The duration of a shooting star

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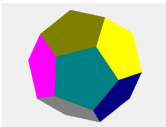


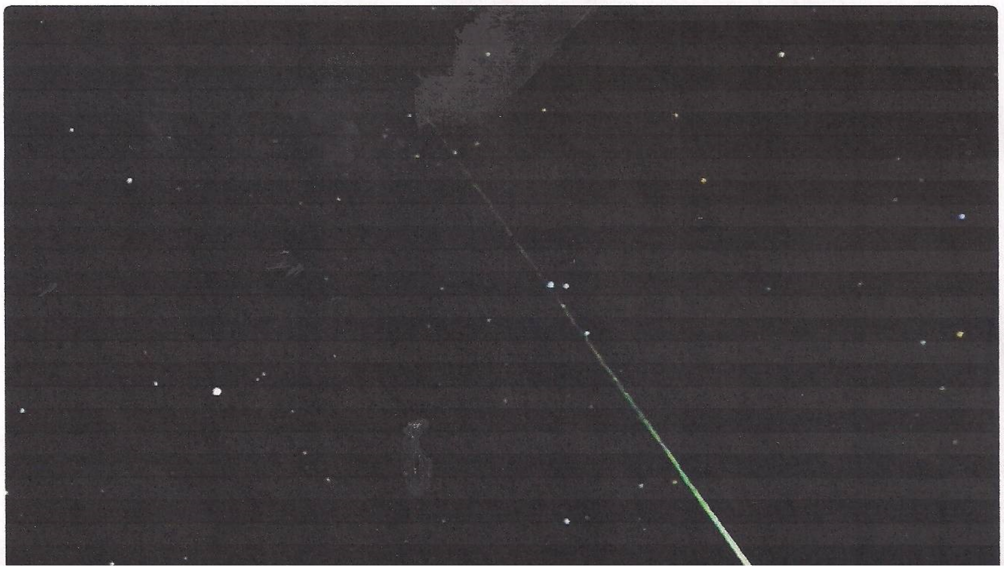
Ole Witt-Hansen 2009 (2016)

Contents

1. A shooting star

Sunday the second of march 2009 a large meteor (shooting star) was observed in many places in Denmark. It is common knowledge that a meteor obtains so high temperatures that it is seen as a shooting star, when it burns out and evaporates within a second on its way down through the atmosphere of the earth.
In my position as a physics teacher in the Danish senior high, I have often encountered the question whether I could explain the phenomena.

My answer is that I possibly can, but it is highly unlikely that the inquisitor will understand the answer.
I have searched the Internet, but I have never encountered a theoretical calculation, which just qualitatively could explain, knowing the mass and the speed of the meteor, what happens to a shooting star on its way down the atmosphere.
My aim is to try to answer some of these questions in the following. I assume that the shooting star is a piece of rock.



A photo of a meteor in the Swam of the Leonides taken i Denmark November 2006

2. Why does a shooting star decelerate?

In analyzing a complex physical problem, it is often necessary to make simplifying assumptions. In this case we assume that the impact of the meteor with the molecules in the atmosphere is completely inelastic. That the meteor and the molecules do not continue as one body (which is actually the definition of an inelastic impact), is of minor importance for the result of the calculations.

First some notations: The initial mass of the meteor is m_0. In the elapsed time dt, it collides with the molecules having the mass dm, and for that reason the velocity decreases an amount dv. According to the conservation of momentum:

(2.1) $$(m_0 + dm)(v + dv) = m_0 v \quad \Rightarrow \quad v + dv = \frac{m_0}{m_0 + dm} v \quad \Rightarrow \quad v + dv = \frac{1}{1 + \frac{dm}{m_0}} v$$

Since $\frac{dm}{m_0} << 1$ we shall use the approximation $\frac{1}{1+h} \approx 1 - h$ in the denominator. Then we get:

(2.2) $$v + dv = \frac{1}{1 + \frac{dm}{m_0}} v \quad \Rightarrow \quad v + dv = (1 - \frac{dm}{m_0}) v \quad \Rightarrow \quad dv = -\frac{dm}{m_0} v$$

The last equation is then rewritten as: $m_0 dv + v dm = 0$

Which expresses a differential conservation of momentum, and we could equally well have written it from the start.
To construct a solvable differential equation we assume, that the density ρ of the atmosphere is constant, and does not decrease exponentially according to the formula:

(2.3) $$\rho(h) = \frac{M}{RT} p_0 e^{-\frac{Mg}{RT} h}$$

Where M is the molar mass, R is the gas constant (from the equation of state for ideal gasses), T is the absolute temperature, g is the gravitational acceleration, p_0 is the pressure at sea level and h is height over sea level

The assumption of a constant density does not offer a serious restriction, since we may replace the real atmosphere with an atmosphere having a constant density, but with the same mass in a smaller volume.

