The peculiar Coriolis force

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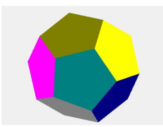

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1. Introduction to the subject

In Denmark the academic level of physics has violently deteriorated since 2005, but at the same time much of the daily teaching has been transformed into individually and larger (2 – 20 pages) examination papers.
And this has turned out to be professional contradiction, since most of the subjects that the students choose, or are presented for, are at a much higher professional level than is reflected in their daily teaching.

This has also presented a professional challenge for the teachers, since often they have had no real knowledge from their education of most of subjects the students choose in their examination papers. And both the teachers and the students have to rely on what they may find on the Internet, let alone university text books.
Furthermore, and utterly absurd, the student papers have to be structured as a scientific article for a magazine

Several times, when both the Internet and university books became inadequate, I had to write some notes myself and leave them to the students to save the students from delivering a disastrous article. Many of these notes have been brought into articles on my home page, from which this article about the Coriolis force is an example.

I was in 2008 the censor at an exam in physics in the Danish high school. One of the questions, the student should respond to, concerned the concept of pressure equalization in the atmosphere.
That is, as I perceived it, something with partial pressure and Dalton's law.
The student was asked to look at a weather chart with isobars, and the student should then indicate the direction of the wind. That seemed easy enough, since I was taught in the geography lessons in grammar school that the wind evidently blows from high pressure to low pressure, which is also the result of Daltons experiment.
Rather surprisingly the student answered that the wind blew along the isobars, with the clock on the northern hemisphere, and against the clock on the southern hemisphere, and where the explanation to this phenomenon was the presence of the Coriolis force.
The examiner did not intervene to the students answer, and for that reason, neither did I.

But it sounded certainly strange since the isobars are curves through areas having the same pressure, which should prevent any movement of the atmosphere.
Naturally the students should not be blamed for the ignorance of their teachers, and the answers were consequently accepted as being correct.

What bothered me more was the old hoax that a vortex always moves clockwise on the northern hemisphere and counter clockwise on the southern atmosphere, and where the explanation allegedly was that it was a consequence of the Coriolis force!

Concerning the Northeast passage and the Southeast passage, which blo from the subtropical high pressure towards the calm belt near equator, then you don't need to include the Coriolis force to account for the clockwise and counter clockwise deviation of the wind.
The angular velocity of the earth is:

$$\omega_{earth} = \frac{2\pi}{T_{earth}} = \frac{2\pi}{24h} = 7{,}27 \cdot 10^{-5} s^{-1}$$

If we calculate the velocity in the circular motion on the surface of the earth at equator and at the 20 latitude, we find:

$v_0 = \omega_{earth} r_{earth} = 72.7 \cdot 10^{-5} \cdot 6{,}370 \cdot 10^6 m/s = 463\ m/s$ and $v_{20} = \omega_{earth} r_{earth} \cos 20 = 435 m/s$

The carrying speed of the atmosphere is therefore less at the 20 latitude than it is near equator.
If the wind furthermore has a component along a meridian, then consequently the wind will be deviated opposite to the rotation of the earth, that is, to the vest, which will be perceived as if the wind is coming from the north east on the northern hemisphere and from south east on the southern hemisphere.

If one tries to calculate the deviation angle of the wind, as a consequence of the rotation of the earth, (where we ignore viscosity, which is quite unrealistic, but the best we can do), then we find a rather surprising result.

