Solution to the diffusion equation

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1. The diffusion equation

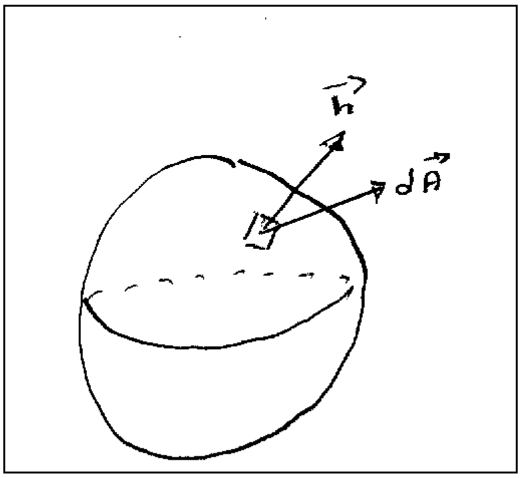
In a heat conducting material, the temperature ψ obeys the diffusion equation:

(1.1) $$\vec{\nabla}^2\psi - \frac{1}{\kappa}\frac{\partial\psi}{\partial t} = 0$$

$\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator, and $\kappa = \frac{k}{c\rho}$, where k is the heat conductivity, c is the specific heat and ρ is the density. The connection between the heat conducting vector $\vec{h}$ and the temperature gradient is:

(1.2) $$\vec{h} = -k\vec{\nabla}T$$

1.1 Derivation of the diffusion equation



In the figure is shown a mathematical surface in a heat conducting body. The flow of heat per second through the surface is equal to the rate of change of heat in the volume enclosed by the surface where T is the temperature.

(1.3) $$\int_{surface} \vec{h}\cdot d\vec{A} = -\frac{\partial}{\partial t}\int_{volume} \rho c T dV$$

If we use Gauss' law on the first integral, and using that $\vec{h} = -k\vec{\nabla}T$, we have:

(1.4) $$\int_{surface} \vec{h}\cdot d\vec{A} = \int_{volume} \vec{\nabla}\cdot\vec{h} dV = -k\int_{volume} \vec{\nabla}^2 T dV$$

We then have the equation:

$$-k\int_{volume} \vec{\nabla}^2 T dV = -\frac{\partial}{\partial t}\int_{volume} \rho c T dV$$

Since this equation must hold for all volumes, it must also hold for the infinitesimal volume dV, therefore:

(1.5) $$k\vec{\nabla}^2 T = \rho c\frac{\partial T}{\partial t} \quad \text{or} \quad \vec{\nabla}^2 T = \frac{\rho c}{k}\frac{\partial T}{\partial t} \quad \Leftrightarrow \quad \vec{\nabla}^2 T = \frac{1}{\kappa}\frac{\partial T}{\partial t}$$

Our aim is then to solve this equation in some special cases, with radial symmetry.

1.2 Solution of the spherical diffusion equation

The spherical coordinates are (r,θ,φ), and if there are no dependence on θ,φ, the Laplace operator reduces to:

(1.6) $$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial U}{\partial r})$$

If we replace the temperature T by ψ, then we have the equation:

(1.7) $$\vec{\nabla}^2\psi = \frac{1}{\kappa}\frac{\partial\psi}{\partial t} \qquad \Leftrightarrow \qquad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) = \frac{1}{\kappa}\frac{\partial\psi}{\partial t}$$

Here $\psi = \psi(r,t)$ depends only on r and t, but dependence on r and t cannot be factorized.
It is practical to eliminate the constant κ, so we make the substitution: $r = \sqrt{\kappa}\rho$, and then the equation becomes:

(1.8) $$\frac{1}{\kappa\rho^2}\frac{\partial}{\sqrt{k}\partial\rho}(\kappa\rho^2\frac{\partial\psi}{\sqrt{\kappa}\partial\rho}) = \frac{1}{\kappa}\frac{\partial\psi}{\partial t} \qquad \Leftrightarrow \qquad \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial\psi}{\partial\rho}) = \frac{\partial\psi}{\partial t}$$

There are (as far as I know) no general methods to solve this partial differential equation but 35 years ago, when I was occupied with this equation, analyzing geothermal heating pipes.
I made my way by an educated guess to find the solution:

(1.8) $$\psi(\rho,t) = \frac{\psi_0}{(4\pi t)^{\frac{3}{2}}}e^{-\frac{\rho^2}{4t}}$$

We shall now demonstrate that this is indeed a solution:

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial\psi}{\partial\rho}) = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(-\frac{\rho^3}{2t}\psi) = -\frac{1}{\rho^2}\frac{1}{2t}(3\rho^2\psi - \frac{\rho^4}{2t}\psi)$$

So

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial\psi}{\partial\rho}) = (\frac{\rho^2}{4t^2} - \frac{3}{2t})\psi$$

And

$$\frac{\partial\psi}{\partial t} = -\tfrac{3}{2}\frac{\psi_0}{(4\pi t)^{\frac{3}{2}}}\frac{1}{t}e^{\frac{-\rho^2}{4t}} + \frac{\psi_0}{(4\pi t)^{\frac{3}{2}}}e^{\frac{-\rho^2}{4t}}(\frac{\rho^2}{4t^2}) = (\frac{\rho^2}{4t^2} - \frac{3}{2t})\psi$$

The solution to (1.7) is then, apart from a constant, obtained by substituting $\rho = \frac{r}{\sqrt{k}}$ in (1.8).

And since $\int_0^\infty e^{-\frac{r^2}{4\kappa t}}dr = \sqrt{4\pi\kappa t}$, we write the expression for dimensional and normalization purposes

(1.9) $$\psi(\frac{r}{\sqrt{\kappa}},t) = \frac{\psi_0 r_0 t_0}{t\sqrt{4\pi\kappa t}}e^{-\frac{r^2}{4\kappa t}}$$

We notice that if $r \neq 0$ *then* $\psi(\frac{r}{\sqrt{\kappa}}, t) \to 0$ *for* $t \to 0$. This is what we may expect if a heat is released at $r = 0$ and $t = 0$.

But $\psi(0,t) = \frac{\psi_0 r_0 t_0}{t\sqrt{4\pi\kappa t}}$ has a pole at $t = 0$, so a more correct description is:

$$\psi(\frac{r}{\sqrt{\kappa}}, t) = \frac{\psi_0 r_0 t_0}{t\sqrt{4\pi\kappa t}} e^{-\frac{r^2}{4\kappa t}} \quad for \quad t > 0 \quad and \quad \psi(r,0) = \delta(r)$$

Where $\delta(r) = \begin{cases} 0 & for \quad r \neq 0 \\ \infty & for \quad r = 0 \end{cases}$ is the Dirac *delta* function.

1.3 Solution of the cylindrical diffusion equation

The cylindrical coordinates are: (r, θ, z). The infinitesimal displacement vector is $d\vec{s} = (dr, rd\theta, dz)$ and the distance element is $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$, (since the three displacements are orthogonal).

In cylindrical coordinates the gradient of a scalar field ψ therefore becomes:

(1.10)
$$\vec{\nabla}\psi = (\frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{\partial\psi}{\partial r}, \frac{\partial\psi}{\partial z})$$

And the Laplace operator can be shown to be. (See e.g. www.olewitthansen.dk vector analysis).

(1.11)
$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r}) + \frac{\partial^2\psi}{\partial z^2}$$

If the field ψ = ψ(r,t) depends neither on θ nor on z, the Laplacian reduces to:

(1.12)
$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r})$$

And the diffusion equation $\nabla^2\psi = \frac{1}{\kappa}\frac{\partial\psi}{\partial t}$ reduces to

(1.13)
$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r}) = \frac{1}{\kappa}\frac{\partial\psi}{\partial t}$$

As we did in the spherical symmetric case, we eliminate the factor κ, by the substitution $r = \sqrt{\kappa}\rho$, which results in the equation:

(1.14)
$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho}) = \frac{\partial\psi}{\partial t}$$

Neither this equation can be solved by traditional methods, but an educated guess, suggests:

(1.15) $$\psi(\rho,t)=\frac{1}{t}e^{-\frac{\rho^2}{4t}}$$

Proof:

$$\frac{\partial\psi}{d\rho}=\frac{1}{t}(-\frac{2\rho}{4t})e^{-\frac{\rho^2}{4t}} \quad => \quad \rho\frac{\partial\psi}{d\rho}=\frac{1}{t}(-\frac{2\rho^2}{4t})e^{-\frac{\rho^2}{4t}}=-\frac{\rho^2}{2t^2}e^{-\frac{\rho^2}{4t}}$$