The resulting thickness can the be calculated from the equation: $\rho_{air} g h = p_0$, where h is the new thickness of the atmosphere $\rho_{air} = 1.29\ kg/m^3$ and $p_0 = 1{,}013\,10^5\ Pa$,
which gives the result *h= 8,0 km*

To establish an expression for *dm*, we consider the tube, with a diameter equal to the cross section, that the meteor ploughs through during the time *dt.* The length of the tube is *ds = vdt,* and the cross section of the tube, (which is the cross section of the meteor), we denote *A*. The volume *dV* is therefore *dV = Avdt*.
We are then able to write an expression for the mass *dm*.
$dm = \rho A v dt$ and therefore $\frac{dm}{dt} = \rho A v$, which is a well known formula for the flow of liquids

Dividing the equation (2.2) $dv = -\frac{dm}{m_0} v$ by dt, we obtain an equation which can be integrated.

$$\frac{dv}{dt} = -\frac{\frac{dm}{dt}}{m_0} v = -\frac{\rho A v}{m_0} v \quad \Leftrightarrow$$

(2.4) $$\frac{dv}{dt} = -\frac{\rho A}{m_0} v^2$$

This equation is already known from turbulent resistance in liquids.
In the equation above, we have not taken into account the increase in velocity caused by the gravitational acceleration however, it is easy to add the term g to the right side.
The equation then reads:

$$\frac{dv}{dt} = -\frac{\rho A}{m_0} v^2 + g$$

This equation can not be integrated directly, but instead we solve the inequality:

$$g < \tfrac{1}{10} a_{air} \quad \Leftrightarrow \quad g < \tfrac{1}{10} \frac{\rho A}{m_0} v^2$$

which gives $v > \sqrt{\dfrac{10 m_0 g}{\rho A}} \quad \Rightarrow v > 241\ m/s.$

As the speed of a meteor usually is much higher than this value, we may safely neglect the gravitational acceleration. Without the term g, the equation (2.4) can be separated and integrated.

$$\int_{v_0}^{v} \frac{1}{v^2} dv = -\frac{\rho A}{m_0} \int_0^t dt \quad \Leftrightarrow \quad -\frac{1}{v} + \frac{1}{v_0} = -\frac{\rho A}{m_0} t \quad \Leftrightarrow$$

(2.5) $$v = \frac{v_0}{1 + \frac{\rho A v_0}{m_0} t}$$

And we may also find an expression for the distance travelled.

$$ds = vdt = \frac{v_0}{1 + \frac{\rho A v_0}{m_0} t} dt \quad \Leftrightarrow \quad \int ds = \int \frac{v_0}{1 + \frac{\rho A v_0}{m_0} t} dt$$

(2.6) $$s = \frac{m_0}{\rho A} \ln\left(1 + \frac{\rho A v_0}{m_0} t\right)$$

When the distance is known, we may find t ” the time for the drop” from the last equation. It can then be inserted in (2.5) to determine the velocity v as a function of the distance s.

The derivations above can not be maintained for several reasons, but we shall proceed to calculate the loss in kinetic energy to get an estimate of the increase in temperature. But first we need some data.

$\rho = \rho_{air} = 1.29\ kg/m^3$.
$v_0 = 10\ km/s = 10^4\ m/s$ (The escape velocity from the earth 11. 2 *km/s*)
$m_0 = 100\ kg$, $\rho_{meteor} = 2.8\ g/cm^3$
$m_{meteor} = V\rho_{meteor} = \frac{4}{3}\pi r_{meteor}^3 \rho_{meteor} = 100\ kg \quad \Rightarrow \quad r_{meteor} = 0.204\ m$ and $A_{meteor} = 0{,}131\ m^2$.