We may find a qualitative result for the deviation angle, if we estimate that velocity of the wind along a meridian is 6.0 m/s. If v denotes the velocity of the wind, R is the radius of the earth and $\Delta s_1 = v\Delta t$ is the distance the wind moves in time Δt, then this correspond to a change in latitude $\Delta b = \Delta s_1/R$. The displacement of the earths surface along the latitudes b and $b + \Delta b$ can then be calculated as:

$$\Delta s_2 = v_{b+\Delta b}\,\Delta t - v_b\,\Delta t = \omega\,(r_{b+\Delta b} - r_b)\Delta t = \omega R\,(\cos(b+\Delta b) - \cos(b))\,\Delta t \approx -\omega R \sin b \Delta b \Delta t$$

The angle between the wind and the meridian which follows the rotation of the earth is then given by:

$$\tan(\beta) = \frac{\Delta s_2}{\Delta s_1} = \frac{\omega R \sin b \Delta b \Delta t}{v \Delta t} = \frac{\omega R \sin b}{v} \Delta b$$

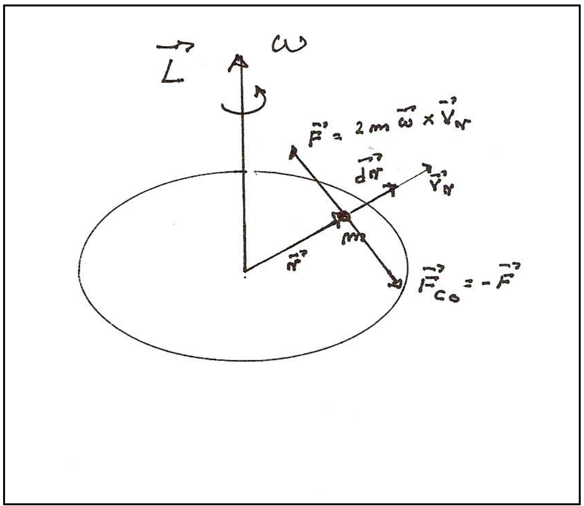
If we insert $R = 6.370\ 10^6$ m, $\omega = \omega_{jord} = 7.27\ 10^{-5}\ s^{-1}$, $v = 6.0$ m/s, $b = 20^0$ and $\Delta b = 1^0 = \frac{\pi}{180}$,
Then one finds somewhat surprisingly $\beta = 24.7^0$. But it probably reflects that it is completely unrealistic to ignore viscosity and turbulence.

Seen from outer space it is relatively easy to understand the motion on the surface of the earth, but it gets more complicated when analyzed, as seen from the earth, which is a rotating coordinate system. In such a system appear fictitious forces, such as the *centrifugal* force and the *Coriolis* force.
The centrifugal force is most easily understood. To uphold a uniform circular motion, it requires a *centripetal* force: $F_c = m\frac{v^2}{r}$.
If, however, you try to follow the circular motion without a centripetal force, you will feel an outward (centrifugal) force equal to the required centripetal force. The centrifugal force is not a physical force, but merely reflects the absence of a centripetal force.

It turns out that the Coriolis force does not depend on the position in a rotating frame, but only on the velocity with which you move relatively to the rotating frame.

2. Description of the Coriolis force

A more comprehensive theoretical derivation for the expression for the Coriolis force, one may for example find in the textbook by Landau and Lifschitz: Mechanics from 1962.
The formula is however derived from a total differential of the Lagrange function together with some theorems from the analytical mechanics.
Somewhat more accessible and more direct is the presentation given in “The Feynmann Lectures I” from 1963. However Feynmann’s derivation is a scalar computation and not vector computation.



We consider a coordinate system, which moves with constant angular velocity.
We assume that we have a mass *m*, at the position vector $\vec{r}$, having the angular momentum $\vec{L} = m\,\vec{r}\times\vec{v}$ because of the collective motion. (See the figure)
Since $\vec{r} \perp \vec{v}$ we must have $\vec{v} = \vec{\omega}\times\vec{r}$, $v = \omega r$, and therefore $L = mrv$.
We consider first the case where the mass *m* moves radially relative to the uniformly rotating system.
Here we have:

$$L = mvr = m\omega r^2 .$$

By differentiating this equation, we find:

(2.1) $$\frac{dL}{dt} = 2m\omega r\frac{dr}{dt} = r\cdot 2m\omega v_r .$$

Since $d\vec{r}$ has the same or the opposite direction to $\vec{r}$, then $\frac{d\vec{L}}{dt}$ must be parallel to $\vec{L}$.
Comparing (2.1) to Newton’s theorem of torque:

(2.2) $$\vec{H} = \frac{d\vec{L}}{dt} = \vec{r}\times\vec{F} \quad\Rightarrow\quad \frac{dL}{dt} = r\cdot 2m\omega v_r = rF$$

The last equation follows, since if $\frac{d\vec{L}}{dt} \perp \vec{r}$ then must also $\vec{F}\perp\vec{r}$, so that $\left|\vec{r}\times\vec{F}\right| = rF$
We can then see that the particle will be influenced by a moment of force (torque), with the magnitude $r\cdot 2m\omega v_r$ and having the same direction as $\frac{d\vec{L}}{dt}$.

If we consider the directions of $\vec{r}$, $\vec{\omega}$ and $\vec{v}_r$, we can see that the particle must be influenced by a force:

$$\vec{F} = 2m\vec{\omega}\times\vec{v}_r .$$

The fictitious force that is felt by the mass moving along a radius is the Coriolis force: $\vec{F}_{Co} = -\vec{F}$.

(2.3) $$\vec{F}_{Co} = 2m\vec{v}_r\times\vec{\omega}$$

As shown in the figure above, a body moving radially in a uniformly rotating system, will be influenced by a fictitious force F_{Co}, which acts transversely to the radial velocity and acting opposite to the circular motion when moving away from the centre, and along the circular motion, when moving towards the centre.

Notice that the fictitious Coriolis force depends only on the relative velocity to the rotating system and that it is independent of the distance to the centre of the circular motion.

Although it is common knowledge, it is remarkable that a particle is subjected to the same Coriolis force, no matter where you are placed in the rotating system and independent of the direction of the relative speed, and that the force in all cases is given by the expression above.

We shall now substantiate this by looking into the case, where the moving body has a tangential velocity $\vec{v}_r$, relative to the circular motion.

Observed from an inertial system, the velocity of the mass is $\vec{v} = \vec{v}_c + \vec{v}_t$, that is the sum of the circular and the tangential motion. The resulting force in a inertial system is equal to the centripetal force.

(2.4) $$F_c = m\frac{v^2}{r} \qquad \text{where} \quad v^2 = \vec{v}^2 = (\vec{v}_c + \vec{v}_t)^2 = v_c^{\ 2} + v_t^{\ 2} + 2\vec{v}_c \cdot \vec{v}_t$$

In the rotating system the first two terms will deliver a contribution to the ”centrifugal force”.
If $v_t << v_c$, the second term will be infinitesimally. To understand the third term, we shall write the contribution to the centripetal force, this term in vector form.

(2.5) $$\vec{F}_c = -2m\frac{\vec{v}_c \cdot \vec{v}_t}{r^2}\vec{r}$$

Since $\vec{v}_c$ and $\vec{v}_t$ are unidirectional (or opposite directed) and $v_c = \omega \cdot r$, and if we write: $\vec{r} = r \cdot \vec{e}_r$.

(2.6) $$\vec{F}_c = -2m\frac{\omega \cdot r \cdot v_t \cdot r}{r^2}\vec{e}_r = -2m\omega \cdot v_t \vec{e}_r$$

As $\vec{v}_t \perp \vec{r}$ and $\vec{v}_t \perp \vec{\omega}$ you may see, that we can write this in vector form as:

$$\vec{F}_c = -2m\vec{v}_t \times \vec{\omega}.$$

The corresponding fictitious Coriolis force is thus given by the expression.

(2.7) $$\vec{F}_{Co} = 2m\vec{v}_t \times \vec{\omega}$$

And this turns out to be the same formula as we found from the radial motion.

Hereafter it is easy to argue that the formula for the Coriolis force is valid no matter where you are placed in the rotating system, and we may replace $\vec{v}_t$ by $\vec{v}$ in the expression.

