1}{2}\pi\mu r \frac{\partial i}{\partial t} = L\frac{\partial i}{\partial t}$$

Which shows that $L = \frac{1}{2}\pi\mu r$. Replacing *R* with $dR = \rho \frac{2\pi r}{w dr}$ together with $L = \frac{1}{2}\pi\mu r$ in the expression for i_{eff}: $i_{eff} = \frac{\sqrt{2}}{2} \frac{\omega \Phi_0}{\sqrt{dR^2 + (\omega L)^2}}$ we have:

$$i_{eff} = \frac{\sqrt{2}}{2} \frac{\omega B_0 \pi r^2}{\sqrt{\left(\rho \frac{2\pi r}{w dr}\right)^2 + \left(\frac{1}{2}\omega\pi\mu r\right)^2}}$$

Which can be reduced to:

$$i_{eff} = \frac{\sqrt{2}}{2} \frac{\omega B_0 w r dr}{2\rho\sqrt{1 + \frac{1}{16}\left(\frac{\omega w \mu}{\rho}\right)^2 dr^2}}$$

In the denominator square root the last term is seen to be vanishing compared to 1, so if we discard that term we simply get:

$$i_{eff} = \frac{\sqrt{2}}{2} \frac{\omega B_0 w}{2\rho} r dr$$

And for the power we find:

$$dP = i_{eff}^2 dR = \rho \frac{2\pi r}{w dr} \cdot \frac{1}{2} \frac{\omega^2 B_0^2 w^2}{4\rho^2} r^2 dr^2 \quad \text{which gives} \quad dP = \frac{\pi}{4} \cdot \frac{\omega^2 B_0^2 w}{\rho} r^3 dr$$

If the metal plate is a metal with the relative permeability μ_r , then B_0 should be multiplied with that factor. We finally end up with exactly the same expression as we got before, though it was derived in a quite different manner. This is of course satisfactory. At the same time we have shown that the self inductance plays a very little role in the calculation of the released power in the induction hub.

$$dP = \frac{\pi}{4} \cdot \frac{\omega^2 \mu_r^2 B_0^2 w}{\rho} r^3 dr$$

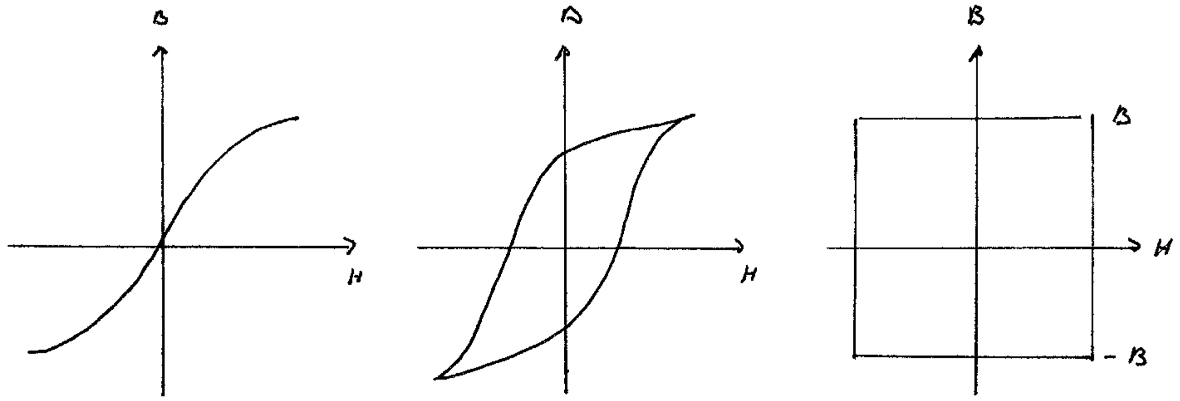
3. The contribution from the magnetic moments

Whereas self-induction gives a minimal contribution to the functionality of the induction hub, we can be almost certain that there comes a contribution to the heating from the rotational energy of the magnetic moments of the iron atoms. The formulas for the potential energy of and the torque on the magnetic moments μ from a magnetic field are:

$$E_{pot} = \vec{\mu} \cdot \vec{B} \qquad \vec{H} = \vec{\mu} \times \vec{B}$$

As we have demonstrated above one can fairly easy calculate the contribution to the heating of the plate from the eddy currents. It is, however, far more complicated to estimate the contribution from the magnetic moments in an oscillating field.

Firstly it is necessary to know the hysteresis curve for the iron as well as the magnetic dipole moments for the atoms, and even when both are well known, a calculation may be unrealistic. Below are sketched three hysteresis curves, where only the first two have anything to do with reality. But for the last two seem to be mathematically equivalent when calculating the loss in potential energy.



The potential energy of a magnetic moment is $E_{pot} = \vec{\mu} \cdot \vec{B}$, and we shall evaluate the loss in potential energy, when a magnetic moment is put through the curve, starting and ending in the same point.
For the first curve the loss will be zero, and for the last curve, the contribution from the horizontal lines also be zero, since the *B*-field is constant. On the vertical lines, however, there will be contributions -2μB and 2μB, so that $\Delta E_{pot} = -4 \mu B$.

In the Feynmann Lectures II (from 1964) there is a rather comprehensive and detailed description of Ferro magnetism. Here you can learn that ferromagnetism is entirely due from the magnetic moment of one electron, having the magnitude of one Bohr magneton.

$$\mu = \frac{e\hbar}{2m_e} = 9.31 \cdot 10^{-24} Am^2$$

Evaluating the number of iron atoms in an iron plate in the bottom of the previously mentioned cooking pot, we find:

$$N = n_M N_A = \frac{m}{M_{Fe}} N_A = \frac{\rho V_{Fe}}{M_{Fe}} N_A = 1{,}33 \cdot 10^{25}$$

We then find that $\Delta E_{pot} = -4\mu NB = -49.7J$. Multiplying with a frequency of 1 kHz, we find in this (presumably unrealistic calculation) a power of 49.7 kW. So something is definitely wrong, at least with the magnitudes of the magnetic field or the oscillating frequency of the external magnetic field.
Also the atoms will rotate, only when the frequency is near resonance of the atoms.

Furthermore, the power from the eddy currents grows with the square of the magnetic field and the square of the frequency, whereas the powers from the magnetic moments grow linearly with these quantities.

It is of course possible to chose an external magnetic field and a frequency that results in a power of about 2 *kW*, but this is not really interesting, as long as we don't know the detailed dynamics of the magnetic moments.
Although we have not reached a quantitative description of the functionality of the induction stove, we have accounted for the physics of the two mechanisms that contribute to an induction hub.