It then follows: $$\frac{\rho A}{m_0} = 1.69 \cdot 10^{-3}\ m^{-1}$$

From (2.5) we may for example investigate how long it takes for the meteor to reduce its speed with 90%. So we solve the equation:

$$v = \frac{v_0}{1 + \frac{\rho A v_0}{m_0} t} = \frac{1}{10} v_0, \qquad \text{when solved for } t: \qquad t = \frac{m_0}{\rho A v_0}(10 - 1) = 0.533 s$$

Even if this is compatible with the duration of a shooting star, we cannot put much credibility in this result. The formula for the turbulent resistance requires namely adding a “form factor” $\alpha < 1$ to the cross section *A* of the meteor. Adding this form factor the time of travelling will be prolonged with the inverse value of the form factor.
The distance the meteor has travelled can then be calculated from (2.6)

$$s = \frac{m}{\rho A} \ln(1 + \frac{\rho A v_0}{m_0} t) = 1{,}36\ km$$

You should notice that this distance is relative to the ”thickness” of the atmosphere with a constant density that we found to be 8.0 *km*.

We shall then evaluate the loss in kinetic energy.
.
(2.7) $$\Delta E = \tfrac{1}{2} m_0 (v^2 - v_0^2) = \tfrac{1}{2} m_0 ((0{,}1 v_0)^2 - v_0^2) = -0{,}99 \tfrac{1}{2} m_0 v_0^2 = -\,0{,}99 \cdot 5{,}0 \cdot 10^9\ J$$

From which we see that it has lost 99% of its energy on a distance of 1.36 *km*.

How much of this energy goes to heating of the meteor is preliminary speculation.

3. The heating of a meteor

We have assumed the meteor is a lump of rock. The specific heat of a rock: $c_{rock} = 800\ J/kgK$, and if we (as a working hypothesis) assume that a fraction $\eta = \frac{1}{10}$ goes to heating of the meteor, we can find the increase in temperature from the caloric equation,

(3.1) $$\Delta E = mc\Delta T$$

The result is: $\Delta E = 0{,}1 \cdot 0{,}99 \cdot 5{,}0 \cdot 10^9 J = 4{,}95 \cdot 10^8 J$, giving a temperature: $\Delta T = 6{,}2\ 10^3$ K

So the temperature is about 6000 K. However we do not have any real possibility to estimate the fraction of energy which goes to heating of the meteor. An estimate $\eta = \frac{1}{100}$ leads to a temperature $\Delta T = 6{,}2\ 10^2\ K$. Since we know, that the meteor usually evaporates, the first estimate is probably the more likely, than the last. And it is in good accordance with the fact that shooting star has the same luminosity as is the case of the stars.
We have used the empiric formula, which is usually used for the drag in turbulent flow, but hitherto without a form factor on the cross section A of the meteor. If we replace the cross section A by αA and put $\alpha = \frac{1}{10}$, then both the time and the distance roughly speaking are increased by a factor 10, and we get $t = 5.9\ s$ and $s = 14\ km$.

Since a meteor hardly is at rest, when it enters the gravitational field of the earth, the velocity when it reaches the atmosphere is more likely to be 20 - 30 *km/s*, than 10 *km/s*. If we do a similar calculation, with the $v_0 = 25\ m/s$, the time is reduced with a factor 3 to 2.0 s and $s = 8.7\ km$.

The problem with these calculations are however, at even if it gives at qualitative understanding of the phenomena of a shooting star, you may “screw on” the parameters α and η, until you get exactly the result you want.
For a more quantitative understanding, the simple model above is simply no good.

3.1 The Duration of a shooting star

The assumption that the meteor as a whole is heated until it evaporates cannot be maintained.
We shall therefore consider the problem from quite another view, as we assume that only the most outer layer of the meteor (in the direction of motion) is heated and evaporates. Consequently the meteor gradually loses its mass, travelling down through the atmosphere.

This model, will certainly build on much better theoretical foundation, but the drawback is that the resulting differential equations can no longer be solved analytically.

To do the calculation we will also have to know the heat of vaporization for the meteor.
As before we assume, that only the fraction η of the loss in kinetic energy goes to heating of the meteor and we have a form factor α so that the now size dependent area $A_r = \alpha A$.
From the previous formula (2.4): $\frac{dv}{dt} = -\frac{\rho\alpha A}{m}v^2$, we may express the meteor’s loss of power

(3.2) $$P = \eta F_{res} v = -\eta m \frac{dv}{dt} v = -\eta m \frac{\rho \alpha A}{m} v^2 v \quad \Rightarrow \quad P = -\eta\alpha\rho A v^3$$

Notice that the loss in power is proportional to v^3.
When the mass is not constant anymore, we can no longer integrate (2.4) for dv/dt .
However it is still possible to establish an equation, which gives the relation between mass and velocity.
The conjecture is namely that it is only a small part of the mass dm, in the outer layer of the meteor, which is heated so violently that it evaporates. To calculate dQ the heat needed, we use the familiar formula where L is the heat of vaporization.