The velocity can namely be resolved in three components: One radial, one tangential and one along the rotating axis: $\vec{v} = \vec{v}_c + \vec{v}_t + \vec{v}_\omega$
Since the expression for the Coriolis force is a vector equation, and the cross product is distributive with respect to addition then the evaluation of

$$\vec{F}_{Co} = 2m\,\vec{v}\times\vec{\omega} = 2m\,(\vec{v}_c + \vec{v}_t + \vec{v}_\omega)\times\vec{\omega}$$

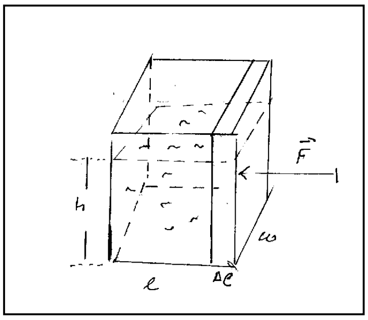
Will give zero for the last term, since $\vec{v}_\omega$ *and* $\vec{\omega}$ are parallel, and we may in any case use

$$\vec{F}_{Co} = 2m\,\vec{v}\times\vec{\omega}$$

For the Coriolis force.

3. The difference in water level on the two sides in a belt

Between the two Danish major island Zealand and Fyn runs a belt with an almost constant current. Due to the rotation of the earth, this should induce a Coriolis force on the water, and we want to calculate the consequences concerning the displacement of water on the two sides of the belt.



To simplify matters, we shall first look at a rectangular container with water, where the upper lid is open, and where one of the sides acts as a piston that can be moved in or out due to a force acting perpendicular to that side.
The volume of the water is length x width x height: $V = lwh$.
If either *l* or *h* change we have:

$$V + \Delta V = (l + \Delta l)(h + \Delta h)w = lwh + (l\Delta h + h\Delta l + \Delta h\Delta l)w$$

$$\Delta V = l\Delta hw + h\Delta lw$$

Since the volume of the water does not change, we must have $\Delta V = 0 \quad \Leftrightarrow \quad \Delta h = -h\dfrac{\Delta l}{l}$

Since $\Delta l < 0$ for a compression then $\Delta h > 0$.

Then we shall make an energy consideration: The work done by the force *F*, must be equal to the change in potential energy of the water in the container.

$$F\Delta l = mg\Delta h \quad \Rightarrow \quad F\Delta l = mgh\frac{\Delta l}{l} \quad \Rightarrow \quad F = \frac{mgh}{l}$$

The expression for the Coriolis force is: $\vec{F}_{Co} = 2m\,\vec{v}\times\vec{\omega}$. As $\vec{v} \perp \omega$, then we have: $F_{Co} = 2mv\omega$.
If we insert this as *F* in the expression above, we find:

$$F_{Co} = F \quad \Rightarrow \quad 2mv\omega = \frac{mgh}{l} \quad \Rightarrow \quad h = \frac{2v\omega l}{g}$$

The width of Storebelt is roughly 20 km. $\omega_{earth} = 7.3 10^{-5}\ s^{-5}$, $v_{stream} = 1.0\ m/s$ (estimated), and then we may calculate *h*.

$$h = \frac{1.0\ m/s\ 7.310^{-5}\ s^{-1}\ 2.0\ 10^{4} m}{9.82\ m/s^{2}} = 14\ cm$$

That is an increase of the water level of 14 cm on the west side.

This does not seem to be very much, but the Coriolis force acting on 1 kg of water is:

$$F_{Co} = 2mv\omega = 2\,kg\,1\,m/s\ 7.310\,10^{-5} = 1.46\,10^{-4}\ N$$

4. Formal theoretical derivation of the expression for the Coriolis force.

The expression for the Coriolis force may be derived in a moreformal theoretical method, by applying the expression

$$\vec{L} = m\,\vec{r}\times\vec{v}$$

for the angular momentum of a particle.