So

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho})=\frac{1}{\rho}\frac{\partial}{\partial\rho}(-\frac{\rho^2}{2t^2}e^{-\frac{\rho^2}{4t}})=\frac{1}{\rho}(-\frac{\rho}{t^2}e^{-\frac{\rho^2}{4t}}-\frac{\rho^2}{2t^2}(-\frac{\rho}{2t})e^{-\frac{\rho^2}{4t}}=\frac{1}{t^2}(\frac{\rho^2}{4t}-1)e^{-\frac{\rho^2}{4t}}$$

And

$$\frac{\partial\psi}{\partial t}=\frac{\partial}{\partial t}(\frac{1}{t}e^{-\frac{\rho^2}{4t}})=-\frac{1}{t^2}e^{-\frac{\rho^2}{4t}}+\frac{1}{t}\frac{\rho^2}{4t^2}e^{-\frac{\rho^2}{4t}}=\frac{1}{t^2}(\frac{\rho^2}{4t}-1)e^{-\frac{\rho^2}{4t}}$$

To obtain the correct dimension of the equation, we add two constants

(1.16) $$\psi(\rho,t)=\psi_0\frac{{\rho_0}^2}{t}e^{-\frac{\rho^2}{4t}}$$

To obtain the solution to the original diffusion equation we reverse the substitution $r=\sqrt{\kappa}\rho$, and we find, when adding two constants.

(1.17) $$\psi(\frac{r}{\sqrt{\kappa}},t)=\frac{\psi_0{r_0}^2}{\kappa t}e^{-\frac{r^2}{4\kappa t}}$$

Or, when written with the temperature T instead of ψ.

$$T(\frac{r}{\sqrt{\kappa}},t)=\frac{T_0{r_0}^2}{\kappa t}e^{-\frac{r^2}{4\kappa t}}$$

Since $\int_0^\infty e^{-\frac{r^2}{4\kappa t}}dr=\sqrt{4\pi\kappa t}$ we shall write the equation for normalization purposes as:

$$T(\frac{r}{\sqrt{\kappa}},t)=\frac{T_0{r_0}^2}{\sqrt{4\pi\kappa t}}e^{-\frac{r^2}{4\kappa t}}$$

As in the spherical symmetric case, we have for $r\neq 0$, $T(\frac{r}{\sqrt{\kappa}},t)\to 0 \;\; for \;\; t\to 0$,

But $T(0,t)=\frac{T_0{r_0}^2}{\sqrt{4\pi\kappa t}}$ has a pole at $t = 0$, so the correct description is:

$$T(\frac{r}{\sqrt{\kappa}},t)=\frac{T_0{r_0}^2}{\sqrt{4\pi\kappa t}}e^{-\frac{r^2}{4\kappa t}} \quad for \;\; t>0 \quad and \quad T(r,0)=\delta(r)$$

Even having found the mathematical solutions to the diffusion equation, it is not so easy to apply it to the boundary conditions of concrete physical examples.

2. Heating of a body, the stationary situation.

Let us again consider a body, in which we have a closed mathematical surface, but now having inside a heat source. We assume that so long time has passed, that we have a quasi stationary situation, in which the heat leaving the closed surface is equal to the heat produced within the surface. We then repeat the steps from section 1. If the power of the heating source per unit volume is w, we must have:

(2.1) $$\int_{surface} \vec{h} \cdot d\vec{A} + \int_{volume} w dV = 0$$

If we use Gauss' law on the first integral using: $\vec{h} = -k\vec{\nabla}T$, we have:

$$\int_{surface} \vec{h} \cdot d\vec{A} = \int_{volume} \vec{\nabla} \cdot \vec{h} dV = -k \int_{volume} \vec{\nabla}^2 T dV$$

We then get the equation:

$$-k \int_{volume} \vec{\nabla}^2 T dV + \int_{volume} w dV = 0 \quad \Leftrightarrow \quad \int_{volume} \vec{\nabla}^2 T dV = \frac{1}{k} \int_{volume} w dV$$

or

(2.2) $$\vec{\nabla}^2 T = \frac{w}{k}$$

This is formally the same equation as Gauss' law for the electric field of a charge distribution of charge (Maxwell's 1. equation): $\vec{\nabla}^2 E = \frac{\sigma}{\varepsilon_0}$, and for spherical symmetric heat source W, placed in the centre of a sphere, it has the solution.

(2.3) $$T = \frac{w}{4\pi k} \frac{W}{r^2}$$

By the same token, we may find the temperature distribution for a cylindrical symmetric heat source:

(2.4) $$T = \frac{1}{2\pi k} \frac{W}{r}$$