(3.3) $$dQ = L dm$$

The energy to the heating is delivered by the impact with the molecules in the air.

(3.4) $$dQ = \eta P dt \quad \Leftrightarrow \quad L dm = \eta m \frac{dv}{dt} v dt \quad \Leftrightarrow \quad \eta m v dv = L dm \quad \Leftrightarrow$$

$$\frac{dm}{m} = \frac{\eta}{L} v dv \qquad \text{which can be integrated to} \qquad \ln\frac{m}{m_0} = \frac{\eta}{2L}(v^2 - v_0^2)$$

The equation may either be solved with respect to m or v^2.

(3.5) $$m = m_0 \exp(\frac{\eta}{2L}(v^2 - v_0^2)) \qquad \text{or} \qquad v^2 = v_0^2 + \frac{2L}{\eta}\ln(\frac{m}{m_0})$$

We may from some simple considerrations obtain a conception of the size of $\frac{\eta}{2L}$

If we tentatively assume that the initial velocity is: $v_0 = 20\ km/s$, and that the mass is reduced to $\frac{1}{100} m_0$ when the velocity is reduced to $\frac{1}{10} v_0$, then we find: $\frac{2L}{\eta} = 8.710^7\ J/kg$, and if we as before put $\eta = \frac{1}{10}$, then it yields *L=4,35 10⁶ J/kg*. The heat of vaporization for a rock: L_{rock} = 6.26 10^6 *J/kg*, so the stipulated values for α and η cannot be entirely discarded.
Depending of the choice of α, η, L we find different results of course.

In the following, we shall apply $\alpha = \eta = \frac{1}{10}$ and $L = 4.35\ 10^6\ J/kg$
We are interested in finding how the mass *m* and the distance *s* depend on time, so we return to the differential equation (2.4) $\frac{dv}{dt} = -\frac{\rho\alpha A}{m} v^2$, with the change that the mass now depends on the velocity according to (3.5) $m = m_0 \exp(\frac{\eta}{2L}(v^2 - v_0^2))$. When inserted in (2.4) we get:

(3.6) $$\frac{dv}{dt} = -\frac{\rho\alpha A}{m_0 \exp(\frac{\eta}{2L}(v^2 - v_0^2))} v^2$$

But when the mass is reduced, we can no longer count on that the area is unchanged.

On the contrary, the mass is proportional to the volume, proportional to r^3, and the cross section A is proportional to r^2, so we must have: $\frac{A_m}{A} = \left(\frac{m}{m_0}\right)^{\frac{2}{3}}$. When this is inserted in: $\frac{dv}{dt} = -\frac{\rho\alpha A_m}{m} v^2$

We get:

$$\frac{dv}{dt} = -\frac{\rho\alpha A (m/m_0)^{\frac{2}{3}}}{m} v^2 \quad \Rightarrow \quad \frac{dv}{dt} = -\frac{\rho\alpha A}{m_0^{\frac{2}{3}} m^{\frac{1}{3}}} v^2 \quad \Rightarrow$$

$$\frac{dv}{dt} = -\frac{\rho\alpha A}{m_0 \exp(\frac{\eta}{2L}(v^2 - v_0^2))^{\frac{1}{3}}} v^2 \quad \Leftrightarrow$$

(3.7) $$\frac{dv}{dt} = -\frac{\rho\alpha A}{m_0} v^2 \exp(-\frac{\eta}{6L}(v^2 - v_0^2))$$

From the equation: $P = -\eta\alpha\rho A v^3$ and $P = L\frac{dm}{dt}$ we may also find a differential equation for the dependence of the mass with time.

$$\frac{dm}{dt} = -\frac{\eta\alpha\rho A_m}{L} v^3 \quad \Rightarrow$$

(3.8) $$\frac{dm}{dt} = -\frac{\eta\alpha\rho(m/m_0)^{\frac{2}{3}} A}{L} v^3$$

If we also want the distance travelled by the meteor, we must use the definition $ds/dt = v$.