For a particle following the uniform circular motion $\vec{L}$ is constant, and thus $\frac{d\vec{L}}{dt} = 0$

If we insert $\vec{v} = \vec{\omega}\times\vec{r}$ into $\vec{L} = m\,\vec{r}\times\vec{v}$ we get $\vec{L} = m\,\vec{r}\times(\vec{\omega}\times\vec{r})$, and then we differentiate this.

(4.1) $$\frac{d\vec{L}}{dt} = m\frac{d\vec{r}}{dt}\times(\vec{\omega}\times\vec{r}) + m\,\vec{r}\times(\vec{\omega}\times\frac{d\vec{r}}{dt})$$

In general the cross product is neither commutative nor associative, but in this case we have $\vec{\omega}\perp\vec{r}\ \ and\ \ \vec{\omega}\perp\frac{d\vec{r}}{dt}$, so the two terms in (4.1) are the same after two permutations of the two factors in the first term.

(4.2) $$\frac{d\vec{r}}{dt}\times(\vec{\omega}\times\vec{r}) = -(\vec{\omega}\times\vec{r})\times\frac{d\vec{r}}{dt} = (\vec{r}\times\vec{\omega})\times\frac{d\vec{r}}{dt} = \vec{r}\times(\vec{\omega}\times\frac{d\vec{r}}{dt})$$

So we have:

(4.3) $$\frac{d\vec{L}}{dt} = 2m\,\vec{r}\times(\vec{\omega}\times\frac{d\vec{r}}{dt})$$

Furthermore we have $\frac{d\vec{r}}{dt} = \vec{v}_c + \vec{v}_{rel}$, and only the relative motion can contribute to $\frac{d\vec{L}}{dt}$. So we get:

(4.4) $$\frac{d\vec{L}}{dt} = 2m\,\vec{r}\times(\vec{\omega}\times\vec{v}_{rel}) = \vec{r}\times 2m\,\vec{\omega}\times\vec{v}_{rel}$$

Which we compare to Newton's second law for rotation (the torque theorem):

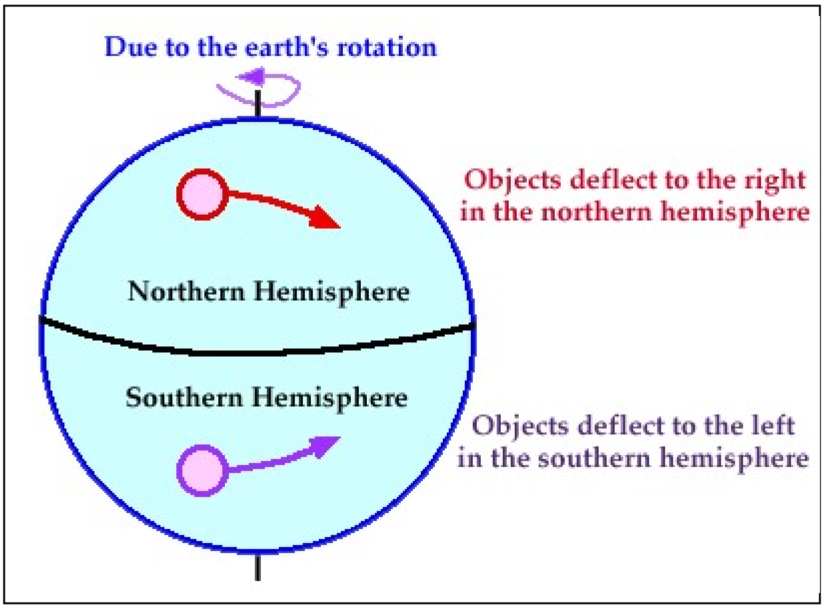
$$\vec{H} = \frac{d\vec{L}}{dt} = \vec{r}\times\vec{F}$$

