Radioactive chains of decay

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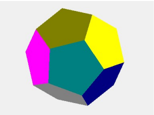

Ole Witt-Hansen 2008

2. Radioactive chains of decay

The differential equation for the number N of radioactive Nuclei, which have not yet decayed is well known from elementary high school.

(2.1) $$\frac{dN}{dt} = -kN$$

It has the equally well known solution

(2.2) $$N(t) = N_0 e^{-kt}$$

The *activity* is the rate of decay

(2.3) $$A(t) = -\frac{dN}{dt} = kN(t)$$

Th constant k is the decay constant, and is equal to ln2 divided by the half life $T_{½}$, since

(2.4) $$\tfrac{1}{2} N_0 = N_0 e^{-kT_{½}} \quad \text{gives:} \qquad k = \frac{\ln 2}{T_{½}}$$

We shall then look at a chain, where the original nucleus decays into another radioactive nucleon. This is well known from the common chains of decay: The Uranium-, the Thorium-, and the Actinium series.

If we denote the two nuclei by (1) and (2), we may establish two differential equations.
The first one is identical to (2.1), with decay constant k_1, whereas the second expresses that nucleus (2) is produced with a speed that is equal the activity of nucleus (1), subsequently decays with the decay constant k_2.

$$\frac{dN_1}{dt} = -k_1 N_1 \quad \textit{and} \quad \frac{dN_2}{dt} = -\frac{dN_1}{dt} - k_2 N_2 \quad \Rightarrow$$

(2.2) $$\frac{dN_2}{dt} = k_1 N_1 - k_2 N_2$$

The last differential equation has the form

(2.3) $$\frac{dy}{dx} = -ky + h(x)$$

It is solved by moving the term $-k \cdot y$ to the left hand side, multiplying the equation by e^{kx} , and rewrite it as a single differential quotient.

(2.4) $$\frac{dy}{dx} = -ky + h(x) \quad \Leftrightarrow \quad \frac{dy}{dx} e^{k \cdot x} + k e^{k \cdot x} y = h(x) e^{k \cdot x} \quad \Leftrightarrow \quad \frac{d(y e^{k \cdot x})}{dx} = h(x) e^{k \cdot x}$$

If $H(x) = \int h(x)e^{k\cdot x}dx$, then the differential equation has the solution:

(2.5) $$ye^{k\cdot x} = H(x) + c \quad \Leftrightarrow \quad y = H(x)e^{-k\cdot x} + ce^{-k\cdot x}$$

The constant *c* is the usual constant of integration, which is to be determined by the initial conditions.

Replacing *x* with *t*, *y* with N_2 , and *h*(*x*) with $N_1(t)$ in (2.3) and following the same manipulations with the new variables, we find:

$$\frac{dN_2}{dt} = k_1 N_1 - k_2 N_2 \quad \wedge \quad N_1 = N_0 e^{-k_1\cdot t} \quad \Rightarrow$$

$$e^{k_2\cdot t}\cdot\frac{dN_2}{dt} + k_2 e^{k_2\cdot t}\cdot N_2 = k_1 N_0 e^{-k_1\cdot t}e^{k_2\cdot t} \quad \Leftrightarrow$$

$$\frac{d(e^{k_2\cdot t}\cdot N_2)}{dt} = k_1 N_0 e^{(k_2-k_1)\cdot t} \quad \Leftrightarrow$$

$$e^{k_2\cdot t}\cdot N_2 = N_0\frac{k_1}{k_2-k_1}e^{(k_2-k_1)\cdot t} + c \quad \Leftrightarrow$$

$$N_2 = N_2(t) = N_0\frac{k_1}{k_2-k_1}e^{-k_1\cdot t} + ce^{-k_2\cdot t} \quad \Leftrightarrow$$

The constant *c* is determined by $N_2(0) = 0$ => $c = -N_0\frac{k_1}{k_2-k_1}$, and the solution is hereafter:

$$N_2(t) = N_0\frac{k_1}{k_2-k_1}e^{-k_1\cdot t} - N_0\frac{k_1}{k_2-k_1}e^{-k_2\cdot t} \quad \Leftrightarrow$$

(2.6)

$$N_2(t) = N_0\frac{k_1}{k_2-k_1}(e^{(k_2-k_1)\cdot t} - 1)e^{-k_2\cdot t}$$

Notice that $N_2 > 0$ for $t > 0$, whether $k_2 > k_1$ or not. (The case $k_2 = k_1$, has only academic interest, but the solution is: $N_2 = k_1 N_0 t e^{-k_2\cdot t}$).

The result (2.6) is relatively easy to interpret, since the first two factors are the number of (1) nuclei, which have decayed to (2) nuclei, but have not yet decayed, and the last factor is the law of decay for the (2) nuclei.

If the chain of decay is longer than three nuclei a solution to the differential equations can in principle be found in the same manner, as one should just replace the expression for $N_1(t)$ with the expression for $N_2(t)$ in the differential equation for $N_3(t)$.

Solutions of the type (2.6) can be applied to determine the age of radioactive materials.
In praxis we know the two decay constants k_1 and k_2 together with the ratio N_2/N_1.
Then the following equation (2.7) can be applied to find the time t which has elapsed since the material N_1 was created. This has been one of the first reliable methods to determine the correct age of the earth.

(2.7) $$\frac{N_2}{N_1} = \frac{N_0 \frac{k_1}{k_2 - k_1}\left(e^{(k_2-k_1)t} - 1\right)e^{-k_2 t}}{N_0 e^{-k_1 t}} = \frac{k_1}{k_2 - k_1}\left(1 - e^{(k_1-k_2)t}\right)$$

If $k_2 > k_1$ then: $$\frac{N_2}{N_1} \rightarrow \frac{k_1}{k_2 - k_1} \quad \textit{for} \quad t \rightarrow \infty$$