3.2 Numerical solutions to the problem

The differential equations for the velocity *v*, the distance *s* travelled by the meteor and the mass of the meteor, are linked to three coupled differential equations. To solve them one has to resort to numerical methods. The graphs shown below, exhibit examples of solutions to the three differential equations, where their dependence on time are depicted in the same figure.
The variables are scaled to fit in the same coordinate system.

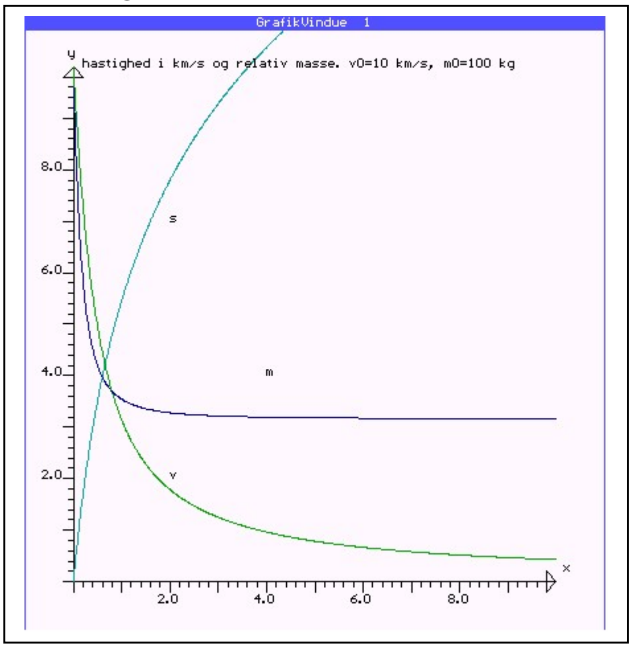
Below are shown some solutions, where the mass is m_0 = 100 *kg* in the first four examples and in the last two examples 10.000 *kg* and 1000 kg respectively. The velocities are 10 , 15, 20, 25, 25 and 25 *m/s*. The graphs show the mass *m*, velocity *v* and distance *s* in the same graph.
If the mass is measured in the unit 10 *kg*, the velocity in *km/s* and the distance in *km*, the scale on the 2. axis will fit for all three variables. The unit on the time axis is 0.1 s.

The figures show among other things, that a mass of 100 *kg* will burn out if the initial velocity exceeds 20 *m/s*, while a meteor of 1000 *kg*, will not burn until the velocity exceeds 25 *m/s*.

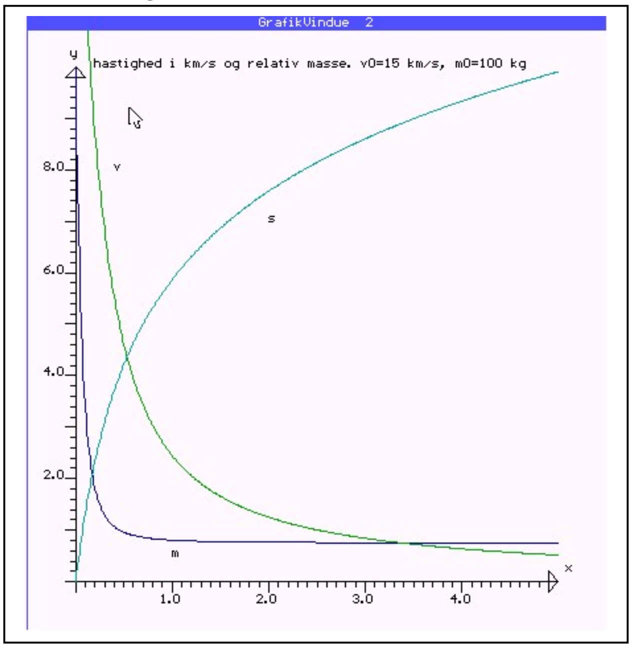
As you can read from the figures, a 100 *kg* meteor looses more than 90% of its energy in less than a second, if the velocity is above 15 *m/s*. We therefore conclude that commonly the duration of a shooting star is about one second, what I believe is in good accordance with everyday experience.

After one second the meteor will either have burned out in the atmosphere or it has hit the ground. In the case, where the meteor dos not burn completely out, it may possibly be seen as a burning ball for a longer time, especially if the trajectory is near to parallel with the surface of the earth.