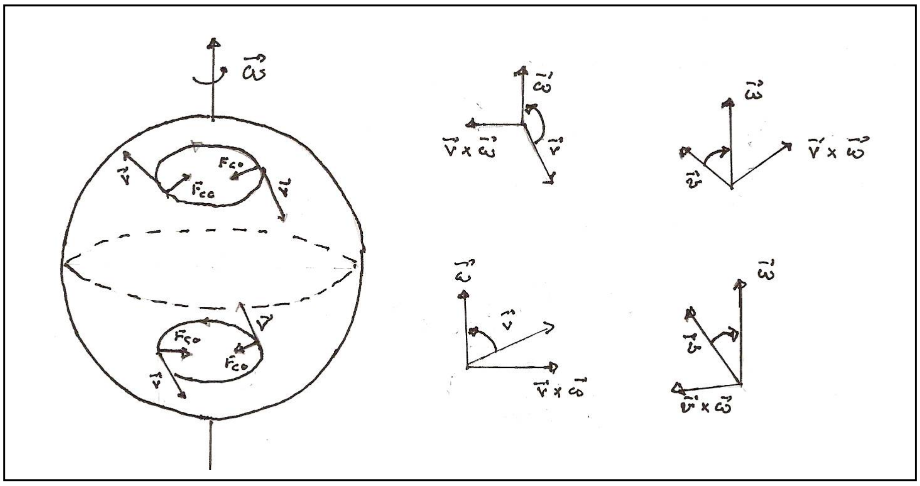
We can see that the particle as a consequence of its relative motion is affected by a force $\vec{F} = 2m\vec{\omega}\times\vec{v}_{rel}$, which induces the fictitious Coriolis force. $\vec{F}_{Co} = -\vec{F}$.

(4.5) $$\vec{F}_{Co} = 2m\,\vec{v}_{rel}\times\vec{\omega}$$

This is the same expression as we derived earlier in a more direct and transparent way.

We notice that the Coriolis force depends only on the relative speed to the rotating system, and the angular speed in the rotation. In contrast to moving in an inertial system the Coriolis force depends neither on the acceleration nor the distance from the rotating axis.
The direction of the Coriolis force is perpendicular to the relative speed and the angular velocity. $\vec{v}_{rel}, \vec{\omega}$ *and* $\vec{F}_{Co}$ form a right hand screw. Therefore the particle will be deflected to the right in a clockwise circle.
This we have tried to illustrate below in the two figures where the motion is along the surface of the earth.





The Coriolis force may deliver an explanation to the myth that vortices rotate clockwise on the northern hemisphere and counter clockwise on the southern hemisphere. But it is a hoax.
From a qualitative point of view, this explanation is however dubious, due to the smallness of the Coriolis force.
As mentioned in the beginning, the deflection of the Northeast and Southeast passage are not due to the Coriolis force but rather due to the fact that the wind blows along a longitude from a latitude with a lower rotational velocity to a latitude with a higher rotational velocity.
The angular velocity of the earth is:

$$\omega_{earth} = \frac{2\pi}{T_{earth}} = \frac{2\pi}{24h} = 7.27 \cdot 10^{-5} s^{-1} \ .$$

If we calculate F_{Co} on a mass of 1.0 kg, which has a relative velocity of 5,0 m/s, on the surface of the Earth, then the result is.

$$F_{Co} = 2m \vec{v}_{rel} \times \vec{\omega} = 2 \cdot 1 \cdot 5.0 \cdot 7.27 \cdot 10^{-5} \text{ N} = 7.27 \cdot 10^{-4} \text{ N}$$

This is less than 1 *mN* per *kg*. This result in an acceleration of $7.27 \cdot 10^{-4}$ m/s^2. Causing a displacement in 5.0 s of $\Delta s = \frac{1}{2} 7.2710^{-5} \cdot 25 = 9.10\,10^{-4} m \approx 1\, mm$

At a glace it is highly improbable that such a infinitesimally force can influence the turbulent motion, neither a massive tornado moving with a velocity more than 200 m/s, nor for the vortex that sometimes is created in a bathroom sink.
I have not found any statistics about the direction of rotation of tornadoes, but as most they certainly rotate clockwise there are observations of some rotating in the opposite direction.