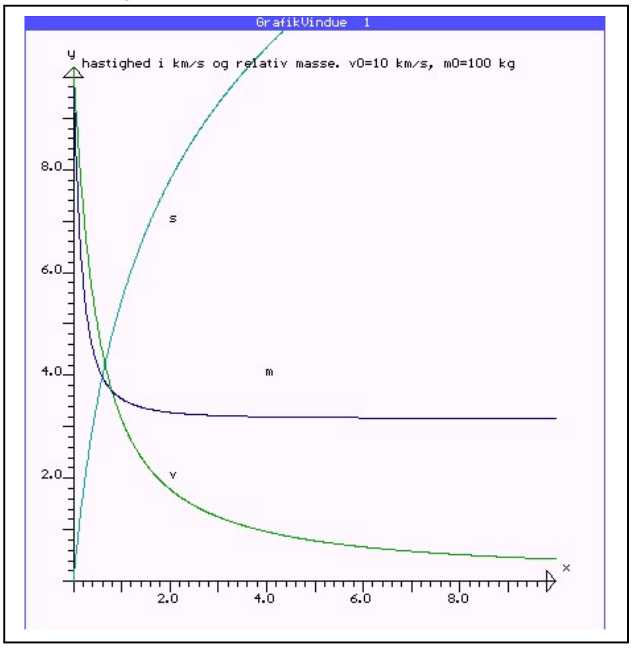
m_0 = 100 *kg*, v_0 = 10 *km/s*



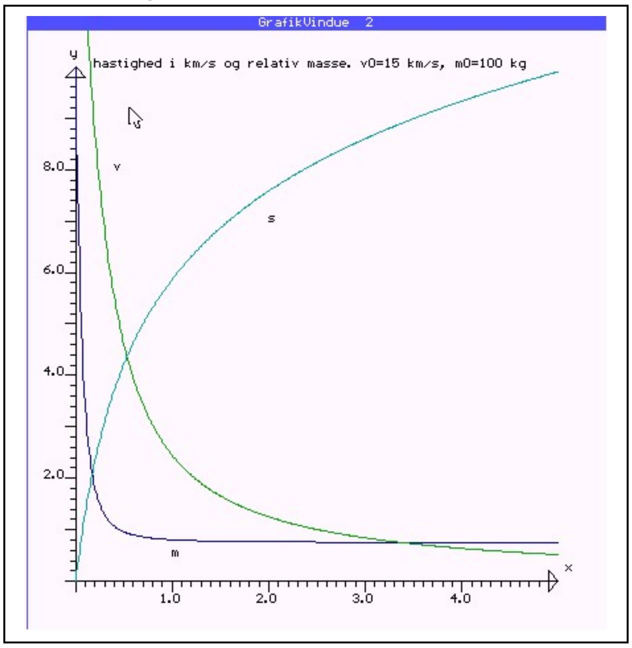
m_0 = 100 *kg*, v_0 = 15 *km/s*



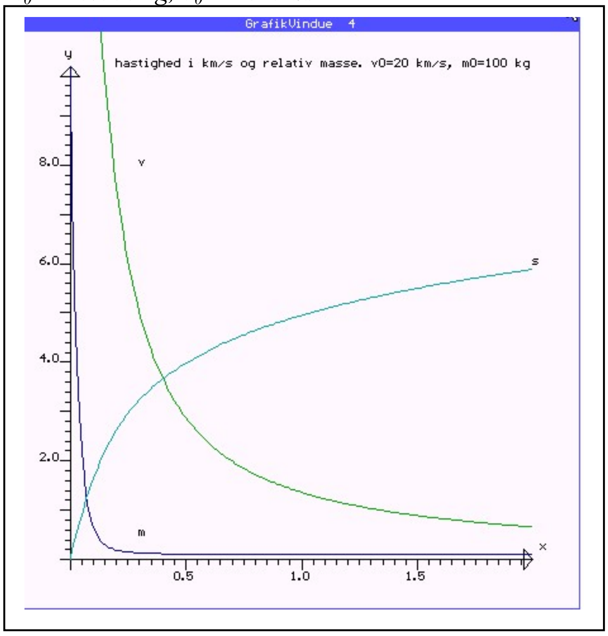
m_0 = 100 *kg*, v_0 = 20 *km/s*



m_0 = 100 *kg*, v_0 = 25 *km/s*



m_0 = 10.000 kg, v_0 = 25 km/s



m_0 = 100 kg, v_0 = 25 km/s